ON THE PROJECTION OF TWO TRIANGLES ON TO THE SAME TRIANGLE

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Abstract.

There are in the theory of projection many constructions for the solution of a problem which contain one or more arbitrary elements. In such cases interesting results are sometimes obtainable by a consideration of the aggregate of all the possible solutions of a problem.

A simple example is the construction for determining the correspondence of two homographic point rows when three points of the one and the three corresponding points of the other are given.

In this case the line joining two centres of projection, from which both point rows may be projected on to the same point row, touches a conic.

The problem considered in this paper is the projection of two triangles on to the same triangle, and the results which may be deduced from a consideration of the aggregate of all the possible solutions.

We are not aware that the same problem has been dealt with by any previous writer. Grassmann in the fourth section of his paper entitled "Die stereometrischen Gleichungen dritten Grades, und die dadurch erzeugten Oberflächen" (which section deals with the projective relation between planes and pencils of rays in space) obtains incidentally a construction for projecting any four points in one plane by three projections and three sections on to any four points in another plane (no three of the four points in either plane being in a straight line). Other writers, of whom we believe the first was Seydewitz, discussed the distinct problem of finding the number of ways in which a plane containing any four fixed points can be moved in space and so placed on another plane that the four fixed points are in perspective with four fixed points in the second plane (no three of the four points in either plane being in a straight line). The number of solutions turns out to be four, and is in contrast with the number—six—of solutions of some of the problems which occur in this

paper. But the problem discussed by Grassmann is of a higher order of complexity than the one from which the present investigation arises. The present investigation leads to a construction which solves Grassmann's problem in a way which at one step is of a more general character than that adopted by Grassmann; and its value consists in the light which it throws on that problem.

The first result that is here obtained is this:—The straight line joining two points from which two triangles in space can be projected on to the same triangle is a generator of the regulus of which the three directing lines are the lines joining corresponding vertices of the triangles (Art. 1).

It is next shown that the two triangles can be projected into the same triangle in any plane in space (Art. 2).

From this the construction for the solution of the problem considered by Grassmann, and referred to above, follows easily (Art. 3).

It is then proved that the projective relation between the planes of the two triangles is unaltered if the centres of projection be moved to other positions on the same generator of the regulus referred to in the first article (Art. 4), and in this case the plane on to which the two triangles are projected passes through a fixed point on the line of intersection of the planes of the two triangles. The fixed point is a self-corresponding point of the planes of the two triangles (Art. 5).

A construction for the determination of this fixed point, when the position of the generator of the regulus is known, is then given (Art. 6).

The next two articles (7 and 8) deal with special positions of the plane on to which the two triangles are projected, which are important for the drawing of the figures required.

The regulus of Art. 1 meets each of the planes of the triangles in a conic. The vertices of one triangle and the point in which the generator joining the centres of projection meets one of the conics correspond to the vertices of the other triangle and the point in which the generator joining the centres of projection meets the other conic. In general no fifth point of either conic can correspond to any point of the other conic. If any fifth point on the first conic correspond to a point on the second conic, then all the points on the first conic correspond to points on the second conic, and the projective relation between the two planes is that which is determined by making those points on the conics correspond which lie on the same generator of the regulus (Art. 9).

It is next shown that, if three points A_1, B_1, C_1 in one plane correspond to three points A_2, B_2, C_2 on another plane, then, as the straight line joining the centres of projection describes the regulus, the point D_2 in the second plane which corresponds to a fixed point D_1 in the first plane describes a nodal cubic passing through A_2, B_2, C_2 (Arts. 10 and 11).

Arts. 12–14 are concerned with the construction of the node and the drawing of the cubic, and the way in which the node arises is explained.

In the last article (15) it is shown that the least number of projections and sections necessary to pass from one set of four points in a plane to another set of four points in a second plane is *in the general case* three. There are, of course, the special cases in which two or one of each are sufficient.

It follows that Grassmann's solution is the simplest.

1. If $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles situated anywhere in space, and if O_1 and O_2 be two points such that O_1 projects $A_1B_1C_1$ and O_2 projects $A_2B_2C_2$ into the same triangle ABC, to prove that the sole condition which must be satisfied by O_1 and O_2 is that O_1O_2 is a generator of the regulus determined by the three directing lines A_1A_2, B_1B_2, and C_1C_2.

Since O_1A_1 meets O_2A_2 at A, therefore the points O_1, A_1, O_2, A_2, A all lie in one plane.

Therefore O_1O_2 meets A_1A_2.

Similarly O_1O_2 meets B_1B_2 and C_1C_2.

Hence O_1O_2 is a generating line of the regulus of which A_1A_2, B_1B_2, and C_1C_2 are the directing lines.

Conversely, if O_1 and O_2 lie on a generating line of the regulus of which A_1A_2, B_1B_2 and C_1C_2 are the directing lines, then O_1O_2 meets A_1A_2, and therefore O_1A_1 meets O_2A_2 in a point which may be called A; similarly O_1B_1 meets O_2B_2 in a point which may be called B, and O_1C_1 meets O_2C_2 in a point which may be called C. Therefore O_1 projects $A_1B_1C_1$ and O_2 projects $A_2B_2C_2$ into the same triangle ABC.

Throughout this paper A_1A_2, B_1B_2, and C_1C_2 will be said to belong to the first system of the generators of the hyperboloid on which the above mentioned regulus lies; and O_1O_2 will be said to belong to the second system of generators, and will be sometimes denoted by the single letter g. The intersection of the two planes $A_1B_1C_1$ and $A_2B_2C_2$ will be called the line i.

2. The triangles $A_1B_1C_1$, $A_2B_2C_2$ having any position in space, and any plane π being chosen arbitrarily, then it is in general possible to find O_1 and O_2 so as to project $A_1B_1C_1$ and $A_2B_2C_2$ into a triangle ABC situated in the plane π.

Let the straight lines B_1C_1, C_1A_1, A_1B_1, B_2C_2, C_2A_2, A_2B_2 meet the plane π in the points J_1, K_1, L_1, J_2, K_2, L_2 respectively. (See Fig. 1.)

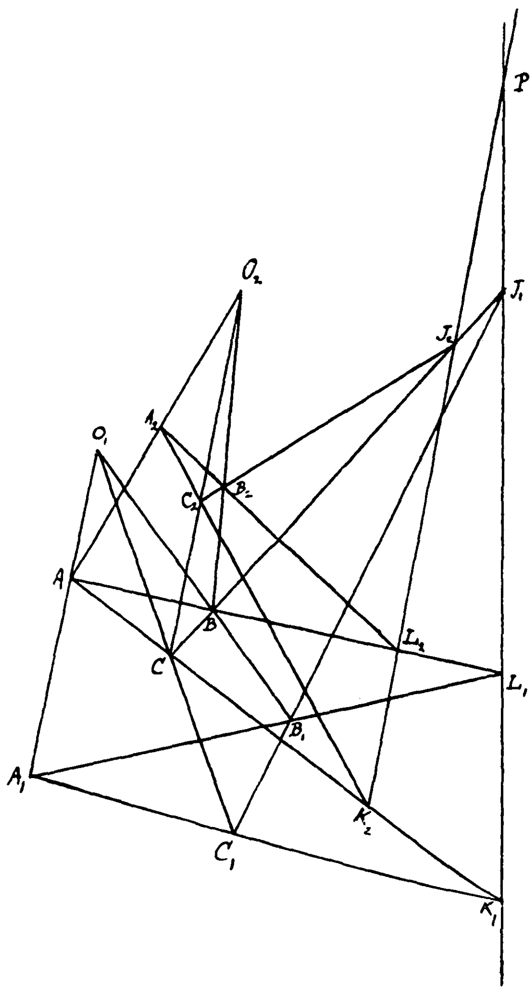

FIG. 1.

The most general case is that in which each of these six points is a single fully determined point and all the six points are distinct.

This case will be taken first.

The points J_1, K_1, L_1 are collinear, because they lie on the intersection of the planes $A_1B_1C_1$ and π.

Similarly J_2, K_2, L_2 are collinear.

The straight lines $J_1K_1L_1$ and $J_2K_2L_2$ are in general distinct.

Since $J_1K_1L_1$ and $J_2K_2L_2$ are in one plane π, therefore K_1K_2 will meet L_1L_2 in a point which may be called A.

Similarly let L_1L_2 meet J_1J_2 at B, and let J_1J_2 meet K_1K_2 at C.

Then BC is the line J_1J_2, and therefore meets B_1C_1 at J_1.

Similarly CA meets C_1A_1 at K_1, and AB meets A_1B_1 at L_1.

Since J_1, K_1, and L_1 are collinear, therefore the triangles ABC and $A_1B_1C_1$ are in perspective.

Therefore AA_1, BB_1, and CC_1 meet in some point O_1.

Similarly AA_2, BB_2, and CC_2 meet in some point O_2.

The positions of O_1 and O_2 have therefore been found which will project $A_1B_1C_1$ and $A_2B_2C_2$ into a triangle ABC in the plane π.

3. The construction given in the last article renders it possible to determine the projective relation between two planes when four points in the one and the corresponding four points in the other are given. [It will be supposed that no three of either set of four points lie on one straight line.]

The construction about to be given is in the case of one step a little more general than that given by Grassmann (*Crelle*, T. XLIX., S. 55)

Suppose that A_1, B_1, C_1, D_1 in the plane π_1 correspond to A_2, B_2, C_2, D_2 in the plane π_2.

Take any point O_2 (distinct from A_1 and A_2) on the straight line A_1A_2.

Through A_1 draw any plane π_3 (distinct from π_1).

From O_2 project $A_2B_2C_2D_2$ into $A_1B_3C_3D_3$ on π_3.*

Draw any plane π (distinct from π_1 and π_3) through A_1. Then, by means of Art. 2, find the centres O_1' and O_3' which will project $B_1C_1D_1$ and $B_3C_3D_3$ into the same triangle BCD in the plane π.

Since the point A_1 is common to the planes π_1, π_3, and π, it follows that the projections from O_1' and O_3' will both leave it unaltered in position.

Thus O_2 projects the points A_2, B_2, C_2, D_2 situated on π_2 into A_1, B_3, C_3, D_3 on π_3.

Then O_3' projects the points A_1, B_3, C_3, D_3 situated on π_3 into A_1, B, C, D on π.

Then O_1' projects A_1, B, C, D on π into A_1, B_1, C_1, D_1 on π_1.

There are, therefore, three projections and three sections, as in Grassmann's method, necessary to establish the projective relation between the two planes when four points in one plane and the corresponding four points in the other plane are given.

* Grassmann's next step is to draw a plane through A_1B_1. It is possible, however, to proceed more generally, as in this paper, by drawing any plane through A_1.

4. The projective relation between the planes of the two triangles is fully determined when the position of the generator g, on which O_1 and O_2 lie, has been fixed; in other words, if O_1 and O_2 be moved to other positions along g, the point E_2 in the plane $A_2B_2C_2$ which corresponds to a fixed point E_1 in the plane $A_1B_1C_1$ remains unaltered.

Let g meet the planes ABC, $A_1B_1C_1$, and $A_2B_2C_2$ at D, D_1, and D_2 respectively. Then O_1 projects A_1, B_1, C_1, D_1 into A, B, C, D, which are projected from O_2 on to A_2, B_2, C_2, D_2.

Hence A_1, B_1, C_1, D_1 in the one plane correspond to A_2, B_2, C_2, D_2 in the other plane for all positions of O_1, O_2 on the same generator g. Now, when four points in the one plane forming a quadrilateral and the corresponding points in the other plane are given, the projective relation between the two planes is determined by the theorem of which a proof is given in the preceding article.

Hence the point E_2 in the plane $A_2B_2C_2$ corresponding to a fixed point E_1 in the plane $A_1B_1C_1$ will not be altered if O_1 and O_2 are moved along the fixed generator g.

[It must be noticed, however, that the plane ABC is altered in position when O_1 and O_2 are moved along the fixed generator g.]

5. To prove that, when O_1 and O_2 lie on a fixed generator g, the plane ABC passes through a fixed point on the line of intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$, and that this is a self-corresponding point of the two planes.

Suppose that, for some given positions of O_1 and O_2 on g, the straight lines $J_1K_1L_1$ and $J_2K_2L_2$, both of which lie in the plane ABC, meet at P (see Fig. 1). Then P is the intersection of the three planes ABC, $A_1B_1C_1$, and $A_2B_2C_2$.

Then O_1 projects P in the plane $A_1B_1C_1$ into P in the plane ABC, and then O_2 projects P in the plane ABC into P in the plane $A_2B_2C_2$.

Therefore P in the plane $A_1B_1C_1$ corresponds to P in the plane $A_2B_2C_2$.

Now, when the generator g is fixed, the projective relation between the two planes is determined.

Hence P in the plane $A_1B_1C_1$ corresponds to P in the plane $A_2B_2C_2$ for all positions of O_1 and O_2 on g.

Hence the plane ABC passes through the point P for all positions of O_1 and O_2 on g.

The point P may be called the self-corresponding point of the two planes when O_1 and O_2 lie on g.

The point P in the intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$ will be said to correspond to the generator g.

6. To show how to construct the position of P when that of g is known.
Let PA_1 meet B_1C_1 at Q_1. (See Fig. 2.)
Let O_1Q_1 meet BC at Q.
Let O_2Q meet B_2C_2 at Q_2.
Then Q_1 in the plane $A_1B_1C_1$ corresponds to Q_2 in the plane $A_2B_2C_2$.
Also O_1Q_1 and O_2Q_2 meet at Q. Therefore Q_1Q_2 meets O_1O_2.
Also A_1A_2 meets O_1O_2.
Again, P, Q_1, A_1 correspond to P, Q_2, A_2 respectively.
Now P, Q_1, A_1 are collinear; therefore P, Q_2, A_2 are collinear.
Therefore A_1A_2 meets Q_1Q_2.
We have also proved that A_1A_2 meets O_1O_2, and Q_1Q_2 meets O_1O_2.

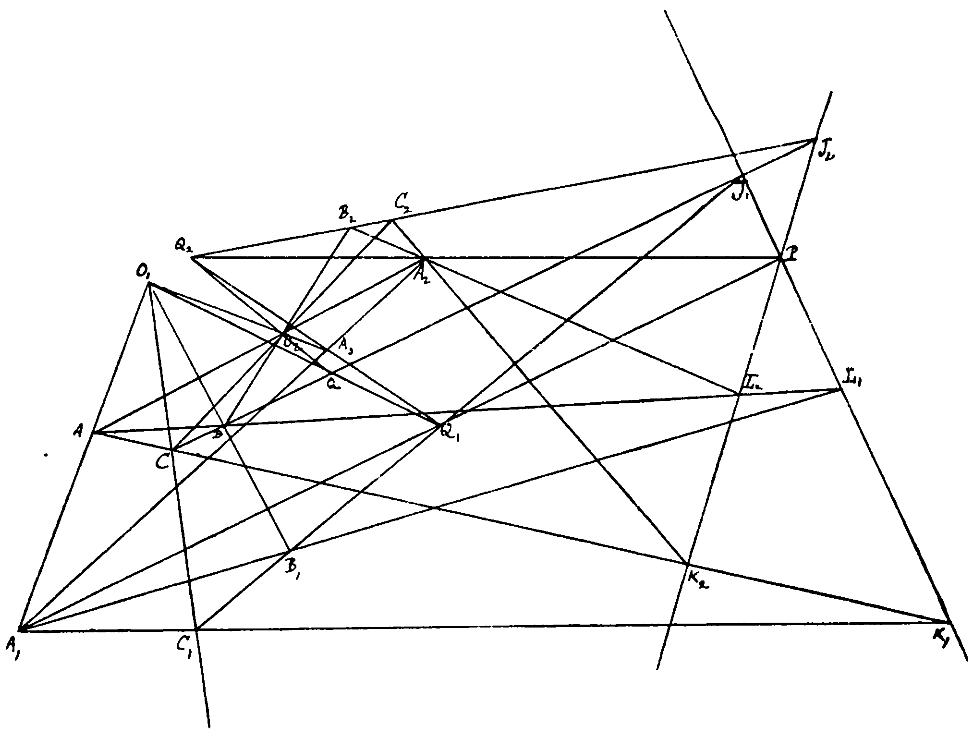

FIG. 2.

Hence there are two alternatives:—

(a) The three straight lines A_1A_2, O_1O_2, and Q_1Q_2 all lie in one plane.
In this case P lies in the plane which contains $O_1O_2A_1A_2$.

Similarly it could be proved that P lies in the plane $O_1O_2B_1B_2$ and in the plane $O_1O_2C_1C_2$.

But these planes meet in O_1O_2.

Hence P lies on O_1O_2, which cannot be generally true.

(b) The lines A_1A_2, O_1O_2, and Q_1Q_2 have one point in common.

This being the only alternative left, we take it to hold.

Let A_1A_2 meet O_1O_2 at A_3.

Therefore Q_1Q_2 passes through A_3.

The line $A_3Q_1Q_2$ meets O_1O_2, B_1C_1, and B_2C_2.

This gives the following construction for the point P:—

Let A_1A_2 meet the generator g, *i.e.* O_1O_2, at A_3. Through A_3 draw the single straight line which meets both B_1C_1 and B_2C_2. Let it meet B_1C_1 at Q_1, and B_2C_2 at Q_2. Then A_1Q_1 and A_2Q_2 meet at P.

Hence $A_3Q_1Q_2$ is a generator of the regulus which has B_1C_1, B_2C_2, and A_1A_2 for directing lines.

Hence the range of A_3 on A_1A_2 is projective to the range of Q_2 on B_2C_2.

Now project the range of Q_2 on B_2C_2 from A_2 on to the line of intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$, *i.e.*, on the line i.

Then the range of Q_2 on B_2C_2 is projective with the range of P on the line i.

Therefore the range of A_3 (*i.e.*, the point in which the selected generator g is met by the fixed line A_1A_2) on the fixed line A_1A_2 is projective with the range of P on the line i.

7. It is necessary now to return to the exceptional case mentioned in Art. 2, viz., that in which the planes $A_1B_1C_1$ and $A_2B_2C_2$ meet the plane π in the same straight line, *i.e.*, $J_1K_1L_1$ and $J_2K_2L_2$ coincide. In this case there are the following classes of cases:—

(a) J_1 and J_2 do not coincide, K_1 and K_2 do not coincide, and L_1 and L_2 do not coincide.

This is the most general case of this kind, and it arises whenever the plane π is drawn through the intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$.

In this case the straight lines J_1J_2, K_1K_2, and L_1L_2 coincide with the line of intersection of the planes; so that the triangle ABC degenerates into a straight line coinciding with the line of intersection of the two planes.

If in this case points O_1 and O_2 exist such that O_1A_1 meets O_2A_2 at A, O_1B_1 meets O_2B_2 at B, and O_1C_1 meets O_2C_2 at C, where A, B, C all lie on the line of intersection of the planes, then A_1A lies in the plane $A_1B_1C_1$, and therefore O_1 also lies in this plane.

Similarly O_2 lies in the plane $A_2B_2C_2$.

As in the general case, O_1O_2 meets A_1A_2, B_1B_2, and C_1C_2. Consequently $O_1 O_2$ is a generator of the regulus determined by $A_1 A_2$, $B_1 B_2$, and $C_1 C_2$.

Hence O_1 is any point on the conic in which this regulus meets the plane $A_1 B_1 C_1$; and O_2 is the point in which the generator g of the second system through O_1 meets the plane $A_2 B_2 C_2$.

Hence in this case the triangles can (in an infinite number of ways) be projected into the same triad of points lying on the line of intersection of the two planes.

[Although O_1 is in the plane $A_1 B_1 C_1$ and O_2 is in the plane $A_2 B_2 C_2$, the points on the plane $A_1 B_1 C_1$ correspond uniquely to the points on the plane $A_2 B_2 C_2$; because the projective relation between the planes is the same as when O_1 and O_2 have any other positions on the generator g.]

(*b*) Let J_1 and J_2 coincide, say at J.

Let K_1 and K_2 not coincide, and let L_1 and L_2 not coincide. (See Fig. 3.)

In this case $K_1 K_2$, which is AC, and $L_1 L_2$, which is AB, coincide with the line i.

The point B is the intersection of $J_1 J_2$ with $L_1 L_2$; hence it is at J.

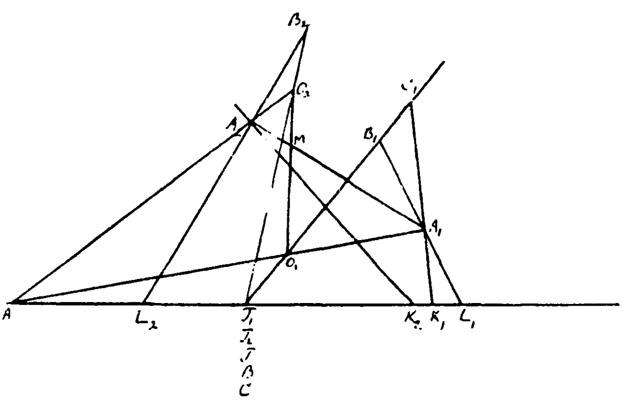

FIG. 3.

The point C is the intersection of $J_1 J_2$ with $K_1 K_2$; hence it also is at J.

The straight line BC, which is $J_1 J_2$, goes through J, but is not otherwise limited; but B and C are not separate points on it. They coincide at J.

In this case the straight lines $B_1 C_1$ and $B_2 C_2$ meet, viz., at J.

Hence $B_1 B_2$ and $C_1 C_2$ meet, and therefore in this case there is no proper regulus* determined by $A_1 A_2$, $B_1 B_2$, and $C_1 C_2$.

Again O_1 lies on BB_1, *i.e.*, on $JB_1 C_1$.

Also O_2 lies on BB_2, *i.e.*, on $JB_2 C_2$.

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* It degrades into a plane. All its generating lines lie in the plane $B_1 C_1 B_2 C_2$ and pass through the point in which $A_1 A_2$ meets this plane. By taking O_1 and O_2 on any generating line it is possible to project $A_1 B_1 C_1$ and $A_2 B_2 C_2$ into the same triangle.

Further O_1O_2 meets A_1A_2.

If, therefore, A_1A_2 meet the plane $C_1B_1JB_2C_2$ in M, and any line be drawn through M in this plane meeting B_1C_1 in O_1 and B_2C_2 in O_2, we obtain positions of O_1, O_2 which project $A_1B_1C_1$ and $A_2B_2C_2$ into three points A, B, C on the line i; but two of these points, B and C, coincide with each other.

In this case there is no proper projection, *i.e.*, so long as the plane ABC is drawn through the line i.

(*c*) J_1 and J_2 coincide, say at J; K_1 and K_2 coincide, say at K; but L_1 and L_2 do not coincide.

Then BC, which is J_1J_2, goes through J, but is not otherwise limited.

Also AC, which is K_1K_2, goes through K, but is not otherwise limited.

Whilst AB is the line L_1L_2, which is the same as JK.

Hence A is at K, B is at J, and C is any point in the plane π.

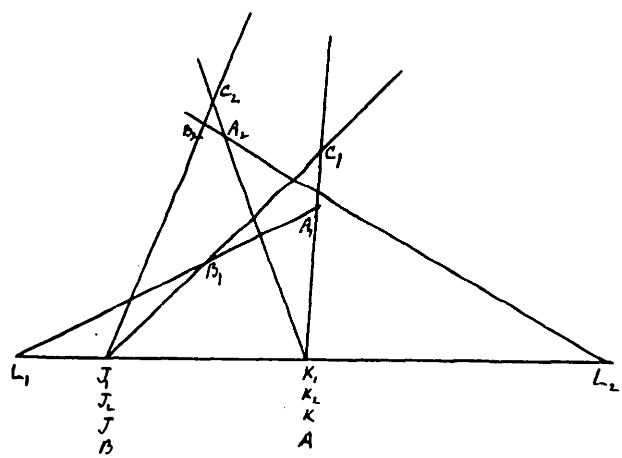

FIG. 4.

In this case AA_1, BB_1, and CC_1 meet at C_1.

Hence O_1 is at C_1, and there is no proper projection of the plane $A_1B_1C_1$ from O_1.

Also AA_2, BB_2, and CC_2 meet at C_2. Hence O_2 is at C_2.

(*d*) J_1 and J_2 coincide, K_1 and K_2 coincide, and L_1 and L_2 coincide.

In this case the triangles are in perspective, and the points O_1 and O_2 coincide at the centre of perspective.

8. It is necessary now to consider a particular case of another kind.

In the preceding cases the six points J_1, K_1, L_1, J_2, K_2, and L_2 are all fully determined. Let us now suppose that the straight lines $J_1K_1L_1$ and $J_2K_2L_2$ are distinct, but let the point J_2 in which B_2C_2 meets the plane π be not fully determined.

This is equivalent to the case in which the plane π is drawn through B_2C_2; and then J_2 can be regarded as any point on B_2C_2. (See Fig. 5.)

Now C_2A_2 meets π at K_2. Therefore K_2 is at C_2.

Also A_2B_2 meets π at L_2. Therefore L_2 is at B_2.

The lines A_1A, B_1B, C_1C meet at O_1.

The lines A_2A, B_2B, C_2C which meet at O_2 are now A_2A, L_2B, K_2C respectively.

These lines meet at A; therefore O_2 is at A. But O_1 is on AA_1; therefore $O_1A_1O_2$ is a straight line; therefore the generator O_1O_2 passes through A_1.

In this case there is no proper projection. For the plane ABC passes through O_2, which is at A.

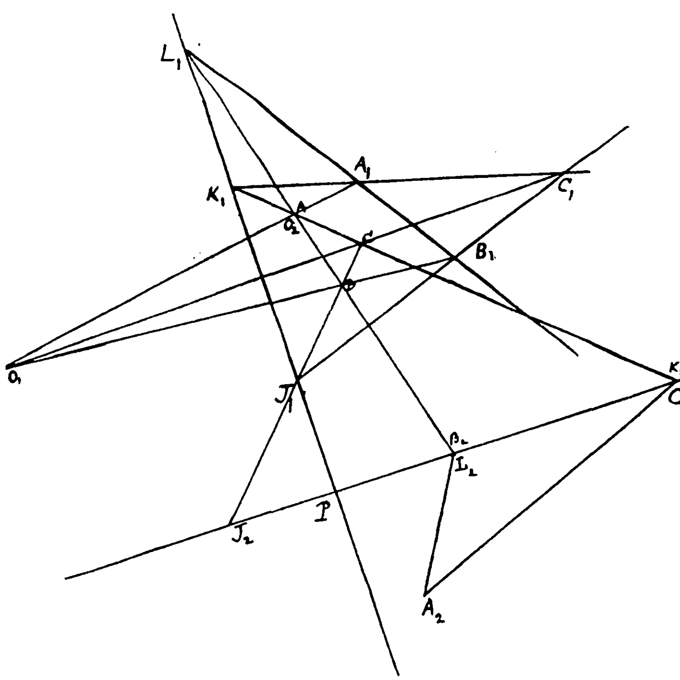

FIG. 5.

The distinguishing feature of this case is that the generator g passes through A_1.

Conversely, let g pass through A_1. Then the straight line O_1A_1 is g. In the general case O_1A_1 meets O_2A_2 at A.

Hence in this special case g meets O_2A_2 at A. But g meets O_2A_2 at O_2. Hence O_2 coincides with A.

Now in the general case $A_2B_2C_2$ is projected from O_2 on to ABC. But in this special case O_2 is the point A in the plane ABC.

In this case there is no proper projection.

Note that in this case, since BC projects into B_2C_2, the plane ABC is the same as the plane $O_2B_2C_2$, and it therefore passes through B_2C_2.

Those points of the plane $A_2B_2C_2$ which are situated on B_2C_2 are projected on to BC, and thence on to B_1C_1.

This would result in making the point in which B_2C_2 meets the line i not a self-corresponding point unless J_2 is taken at that point. This will therefore be supposed to be done in the rest of this section.

All points on the plane $A_2B_2C_2$ not on B_2C_2 are projected from O_2 (which is A) on to A, and thence from O_1 on to A_1.

Hence when g passes through A_1 all points on $A_2B_2C_2$ which are not on B_2C_2 correspond to A_1. The points on B_2C_2 correspond to points on B_1C_1.

It is required to determine the self-corresponding point when the generator g passes through A_1.

The self-corresponding point is the intersection of $J_1K_1L_1$ and $J_2K_2L_2$. Now in this case $J_2K_2L_2$ is B_2C_2. Hence the self-corresponding point lies on B_2C_2. It also lies on the intersection of the two planes $A_1B_1C_1$ and $A_2B_2C_2$. Hence it is the point in which B_2C_2 meets the line of intersection of the two planes.

Hence, if the generator g pass through A_1, the self-corresponding point P is the point in which B_2C_2 meets the line of intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$.

It should be noticed that in this case the points B_1, C_1, and P which are not in a straight line correspond to B_2, C_2, and P which are in a straight line and all points on $A_2B_2C_2$ which are not on B_2C_2 correspond to A_1. This confirms what was shown before, that there is no proper projection in this case.

There are six cases of this class: viz., when g passes through A_1, B_1, C_1, A_2, B_2, C_2 the self-corresponding points are those in which the line of intersection of the two planes meets the lines B_2C_2, C_2A_2, A_2B_2, B_1C_1, C_1A_1, A_1B_1 respectively.

Although in these six cases there is no proper projection, it is possible to make the relation between the two planes quite definite in each case. As the self-corresponding point P moves along the line of intersection i of the two planes to each point E_1 in one plane corresponds a definite point E_2 in the other plane for every position of P, except when it takes up one of the six exceptional positions noted above. As P approaches one of these exceptional positions E_2 approaches a perfectly definite position, and this definite position may be said to correspond to E_1 when P occupies the exceptional position. That this is so follows from Art. 10 below.

9. The regulus whose directing lines are A_1A_2, B_1B_2, C_1C_2 cuts the two planes $A_1B_1C_1$, $A_2B_2C_2$ in two conics.

If g meet these planes at D_1 and D_2, then A_1, B_1, C_1, D_1 on the one conic correspond to A_2, B_2, C_2, D_2 respectively on the other conic.

It will now be proved :—

(i.) That *in general* no fifth point of the first conic can correspond to a point on the second conic.

(ii.) That, if any fifth point on the first conic correspond to a point on the second conic, then all the points on the first conic correspond to points on the second conic, and the projective relation between the two planes is that which is determined by making those points on the conics correspond which lie on the same generator of the regulus.

To prove (i.), suppose, if possible, that a point E_1 of the first conic corresponds to a point E_2 of the second conic.

The most general supposition that can be made is that E_1 and E_2 are distinct points. This will be taken first.

O_1E_1 meets O_2E_2 in a point E of the plane $ABCD$.

Hence E_1E_2 meets O_1O_2.

Hence E_1E_2 has in common with the hyperboloid three distinct points, viz., E_1, E_2, and the point where E_1E_2 meets O_1O_2.

It must therefore be a generator of the same system as A_1A_2, B_1B_2, and C_1C_2.

Now O_1 projects $A_1B_1C_1D_1E_1$ and O_2 projects $A_2B_2C_2D_2E_2$ into $ABCDE$.

Draw the conics through $ABCDE$, $A_1B_1C_1D_1E_1$, and $A_2B_2C_2D_2E_2$.

Take any point F on the conic $ABCDE$.

Let O_1F cut the first plane at F_1; then F_1 is on the conic $A_1B_1C_1D_1E_1$.

Let O_2F cut the second plane at F_2; then F_2 is on the conic $A_2B_2C_2D_2E_2$.

Also F_1 and F_2 correspond to each other.

As F may be any point on the conic $ABCDE$, it follows that every point on the first conic corresponds to a point on the second conic, and the line joining two corresponding points is a generator of the first system.

This conclusion is in the general case impossible, because D_1D_2, which joins corresponding points, is a generator of the second system.

The exceptional cases are (*a*) when D_1D_2 is a generator of both systems, *i.e.*, the hyperboloid is a cone; (*b*) when D_1 and D_2 coincide in one point, so that D_1 and D_2 both lie on a generator of each system.

Putting aside for the present these exceptional cases, it follows that only the points $A_1B_1C_1D_1$ on the one conic correspond to points on the other conic, viz., to $A_2B_2C_2D_2$.

In the exceptional case (*a*) the points O_1 and O_2 coincide with the vertex of the cone and the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are in perspective.

The second exceptional case (*b*) is important. It brings us, in fact, to the case marked (ii.) above.

Suppose that the hyperboloid meets the line of intersection of the two planes at the points T and U. Then T and U lie on both the conics. Suppose that g is taken to pass through T. Then D_1 and D_2 coincide at T. In this case $A_1B_1C_1T$ corresponds to $A_2B_2C_2T$. Therefore the projective relation between the two planes is fully determined. It will now be proved that U corresponds to U. Let the rays of the regulus containing A_1A_2, B_1B_2, and C_1C_2 which pass through T and U be called t and u respectively. Then the points T, A_1, B_1, C_1 on one conic fall on the rays t, A_1A_2, B_1B_2, C_1C_2 of the regulus. The conic and the regulus are therefore in perspective, each ray of the regulus cutting the conic in the point corresponding to it.

In like manner the regulus and the second conic are in perspective. Hence the point U of the first conic corresponds to the generator u of the regulus, and this corresponds to the point U of the second conic. Therefore U corresponds to U.

It remains only to find the position of the self-corresponding point when g passes through T. It will be proved that this is at U.

For, if the plane ABC do not pass through U when g passes through T, then U will not correspond to itself.

Hence, when g passes through T, the self-corresponding point is U.

Also, when g passes through U, the self-corresponding point is T.

Further, these two positions of g determine the same projective relation between the two planes, viz., all the points on the first conic correspond to points on the second conic, corresponding points lying on generators of the first system.

10. Given that A_1, B_1, C_1 correspond to A_2, B_2, C_2, to prove that as g describes the regulus the point D_2 corresponding to a fixed point D_1 in the plane $A_1B_1C_1$ describes a cubic curve, having a double point and passing through A_2, B_2, C_2.

Suppose that, in one position of g, P is the self-corresponding point.

Let PA_1 meet B_1C_1 at Q_1; let PA_2 meet B_2C_2 at Q_2.

Then Q_1Q_2 meets A_1A_2 at A_3, where A_3 is the point at which g meets A_1A_2.

Let PA_1 meet B_1D_1 at R_1 and C_1D_1 at S_1.

Since A_1, B_1, C_1, P correspond to A_2, B_2, C_2, P respectively, therefore

A_1P, B_1C_1 correspond to A_2P, B_2C_2 respectively; therefore Q_1, the intersection of A_1P, B_1C_1, corresponds to Q_2, the intersection of A_2P, B_2C_2.

Since A_1, Q_1, P correspond to A_2, Q_2, P, the ranges on A_1Q_1P and A_2Q_2P are in perspective, and the centre is A_3 (see Fig. 6).

Hence, to get the points corresponding to R_1, S_1, it is only necessary to make A_3R_1 meet A_2P at R_2, and A_3S_1 meet A_2P at S_2.

Then, since D_1 is the intersection of B_1R_1 and C_1S_1, therefore D_2 is the intersection of B_2R_2 and C_2S_2.

As P moves along the line of intersection of the two planes, Q_1, R_1, S_1 describe respectively the straight lines B_1C_1, B_1D_1, and C_1D_1.

Q_2 describes the fixed line B_2C_2.

Projecting from the fixed point A_1, $(P)\barwedge(Q_1)\barwedge(R_1)\barwedge(S_1)$.*

Projecting from the fixed point A_2, $(P)\barwedge(Q_2)$; therefore $(Q_1)\barwedge(Q_2)$.

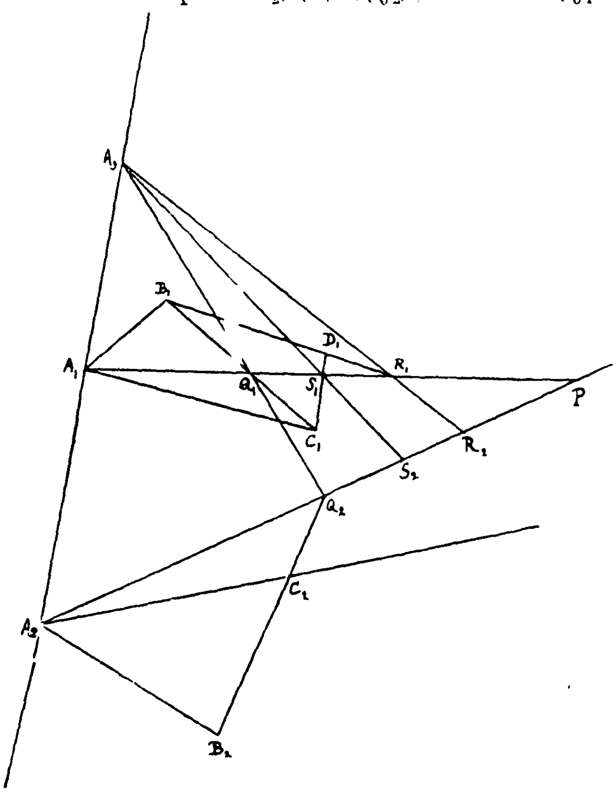

FIG. 6.

Hence Q_1Q_2 describes a regulus, having B_1C_1, B_2C_2, and A_1A_2 for directing lines; therefore $(Q_1)\barwedge(Q_2)\barwedge(A_3)$; therefore $(A_3)\barwedge(R_1)$.

* The symbol $\barwedge$ is used to denote "projective with" and $(P)\barwedge(Q_1)$ means "any four positions of P are projective with the four corresponding positions of Q_1."

Hence A_3R_1 is a generator of a regulus having A_1A_2 and B_1D_1 for directing lines.

This regulus meets the plane $A_2B_2C_2$ in a conic on which R_2 lies. This conic passes through A_2, because A_2 is a possible position of A_3. It also passes through the point where B_1D_1 meets the line of intersection of the two planes.

Similarly, $(A_3)\barwedge(S_1)$; therefore A_3S_1 is a generator of a regulus which has A_1A_2 and C_1D_1 for directing lines.

This regulus meets the plane $A_2B_2C_2$ in a conic on which S_2 lies. This conic passes through A_2. It also passes through the point where C_1D_1 meets the line of intersection of the two planes.

There are, therefore, two conics in the plane $A_2B_2C_2$, both passing through A_2. The points R_2 and S_2 on these conics are projectively related by the lines drawn from A_2. Each such line meets one conic in R_2 and the other in S_2.

The point D_2 is the intersection of B_2R_2 and C_2S_2.

If, now, we draw the sheaf of planes of the second order, whose centre is B_2, and which passes through the generating lines $A_3R_1R_2$, it is projective to the regulus generated by $A_3R_1R_2$, and therefore to the points R_1.

Similarly, the sheaf of planes of the second order, whose centre is C_2, and which passes through the generating lines $A_3S_1S_2$, is projective to the regulus generated by $A_3S_1S_2$, and therefore to the points S_1, and therefore to the points R_1, and therefore to the sheaf of planes of the second order whose centre is B_2 and which passes through the generating lines $A_3R_1R_2$.

Now the surface generated by the intersections of corresponding planes of two projective sheaves of planes of the second order is, *in general*, a ruled surface of the fourth order having a nodal line, which is a curve in space of the third order (Reye, *Geometrie der Lage* (1886), Zweite Abtheilung, Fünfzehnter Vortrag, S. 115).

The section by any plane is, therefore, a curve of the fourth order with three double points.

In this case, however, there is a simplification.

It will now be proved that the two projective sheaves of planes have a self-corresponding plane. The ruled surface of the fourth order will then reduce to this plane and a cubic surface of the third order.

Consider what happens when the self-corresponding point P is at the point where B_2C_2 meets the line of intersection of the planes $A_1B_1C_1$ and $A_2B_2C_2$. Then (Art. 8) g passes through A_1, and therefore A_3 is at A_1. In this case both R_2 and S_2 coincide with P.* Thus the locus of R_2 is a

* It should be observed that, when R_2 and S_2 coincide at the point where B_2C_2 meets the line of intersection of the two planes, it does not follow that D_2, which is the intersection of B_2R_2 and

conic through A_2 and the point where B_2C_2 meets the line of intersection of the two planes. This is the point J_2. In like manner the locus of S_2 is a conic through A_2 and J_2. Since R_1 is on A_1P, both R_1 and R_2 are on A_1P, and therefore the plane $B_2R_1R_2$ is the plane B_2A_1P which contains C_2, since P is on B_2C_2. It is therefore the plane $A_1B_2C_2$. In like manner the plane $C_2S_1S_2$ is the plane $A_1B_2C_2$. Hence the two sheaves of planes of the second order $B_2R_1R_2$ and $C_2S_1S_2$ have a self-corresponding plane.

Hence the ruled surface of the fourth order reduces to this plane and a cubic surface.

Hence the locus of the intersection of B_2R_2 and C_2S_2 is a cubic curve.

Since the locus of R_2 is a conic through A_2 and the locus of S_2 a conic through A_2, therefore A_2 is a point on the cubic locus.

By symmetry B_2 and C_2 lie on the cubic locus.

Hence the locus of D_2 is a cubic curve through A_2, B_2, C_2.

It has a double point at the point which corresponds to D_1 when the projective relation is that which is determined by making the regulus containing A_1A_2, B_1B_2, C_1C_2 perspective to the two conics in which it meets the two planes. For, if the regulus meet the line of intersection of the two planes in T and U, then we get the same projective relation between the planes when P is at T as we do when P is at U. Hence the same point will correspond to D_1 when P is at T as when P is at U. This point will therefore be a double point on the cubic locus.

11. We might also have completed the proof thus:—The conic which is the locus of R_2 and the conic which is the locus of S_2 intersect at A_2. The straight line B_2C_2 passes through one of the intersections of these two conics, viz., the point J_2. The points A_2, R_2, S_2 are on a straight line. Therefore R_2 on its conic is projectively related to S_2 on its conic. So the pencils B_2R_2 and C_2S_2 are projectively related, and they have a self-corresponding ray, viz., B_2C_2 (for when P coincides with the point in which B_2C_2 meet the line of intersection of the two planes R_2 and S_2 coincide with P). Hence the locus of the intersection of B_2R_2 and C_2S_2 is a cubic curve. Hence the locus of D_2 is a cubic curve. (Reye, Holgate's Translation, Art. 205; or the German edition, Erste Abtheilung, S. 137.)

12. In order to draw the cubic, a further examination of the loci of R_2 and S_2 is necessary.

It will be proved in the first place that the locus of R_2 touches the locus of S_2 at A_2.

C_2S_2, also coincides with this point, for B_2R_2 and C_2S_2 have in this case the same direction, and the limiting point of their intersection may be different from the point at which B_2C_2 meets the line of intersection of the planes.

The locus of R_2 is the section of the hyperboloid generated by $A_3 R_1 R_2$ by the plane $A_2 B_2 C_2$.

The tangent plane to the hyperboloid at A_2 is the plane which contains $A_1 A_2$ and the position of $A_3 R_1 R_2$ when A_3 reaches A_2.

But, when A_3 is at A_2, P is at the point where $B_1 C_1$ cuts the line i. This is the point we have previously called J_1.

Hence R_1 is at the intersection of $A_1 J_1$ and $B_1 D_1$, say at R_1'.

Hence the tangent plane at A_2 is the plane containing $A_1 A_2$ and the line joining A_2 to the intersection of $A_1 J_1$ and $B_1 D_1$. But this is the plane $A_1 A_2 J_1$.

Hence the tangent at A_2 to the locus of R_2 is $A_2 J_1$.

By similar reasoning the tangent at A_2 to the locus of S_2 is also $A_2 J_1$.

Hence the two conics touch at A_2.

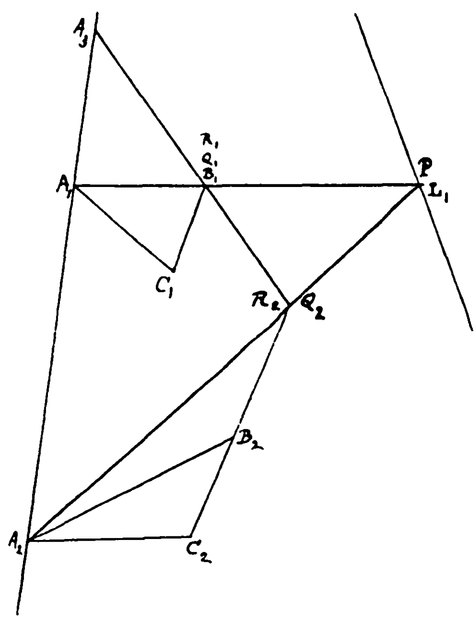

FIG. 7.

The points in which $B_2 C_2$ meets each conic will next be found.

Take P (see Fig. 7) where $A_1 B_1$ meets the line i, *i.e.*, take P at L_1.

Then PA_1, *i.e.*, $A_1 L_1$, meets $B_1 C_1$ at Q_1, and $B_1 D_1$ at R_1 in the general case.

Hence Q_1 and R_1 coincide with B_1.

The point Q_2 is the intersection of $B_2 C_2$ and $A_2 L_1$.

Hence $Q_1 Q_2$ is $B_1 Q_2$, and this meets $A_1 A_2$ in A_3.

Now $A_3 R_1$ meets $A_2 P$ in R_2 in general.

But R_1 coincides with Q_1.

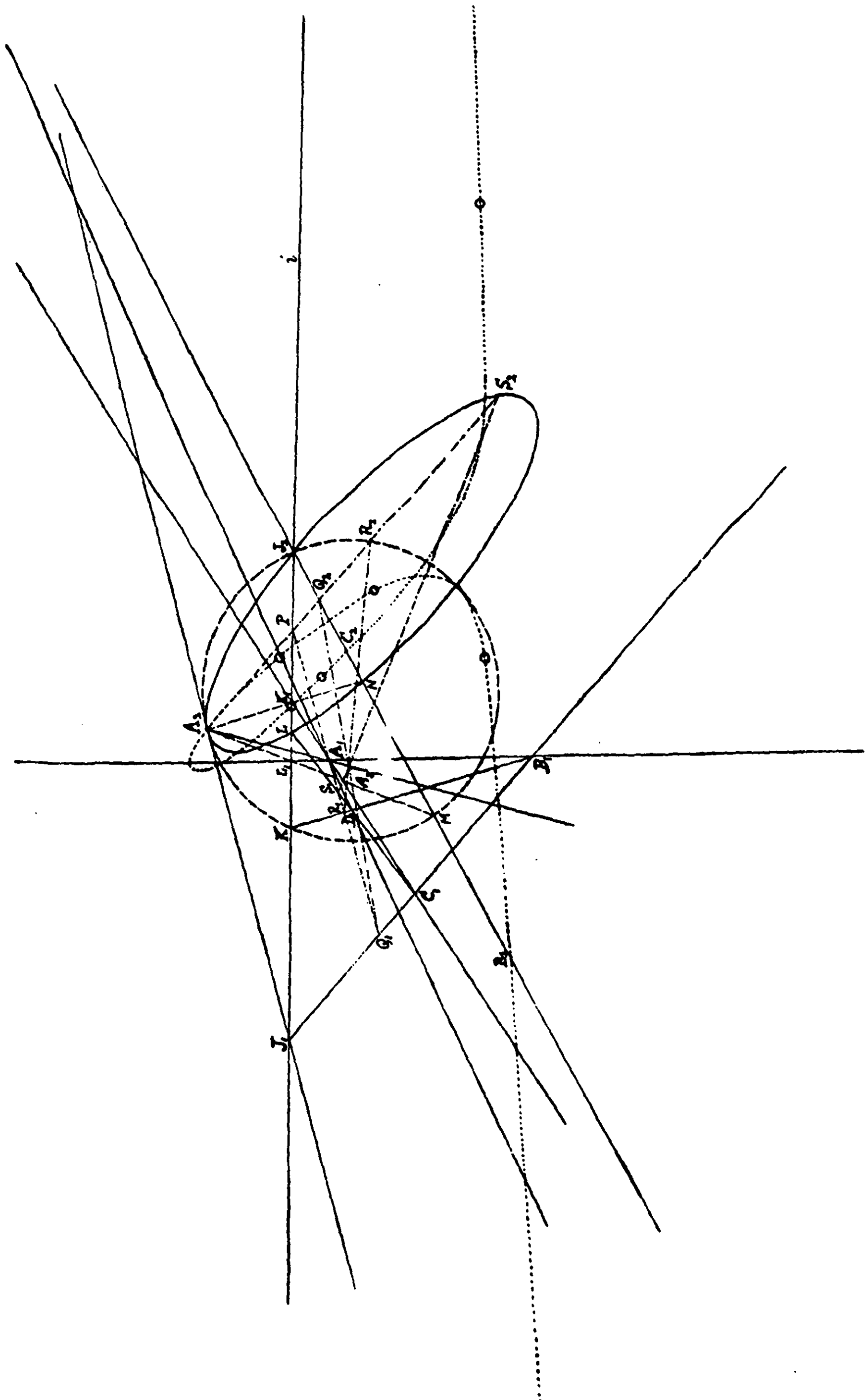

FIG. 8.

Consequently R_2 coincides with Q_2.

Hence one position of R_2 is the intersection of A_2L_1 and B_2C_2.

Similarly one position of S_2 is the intersection of A_2K_1 and B_2C_2.

The loci of R_2 and S_2 are now completely determined.

(1) Each locus passes through A_2 (Art. 10).

(2) Each locus is touched by A_2J_1 at A_2 (Art. 12).

(3) Each locus passes through J_2 (Art. 10).

(4) The locus of R_2 passes through the intersection of B_1D_1 and the line i (Art. 10). Call this the point K. (See Fig. 8.) The locus of S_2 passes through the intersection of C_1D_1 and the line i (Art. 10). Call this the point L.

(5) The locus of R_2 passes through the intersection of B_2C_2 and A_2L_1. Call this point M.

The locus of S_2 passes through the intersection of B_2C_2 and A_2K_1. Call this point N.

13. The conics are determined when the positions of A_1, B_1, C_1, D_1, A_2, B_2, C_2 and the line i are given.

But the converse is not true.

The conics are determined when the positions of J_1, K_1, L_1 on the line i; the position of J_2 on the line i; the direction of the straight line B_2C_2 through J_2 (the positions of B_2 and C_2 on this line need not be known); the position of A_2 and the position of D_1 are known.

If the positions of the points are assumed arbitrarily, it is often very difficult to obtain a sufficient length of the cubic to show its characteristics.

It is, however, possible to construct the cubic in a convenient manner by first drawing two conics having a real point of contact and two real intersections, and then to construct positions of A_1, B_1, C_1, D_1, A_2, B_2, C_2, and i which will lead to these two conics.

Draw two conics touching at A_2 and having two real intersections. Call either of these J_2.

Draw two arbitrary lines through J_2. Call one of these i, and the other B_2C_2 (B_2 and C_2 may be anywhere on this line).

Let $J_2B_2C_2$ meet the locus of R_2 in M and the locus of S_2 in N.

Let the line i meet the locus of R_2 in K and the locus of S_2 in L. Then on referring to Art. 12 it is seen that it is necessary to take J_1 at the intersection of the common tangent at A_2 to the conics with the line i, K_1 at the intersection of A_2N with i, L_1 at the intersection of A_2M with i.

It is possible now to draw any three lines through J_1, K_1, L_1, viz., $J_1B_1C_1$, $K_1C_1A_1$, $L_1A_1B_1$, which will give the positions of A_1, B_1, C_1.

Then D_1 must be taken at the intersection of B_1K and C_1L.

It will now be proved that the positions assigned to A_1, B_1, C_1, D_1, A_2, the line i, and the line B_2C_2 lead uniquely to the conics selected.

For (1) both conics pass through A_2;

(2) both conics touch A_2J_1;

(3) both conics pass through J_2;

(4) the locus of R_2 passes through the point K, which is the intersection of B_1D_1 and i, and the locus of S_2 passes through the point L, which is the intersection of C_1D_1 and i;

(5) the locus of R_2 passes through the point M, which is the intersection of B_2C_2 and A_2L_1; and the locus of S_2 passes through N, which is the intersection of B_2C_2 and A_2K_1.

Hence the conics are uniquely determined by the positions assigned to A_1, B_1, C_1, D_1, A_2 and the lines B_2C_2 and i.

In order to draw the cubic, all that is necessary is to draw the conics; then any line through A_2 meets the conics in corresponding points R_2 and S_2.

Then B_2R_2 and C_2S_2 meet on the cubic.

14. *Construction for the Double Point of the Cubic.*

Take any point R_2 on the locus of R_2; let A_2R_2 cut the locus of S_2 in S_2.

Let B_2R_2 cut the locus of R_2 in R_2', and let A_2R_2' cut the locus of S_2 in S_2'.

Then, if S_2S_2' passes through C_2, the double point of the cubic is the intersection of R_2R_2' and S_2S_2'.

For call this point of intersection δ. Then, when $A_2R_2S_2$ occupies its first position, B_2R_2 and C_2S_2 intersect at δ, and therefore δ is a point on the curve.

Now let $A_2R_2S_2$ move into the position $A_2R_2'S_2'$. Then B_2R_2' and C_2S_2' intersect at δ, and therefore the curve passes a second time through δ, and therefore δ is the double point of the cubic.

It is necessary to find a construction for the double point δ.

Draw any straight line through A_2 meeting the conics at R_2 and S_2. Then $(R_2) \barwedge (S_2)$.

Let B_2R_2 meet the locus of R_2 in R_2'. Then the points R_2, R_2' are corresponding points of an involution on the locus of R_2. Therefore $(R_2) \barwedge (R_2')$.

Let A_2R_2' cut the locus of S_2 in S_2'. Then $(R_2') \barwedge (S_2')$. Therefore $(S_2) \barwedge (S_2')$.

The points S_2' are determined uniquely from the points S_2, and the construction is reversible.

Hence the points S_2 and S_2' form an involution on the conic.

Hence $S_2 S_2'$ always passes through a fixed point W. (See Fig. 9.)

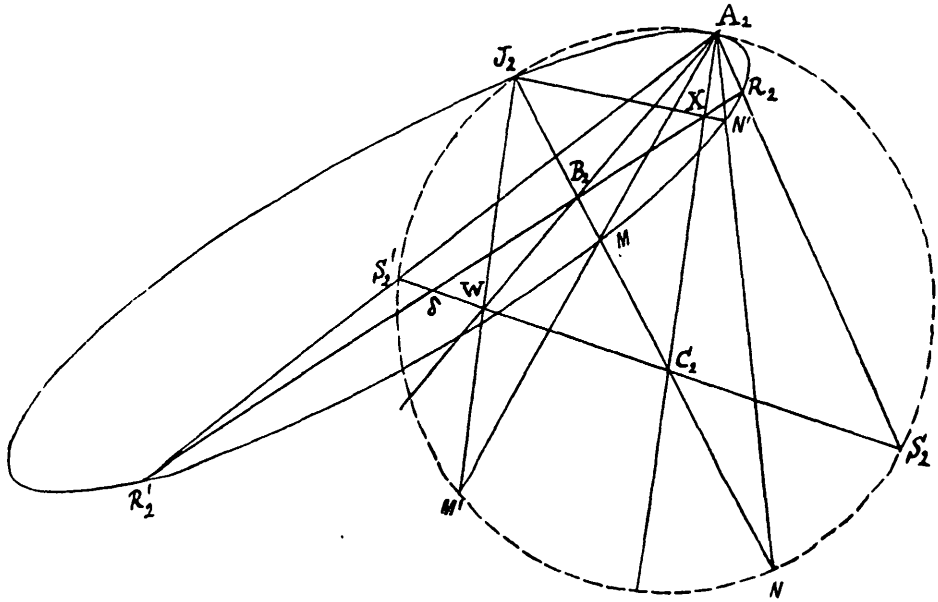

FIG. 9.

Hence $C_2 W$ will meet the locus of S_2 in corresponding points S_2 and S_2', such that the line joining the corresponding positions of R_2 and R_2' will pass through B_2. Then $R_2 R_2'$ meets $C_2 W$ in the double point δ.

This construction can be completed if $C_2 W$ meets the locus of S_2 in real points, but, if $C_2 W$ meets the locus of $S_2 S_2'$ in imaginary points, then it is not immediately apparent how to construct the corresponding positions R_2, R_2'.

It has, however, been shown that the double point δ lies on the straight line $C_2 W$.

It is now possible to get another line which passes through δ thus:— Draw through C_2 any straight line meeting the locus of S_2 in S_2, S_2'. Let $A_2 S_2$ and $A_2 S_2'$ cut the locus of R_2 in R_2 and R_2'. Then R_2, R_2' are corresponding points of an involution on the locus of R_2.

Therefore $R_2 R_2'$ passes through a fixed point X (see Fig. 9), and the double point lies on $B_2 X$.

The double point is therefore the intersection of the fully determined straight lines B_2X and C_2W, and the construction is one which does not fail when C_2W cuts the locus of S_2 in imaginary points.

To find the position of W, take S_2 at A_2. Then R_2 is also at A_2.

Therefore R_2' is the other point in which A_2B_2 cuts the locus of R_2, and S_2' is the other point in which A_2B_2 cuts the locus of S_2. Hence in this position S_2S' coincides with A_2B_2.

Hence W lies on A_2B_2.

Next take S_2 at J_2. Then R_2 is also at J_2. Then R_2' is at the other point in which B_2J_2, *i.e.*, B_2C_2, cuts the locus of R_2. Call this point M. Then A_2M cuts the locus of S_2 in S_2'. Call this point M'. Then S_2S_2' is now in the position J_2M'. Hence W lies on J_2M', and, as it also lies on A_2B_2, it is fully determined.

Similarly X can be determined. In the first place, by taking R_2 at A_2, S_2 coincides with A_2. Therefore S_2' is the other point in which A_2C_2 cuts the locus of S_2, and R_2' is the other point in which A_2C_2 cuts the locus of R_2. Hence in this position R_2R_2' coincides with A_2C_2. Hence X lies on A_2C_2.

Take R_2 at the point J_2 in which B_2C_2 meets both conics. Then S_2 is also at J_2. Then S_2' is at the other point where C_2J_2, *i.e.*, B_2C_2, meets the locus of S_2. Call this point N. Then R_2' is at the other point in which A_2Z meets the locus of R_2. Call this point N'. Hence R_2R_2' is now J_2N'. Hence X lies on J_2N', and also on A_2C_2. It is therefore fully determined.

15. It has been shown that the points A_1, B_1, C_1 may be projected on to the points A_3, B_3, C_3 in an infinite number of ways by two projections and two sections. The locus of the point D_3 in the second plane corresponding to a fixed point D_1 in the first plane has been shown to be a cubic.

Hence it is in general impossible to project four arbitrary points A_1, B_1, C_1, D_1 in one plane into four arbitrary points A_3, B_3, C_3, D_3 in another plane by only two projections and two sections.

Hence it follows that the fewest number of projections and sections necessary to project four arbitrary points A_1, B_1, C_1, D_1 in one plane on to four arbitrary points A_3, B_3, C_3, D_3 in another plane is three, which is the number given by Grassmann.

Suppose now that a third plane is drawn, and on it three points A_2, B_2, C_2 are taken arbitrarily.

Now let $A_1B_1C_1$ be projected by two projections and two sections on to $A_2B_2C_2$. Then the locus of the projection of D_1 will be a cubic through A_2, B_2, C_2.

Now let $A_3B_3C_3$ be projected by two projections and sections on to $A_2B_2C_2$. Then the locus of the projection of D_3 is another cubic through A_2, B_2, C_2. These two cubics intersect on $A_2B_2C_2$ and six other points.

Let D_2 be one of these six points. Then $A_1B_1C_1D_1$ are projected by two projections and two sections into $A_2B_2C_2D_2$, and $A_2B_2C_2D_2$ are projected into $A_3B_3C_3D_3$ in the same way.

Hence A_1, B_1, C_1, D_1 can be projected into $A_3B_3C_3D_3$ by four projections and four sections in six ways, as there are six positions of D_2.

This case is illustrated by a comparison of Figs. 8 and 10.

Fig. 8 shows the projection of A_1, B_1, C_1, D_1 on to A_2, B_2, C_2, D_2; whilst Fig. 10 shows the projection of A_3, B_3, C_3, D_3 on to the same four points A_2, B_2, C_2, D_2. There are then six points any one of which may be taken as D_2. These are marked on each figure by a circle, but are not lettered, and the two figures 8 and 10, which are drawn separately, must be superposed. The points A_1, B_1, C_1, D_1, J_1, K_1, L_1 of Fig. 8 correspond to A_3, B_3, C_3, D_3, J_3, K_3, L_3 of Fig. 10 respectively.

In Fig. 11 the position of the points is such that the cubic has a cusp. This is due to the fact that corresponding positions of B_2R_2, C_2S_2 simultaneously touch the conics. This figure was constructed by drawing a ray $A_2R_2S_2$. Then tangents were drawn to the conics at R_2 and S_2 to meet the line taken for B_2C_2 in B_2, C_2. The intersection of the tangents gives the cusp at δ.

In Fig. 12 the position of the points is such that the cubic has an acnode at δ. The figure is obtained thus:—The conics and the positions of J_2 and B_2 are first selected. J_2B_2 meets the locus of R_2 in M, A_2M meets the locus of S_2 in M', J_2M' meets A_2B_2 in W. Then C_2 is so chosen that C_2W does not cut the locus of S_2 in real points. Then X and therefore δ are determined. T and U are conjugate imaginary points on the line of intersection of the two planes.

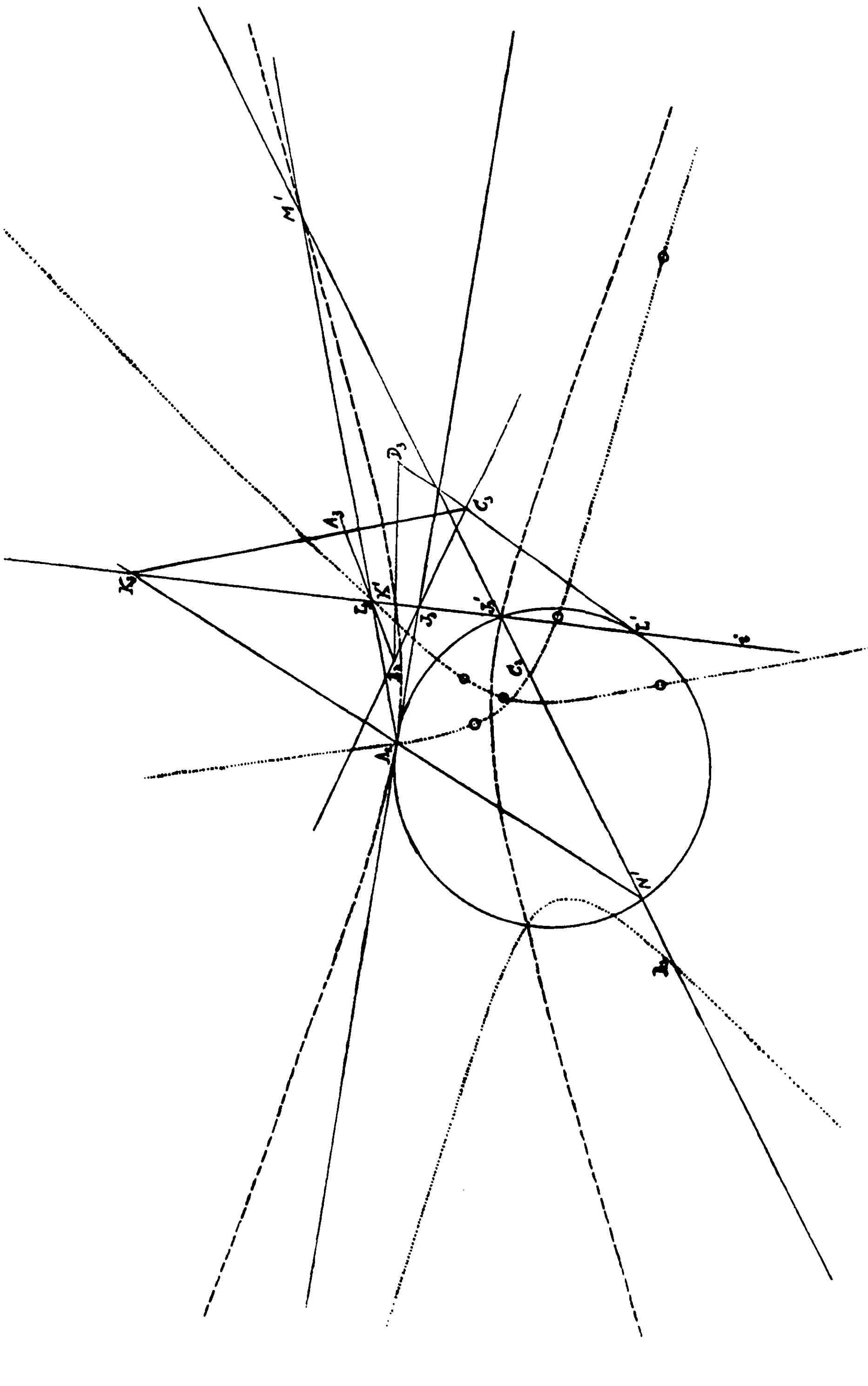

FIG. 10.

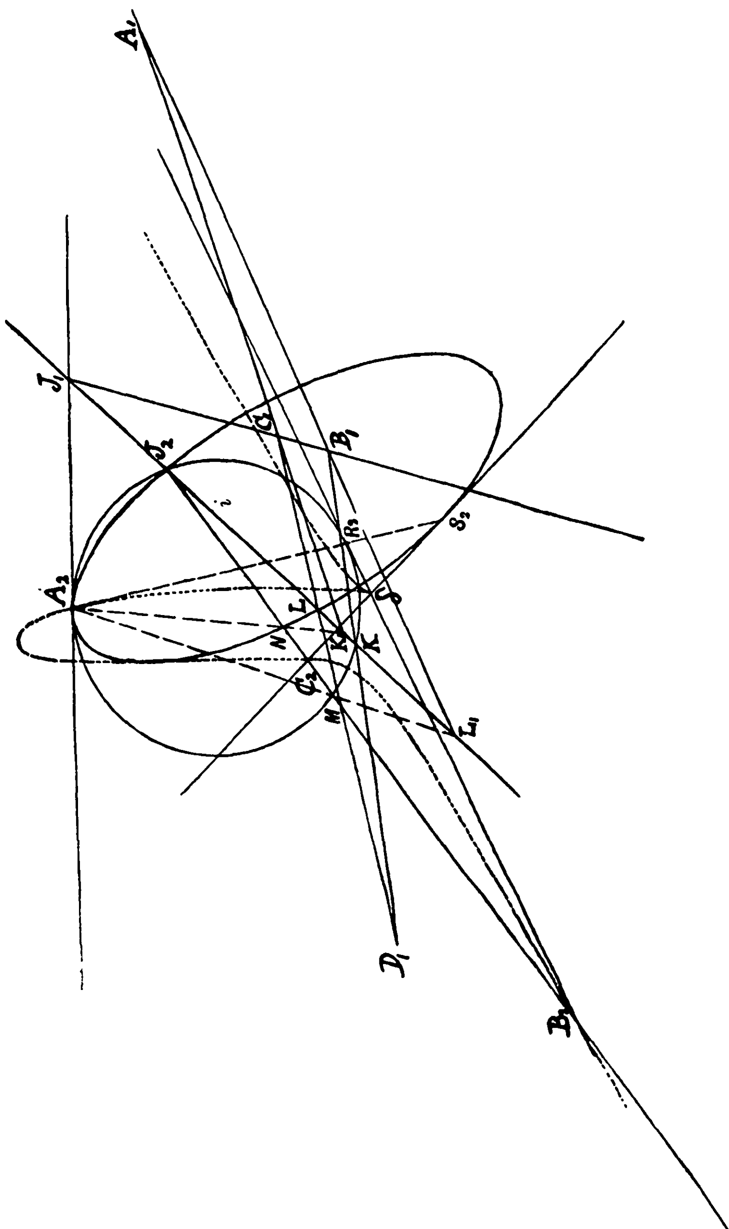

FIG. 11.

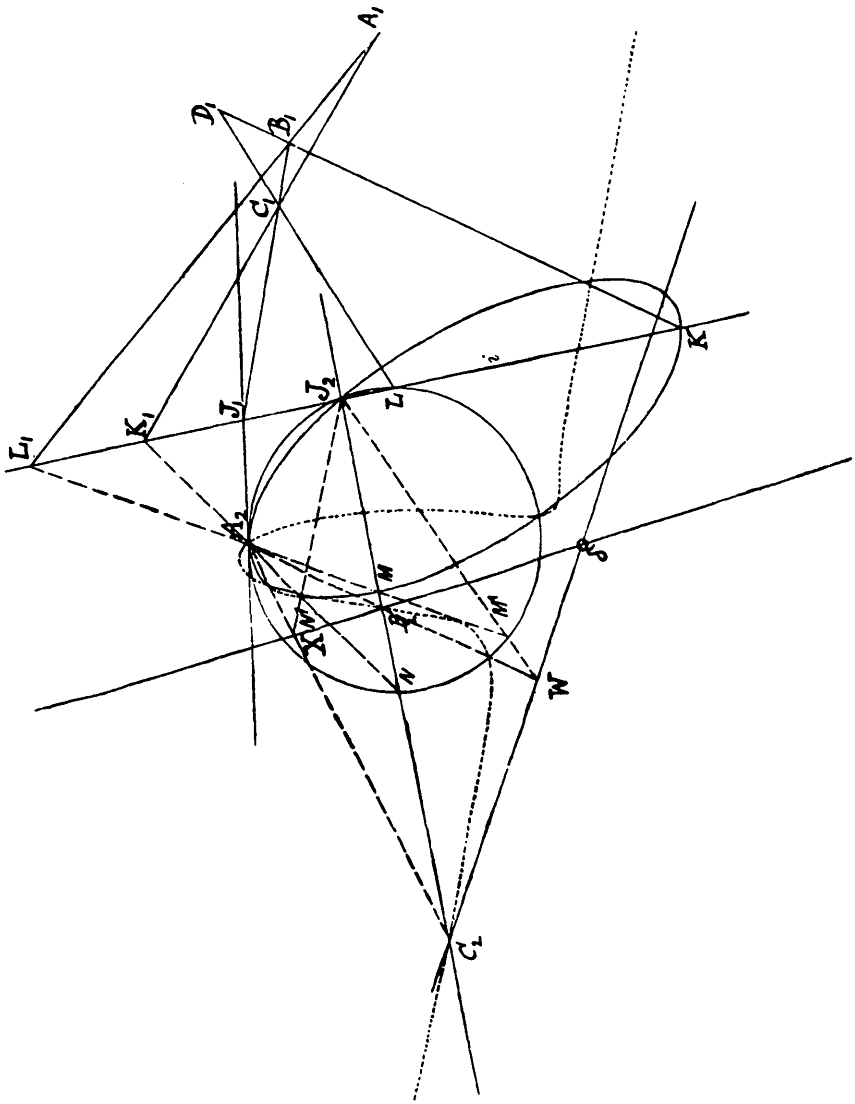

FIG. 12.