## *On the Motion of a Plane under Certain Conditions.*

By S. Roberts, M.A.

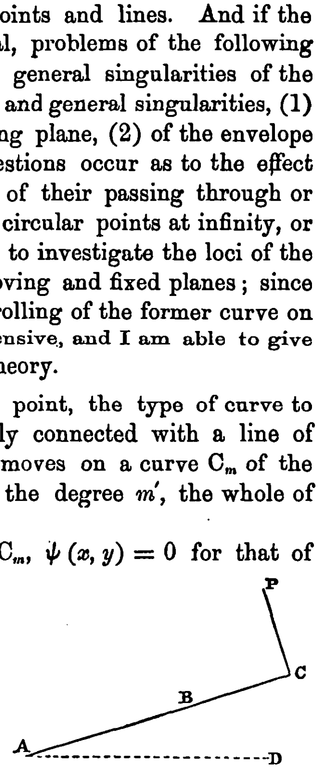
### *Point directrices*

1. Imagine an indefinite plane inclined at an invariable angle to a fixed plane, in which two of the points of the former plane describe given curves. Every point of the moving plane will describe a curve lying in a plane parallel to the fixed plane, and the general trajectory will be the locus of a point rigidly connected with a line of constant length moving with its extremities on the given curves as directrices. With reference to space of three dimensions, the motion constitutes a very restricted case, and the moving plane envelopes a developable.* But if the angle of inclination is evanescent, the motion is that of a plane in plane space, determined by the motions of two points of the plane, and is therefore general.

The primary elements of a plane are points and lines. And if the directrices above mentioned are algebraical, problems of the following kind arise. Given the degree, class, and general singularities of the directrices, to determine the degree, class, and general singularities, (1) of the trajectory of any point of the moving plane, (2) of the envelope of any right line in the plane. Other questions occur as to the effect of the directrices becoming identical, and of their passing through or being otherwise specially related to the circular points at infinity, or to the line at infinity. It is also desirable to investigate the loci of the instantaneous centres relative to the moving and fixed planes; since the motion may also be generated by the rolling of the former curve on the latter. The subject is evidently extensive, and I am able to give little more than the commencement of a theory.

2. With reference to the trajectory of a point, the type of curve to be considered is the locus of a point rigidly connected with a line of constant length, one extremity of which moves on a curve $C_m$ of the degree $m$, and the other on a curve $C_{m'}$ of the degree $m'$, the whole of the conditions relating to plane space.

Write $\phi(x, y) = 0$ for the equation of $C_m$, $\psi(x, y) = 0$ for that of $C_{m'}$. Let $x_1, y_1$ be the coordinates of the point A on $C_m$; $x_2, y_2$ those of the point B on $C_{m'}$; $x, y$ those of P, the describing point. Also let PC (perpendicular to AB at C) $= c$, AC$=a$, BC$=b$, $\angle$BAD (the angle which AB makes with the axis of $x$) $= \theta$.




* The chief characteristics of this developable can be determined by means of the results hereafter obtained.

Then, the axes being rectilinear, we have

$$x_1 = x - a\cos\theta + c\sin\theta,$$
$$y_1 = y - a\sin\theta - c\cos\theta,$$
$$x_2 = x - b\cos\theta + c\sin\theta,$$
$$y_2 = y - b\sin\theta - c\cos\theta.$$

Consequently, by the conditions

$$\phi(x_1, y_1) = 0, \quad \psi(x_2, y_2) = 0,$$

the problem is reduced to the elimination of $\theta$ from

$$\left.\begin{array}{l} \phi\,(x - a\cos\theta + c\sin\theta,\ \ y - a\sin\theta - c\cos\theta) = 0 \\ \psi\,(x - b\cos\theta + c\sin\theta,\ \ y - b\sin\theta - c\cos\theta) = 0 \end{array}\right\} \ldots\ldots \ (1).$$

In fact, these two equations may be taken to represent the curve.

We may also write the equivalent system,

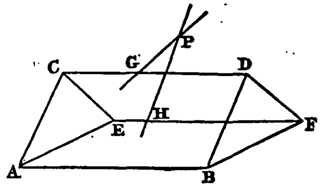
$$\left.\begin{array}{c} \phi\,(x - a\mathrm{X} + c\mathrm{Y},\ \ y - a\mathrm{Y} - c\mathrm{X}) = 0 \\ \psi\,(x - b\mathrm{X} + c\mathrm{Y},\ \ y - b\mathrm{Y} - c\mathrm{X}) = 0 \\ \mathrm{X}^2 + \mathrm{Y}^2 = 1 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots \ (2).$$

3. It is worth while to note an immediate consequence of the forms of the equations (1).

The quantities $a\cos\theta - c\sin\theta$, $a\sin\theta + c\cos\theta$ may be taken as coordinates of a point of the circle $x^2 + y^2 = a^2 + c^2$, and in like manner the quantities $b\cos\theta - c\sin\theta$, $b\sin\theta + c\cos\theta$ may be taken as coordinates of a point of the circle $x^2 + y^2 = b^2 + c^2$. The angle between the radii drawn from the common centre of the circles to two corresponding points (corresponding, that is to say, to the same value of $\theta$) remains constant, while $\theta$ varies. In fact, the angle is equal to the angle subtended by the moving line AB at P the describing point.

Hence, we have the following result :—If, in a given position of the radii, the curve $C_m$ be translated so that the point (in its plane and carried with it) which was at the origin may move to the extremity of the radius whose length is $\sqrt{a^2+c^2}$, and in like manner he curve $C_{m'}$ be translated so that the point (in its plane and carried with it) which was at the origin may move to the extremity of the radius whose length is $\sqrt{b^2+c^2}$; and if then the radii be moved round the centre, maintaining their constant angle, while the curves $C_m$, $C_{m'}$ are made to move without rotation, the locus of the intersections of $C_m$, $C_{m'}$ in successive corresponding positions will be a curve of the kind in question.

To take a simple instance which can readily be verified, let ABCDEF represent a double parallel ruler, the bar AB being fixed, while CD, EF move parallel to one another and to AB. The angles CAE, DBF are equal and constant; in fact, we have two equal and similar tri-



angles ACE, BDF moving about A, B as centres, instead of the usual simple radii.

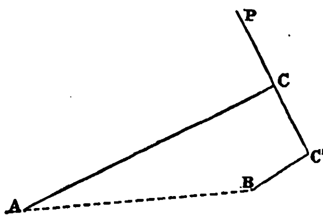
Then, if GP is a right line rigidly attached to CD, and HP is a right line rigidly attached to EF, the locus of P, their intersection, is an ellipse. We have here the locus of a point rigidly connected with a limited line which moves between two linear directrices. If we replace GP, HP by curves, we have the general locus.

It will be observed that the general forms of the equations of two curves, translated in the manner above described, are

$$\left.\begin{array}{l}\phi\,(x-a\cos\theta+c\,\sin\theta,\;\; y-a\sin\theta-c\cos\theta)=0\\ \psi\,(x-b\cos\theta+c'\sin\theta,\;\; y-b\sin\theta-c'\cos\theta)=0\end{array}\right\}\;\ldots\ldots\;(3).$$

These equations correspond to the following construction:—

Let A be a point on $C_m$, B a point on $C_{m'}$. From A, B draw two parallel lines AC, BC′ = $a$, $b$ respectively, and so that the line PCC′ is perpendicular to both; then $x_1$, $y_1$ being, as before, the coordinates of A; $x_2$, $y_2$ those of B; $x$, $y$ those of P; and writing $c$ for PC, $c'$ for PC′, and $\theta$ for the angle made by AC and BC′ with the axis of $x$; we have




$$\begin{aligned} x_1 &= x-a\cos\theta+c\sin\theta,\\ y_1 &= y-a\sin\theta-c\cos\theta,\\ x_2 &= x-b\cos\theta+c'\sin\theta,\\ y_2 &= y-b\sin\theta-c'\cos\theta. \end{aligned}$$

Since the equations (3) represent, therefore, the locus of a point P rigidly connected with AB, which is constant, they are not intrinsically more general than the system (1).

4. If there are three equations homogeneous in $x$, $y$, $z$, and of the forms

$$\begin{aligned} &A\,z^m+(B\,x+C\,y)\,z^{m-1}+(D\,x^2+E\,xy+F\,y^2)\,z^{m-2}+\&c.=0,\\ &A'z^n+(B'x+C'y)\,z^{n-1}+(D'x^2+E'xy+F'y^2)\,z^{n-2}+\&c.=0,\\ &A''z^p+(B''x+C''y)\,z^{p-1}+(D''x^2+E''xy+F''y^2)\,z^{p-2}+\&c.=0, \end{aligned}$$

where A, A′, A″ are of the orders $\mu$, $\nu$, $\pi$ in uneliminated variables, B, C; B′, C′; B″, C″ of the orders $\mu+\alpha$, $\mu+\beta$; $\nu+\alpha$, $\nu+\beta$; $\pi+\alpha$, $\pi+\beta$, &c., so that the orders of the coefficients proceed in arithmetical progression, the order of the resultant of the system is (Salmon's Higher Algebra, 2nd edition, p. 281)

$$mnp\left\{\frac{\mu}{m}+\frac{\nu}{n}+\frac{\pi}{p}+\alpha+\beta\right\};$$

referring to the system (2), we may put $\mu=m$, $n=\nu=m'$, and $\alpha=\beta=-1$. Also, with respect to the third equation, we may consider it of the type $p = \pi = 2$, $\alpha = \beta = -1$. Hence, as a major limit of the degree of the locus, we have $2mm'$; and this degree is attained in the case of two directrices made up of right lines, since the locus then consists of $mm'$ conics.

5. In order to determine the class, we obtain the number of tangents parallel to a given line.

Writing $k$, a constant, for $\frac{dy}{dx}$, and substituting $\phi$, $\psi$ for the full expression of the functions, we have the equations

$$\left.\begin{aligned} \frac{d\phi}{dx}+k\frac{d\phi}{dy} + \left\{\frac{d\phi}{dx}(a\sin\theta+c\cos\theta)+\frac{d\phi}{dy}(-a\cos\theta+c\sin\theta)\right\}\frac{d\theta}{dx} &= 0 \\ \frac{d\psi}{dx}+k\frac{d\psi}{dy} + \left\{\frac{d\psi}{dx}(b\sin\theta+c\cos\theta)+\frac{d\psi}{dy}(-b\cos\theta+c\sin\theta)\right\}\frac{d\theta}{dx} &= 0 \end{aligned}\right\} (4);$$

or, eliminating $\frac{d\theta}{dx}$, and considering $x$, $y$ as variables to be eliminated, we have an equation of the type

$$p = m+m'-2, \quad \pi = m+m'-1, \quad \alpha = \beta = -1;$$

the orders of the coefficients relating to $\cos\theta$, $\sin\theta$; the two equations (1) in the same sense correspond to

$$\mu = m, \quad \nu = n = m', \quad \alpha = \beta = -1.$$

Consequently the order of the resultant in $\cos\theta$, $\sin\theta$ is $mm'(m+m'-1)$; and in $\cos\theta$ or $\sin\theta$, $2mm'(m+m'-1)$. This, therefore, is the class of the locus. We get the same result by taking the order of the conditions that (1) and (4) may coexist, $\theta$ being eliminated.

Again, the order of the conditions that (1) may have two common values of $\theta$, or that (2) may have a double set of values of X, Y, is (Salmon, loc. cit. p. 281)

$$\frac{2mm'}{2}\{2(m-1)(m'-1)+(m-1)+(m'-1)\} = mm'\{2mm'-m-m'\},$$

which gives the number of double points, or, rather, the multiplicity (including cusps, if any) of the locus.

In the present case, we have

$$2mm'(2mm'-1)-2mm'(2mm'-m-m') = 2mm'(m+m'-1).$$

Hence there are generally no cusps.

6. It would probably be difficult to apply this analytical method to the case of directrices possessing singularities; but, as in several analogous theories, there are some simple geometrical considerations which enable us to verify and generalize the preceding results.

To a double point on the directrix $C_m$ will correspond $2m'$ double points on the locus; for a circle of given radius described about the double point as centre will cut the other directrix in $2m'$ points; and each position of the connecting radius will give a double point on the locus. In like manner, a cusp on the directrix $C_m$ will give rise to $2m'$ cusps on the locus. We have, of course, corresponding results relative to double points or cusps on $C_{m'}$. These conclusions are independent of the number of double points and cusps on the directrices, and hold good therefore for compound curves.

Consider, then, two directrices $L_m$, $L_{m'}$ made up of $m$ and $m'$ right lines respectively. The degree of the corresponding locus is $2mm'$, since it consists of $mm'$ conics.

The class is also $2mm'$, but it has been reduced by the effect of $\frac{m(m-1)}{2}$ double points on $L_m$ and $\frac{m'(m'-1)}{2}$ double points on $L_{m'}$. Therefore, for the general locus, when the directrices are without singularities, the class is

$$2mm'+4m'\frac{m(m-1)}{2}+4m\frac{m'(m'-1)}{2} = 2mm'(m+m'-1).$$

If there are $\delta$ double points and $\kappa$ cusps on $C_m$, and $\delta'$ double points and $\kappa'$ cusps on $C_{m'}$, the class is

$$2mm'(m+m'-1)-4\{\delta m'+\delta' m\}-6\{\kappa m'+\kappa' m\};$$

the number of double points being

$$mm'(2mm'-m-m')+2(\delta m'+\delta' m);$$

and of cusps $\quad 2(\kappa m'+\kappa' m).$

If $n$ is now the class of $C_m$ and $n'$ that of $C_{m'}$, the class of the locus may be written $\quad 2\{nm'+n'm+mm'\}.$

To connect the foregoing processes with the fundamental equations (1) or (2), we may observe that if $C_m$ has a double point whose coordinates are $\alpha$, $\beta$, then the first equation of (2) has a double set of roots,

$$x-a\mathrm{X}+c\mathrm{Y} = \alpha,$$
$$y-a\mathrm{Y}+c\mathrm{X} = \beta.$$

Combining these with the second and third equations, and considering $x$, $y$, X, Y as independent variables, we have $2m'$ corresponding double points on the locus; and similar reasoning applies to cusps. The double points and cusps are supposed not to lie on the line at infinity.*

---

* Since this paper was read before the Society, I have observed, in a communication by M. Chasles to the *Comptes Rendus* for May, 1871, several results which bear upon the subject. The author gives, without actual demonstration, numerous theorems which he has obtained by the principle of correspondence. The mode of generation mentioned in §3 enables us to obtain the degree of the locus with which we are concerned by means of the correspondence of points. For consider the linear transversal L cutting the moving curves in any position. It will cut $C_m$ in $m$ points, and $C_{m'}$ in $m'$ points. To a given point on L, considered as an intersection with $C_m$ in $m$ of its positions, will correspond $mm'$ points on L considered as intersections with $C_{m'}$, and *vice versâ*. Hence the number of united points, or the degree of the locus, is $2mm'$. The power of the principle is unfortunately often one-sided, and when it gives the degree with ease, yields the class with difficulty, and conversely.

*Point directrices coincident.*

7. If the two directrices coincide, the formula must be modified. Take a directrix consisting of $m$ right lines. We have as the locus $2m$ right lines and $\frac{m(m-1)}{2}$ conics, which last number is to be doubled, since the ends of the moving line may be reversed. Hence we have for the general degree $2m^2$; or we must simply make $m' = m$ in the previous formula. But for the special class we have $2m(m-1)$, while each double point or cusp on the directrix gives rise to $4m$ double points or cusps on the locus, because each end of the traversing chord may be placed at the point in question. So that we have for the general class

$$2m(m-1)+8m.\frac{m(m-1)}{2}-8\delta m-12\kappa m,$$

or

$$4m^3-2m^2-2m-8\delta m-12\kappa m,$$

or

$$4mn+2m(m-1),$$

where $\delta$ and $\kappa$ are the numbers of double points and cusps, and $n$ is the class of the directrix.

The number of double points on the locus is $2m^3(m-1)+(4\delta+1)m$, and the number of cusps is $4\kappa m$. Of the double points, $m$ are at infinity, and coincide with the points where the directrix meets the line at infinity.

*Point directrices having common points at infinity.*

8. If there are two distinct directrices having common points at infinity, these points are multiple points on the locus. For take a system of right lines $L_m$ having $p$ lines parallel to $q$ lines of the second directrix $L_{m'}$, also composed of right lines; we shall have as part of the locus $2pq$ right lines having the same direction as the parallel lines of the directrices; that is to say, there is a multiple point of the order $2pq$ on the locus. Hence, if there are common multiple points at infinity of the orders $p$, $q$; $p_1$, $q_1$, &c., on the two directrices, the same points are multiple points on the locus of the orders $2pq$, $2p_1q_1$, &c., and the class suffers a corresponding reduction, its value being

$$2mm'(m+m'-1)-4(m\delta'+m'\delta)-6(m\kappa'+m'\kappa)-2\Sigma pq.$$

If the two directrices are of the degrees $m$, and have $m$ common points at infinity, the degree of the locus is $2m^2$, and the class

$$4m^3-2m^2-2m-8\delta m-12\kappa m,$$

agreeing with the formula for a single directrix.

If the single directrix has a multiple point of the order $p$ at infinity, the same point is of the order $2p^2$ on the locus.

Take as directrices the parabola $y^2-kx=0$ and the line $y-l=0$, we have

$$(y-a\sin\theta-c\cos\theta)^2-k(x-a\cos\theta+c\sin\theta)=0 \;\ldots\ldots (a),$$

$$y-b\sin\theta-c\cos\theta-l=0 \;\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

From ($b$) we get

$$\sin\theta = \frac{b(y-l)+c\sqrt{b^2+c^2-(y-l)^2}}{b^2+c^2},$$

$$\cos\theta = \frac{c(y-l)-b\sqrt{b^2+c^2-(y-l)^2}}{b^2+c^2};$$

and, by ($a$) and ($b$),

$$\{l-(a-b)\sin\theta\}^2-k(x-a\cos\theta+c\sin\theta)=0.$$

Hence, by substitution,

$$\left\{\begin{matrix}(a-b)^2\{b^2(y-l)^2+c^2[b^2+c^2-(y-l)^2]\}+(kc-2lb)(a-b)(b^2+c^2)(y-l)\\ -(kx-l^2)(b^2+c^2)^2\end{matrix}\right\}^2$$
$$-[b^2+c^2-(y-l)^2]\,\{2(a-b)bc(y-l)-[k(ab+c^2)+2bc(a-b)](b^2+c^2)\}^2 = 0.$$

This is of the form

$$(\mathrm{A}y^2+\mathrm{B}yz+\mathrm{C}xz+\mathrm{D}z^2)^2-(\mathrm{E}y^2+\mathrm{F}yz+\mathrm{G}z^2)(\mathrm{H}y+\mathrm{K}z)^2=0;$$

which has a finite double point on $\mathrm{H}y+\mathrm{K}z$, and two adjacent double points at $y=0$, $z=0$.

If the second directrix $\mathrm{C}_{m'}$ is a translation of the first directrix $\mathrm{C}_m$, and the length of the moving line is the distance of translation, the locus will contain a curve which also is identical with $\mathrm{C}_m$ translated; and if we exclude it, the locus is of the degree $2m^2-m$. We have a case of this when the directrices are two equal circles, and the chord is the distance between their centres. The locus then consists of a circle equal to the directrices and a nodal bicircular quartic.

## *Conditions of mechanical possibility.*

9. The nature of the foregoing constructions suggests some general remarks. Mechanical possibility introduces obvious limitations on the analytical formulæ. These include in their form the results of mechanically impossible constructions, and these results may be real. Of course, among the trajectories of points of the moving plane, some possess singularities in excess of the numbers which we have obtained. The directrices themselves are extreme cases; but it must be remembered that the whole of a directrix cannot generally be traversed by the describing point, in this case an extremity of the finite line. Hence an arc of a directrix becomes the limit of a locus, or a degenerate curve of higher degree. The conditions may even be such that the plane is practically fixed, and each point represents a locus; or the conditions may be inconsistent, as when the length of the moving bar is less than the shortest distance between the directrices. By a special arrangement of the constants we may obtain a directrix as the locus corresponding to a possible combination, but not generally. When, however, we consider unreal combinations, the directrices are always included among the trajectories.

Thus, if we have two linear directrices, the loci arising from possible movement are ellipses, since, except for a pair of parallel lines, the moving line cannot reach infinity. Considering, therefore, the motion of an extremity, we have a limited line on one of the directrices, representing a flattened ellipse. But, analytically, a limited line may move with a real extremity on one directrix, and an unreal one on the other; and then the real extremity moves along the whole of a directrix. In the same way, generally, while an arc of a directrix is a limiting trajectory corresponding to a possible construction, the whole of the directrix is a trajectory corresponding to unreal constructions. Practically a locus cannot have real points at infinity, unless the directrices have common points at infinity; for if one curve is at infinity finitely distant from another, the distance is analytically evanescent, and the curves meet at infinity. If the directrices have common points at infinity, but the traversing bar cannot pass continuously towards infinity, the multiple points at infinity will be conjugate.

*Case of $m'=1$.*

10. There are several particular cases which merit attention. If $m=2$, $m'=1$, the locus is of the fourth degree with two double points. There are 10 disposable constants. The conic gives 5, the line gives 2; the length of the moving line, the ratio in which it is divided by the perpendicular from the describing point, and the length of that perpendicular, supply 3 more. The curves resulting from the construction are analytically subjected to only two conditions beyond those implied in the existence of the double points. There are two parallel double tangents whose points of contact lie with the two double points on a determinate conic.*

When $m'=1$, we can readily eliminate. There will be no real loss of generality in taking $y=0$ for the line, the other equation remaining general.

The system is reducible to

$$y-b\sin\theta-c\cos\theta=0,$$

$$\phi\,\{x-a\cos\theta+c\sin\theta,\ \ (b-a)\sin\theta\}=0.$$

From the first of these equations we get

$$\sin\theta=\frac{by+c\sqrt{b^2+c^2-y^2}}{b^2+c^2},$$

$$\cos\theta=\frac{cy-b\sqrt{b^2+c^2-y^2}}{b^2+c^2}.$$

* For the case of a circle and line as directrices, forms of these curves are given in connection with a paper "On the Mechanical Description of some species of Circular Curves," *Proceedings*, vol. ii., p. 128.

Substituting these values in the second equation, we get

$$\phi\left\{x+\frac{(b-a)\,c+(ab+c^2)\sqrt{b^2+c^2-y^2}}{b^2+c^2},\ (b-a)\frac{by+c\sqrt{b^2+c^2-y^2}}{b^2+c^2}\right\}=0.$$

Rationalised, this equation is of the form

$$\Phi^2-\Psi^2\,(b^2+c^2-y^2)=0,$$

and shows that there are two multiple tangents parallel to the linear directrix, and having in general $m$ points of contact, and that there are $m\,(m-1)$ double points which form the intersect of two curves of the $m$th and $(m-1)$th degrees respectively. The $2m$ points of contact of the multiple tangents, and the $m\,(m-1)$ double points, lie on a curve of the $m$th degree. The $mm'\,(2mm'-m-m')$ double points of the general locus are also connected in some way, and it would be interesting to know what the relation is.

When the general motion of the plane is determined, we have only two constants at our disposal, viz., $a$ and $c$, or $b$ and $c$. Hence the double points and cusps of all the trajectories will form loci relative to the fixed plane. I suppose that the locus of cusps will, in general, be the locus of instantaneous centres relative to the fixed plane, and will be due to points situated on the corresponding locus relative to the moving plane; but I have not determined these loci. In the case of two lines as directrices, points on the moving locus of instantaneous axes describe flattened ellipses, and not, of course, properly cusped curves; but the exception is of the kind which proves the rule, for the trajectories are doubly orthogonal on the same side to the fixed locus of instantaneous centres. The double point of a trajectory is always the intersection of the lines.

*Case of one point directrix, $a=-b$, $c=0$.*

11. There is a reduction of one-half in the order of the locus when there is only one directrix, and the describing point is the middle point of a constant chord. We can see this by taking a directrix composed of conics and lines. The degree of the locus of the middle point of a constant chord moving about a conic is 4, and the locus of the middle point of a limited line moving along an indefinite line is the latter line. Now, in the case in question, the reversing of the ends of the moving line when the directrices are two conics or lines, or a line and conic, reproduces the same locus. Hence we have not to double the order in the corresponding cases. Thus, if the directrix consist of $\frac{m}{2}$ conics, we have for the degree $\dfrac{8\frac{m}{2}\left(\frac{m}{2}-1\right)}{2}+\dfrac{4m}{2}=m^2$;

or if the directrix consists of $m$ lines, we have $m\,(m-1)+m=m^2$.

12. The result may be established analytically.

We have, since $a-b=2a$,

$$\phi(x-a\cos\theta,\quad y-a\sin\theta)=0,$$
$$\phi(x+a\cos\theta,\quad y+a\sin\theta)=0;$$

or, putting $\phi$ for $\phi(x,y)$,

$$\left.\begin{array}{r}\phi+\left\{\frac{d^2\phi}{dx^2}\cos^2\theta+\frac{d^2\phi}{dy^2}\sin^2\theta+2\frac{d^2\phi}{dxdy}\sin\theta\cos\theta\right\}\frac{a^2}{1.2}+\ldots=0\\ \left(\frac{d\phi}{dx}\cos\theta+\frac{d\phi}{dy}\sin\theta\right)+\ \ldots\ldots=0\ldots\ldots\ldots\ldots\ldots\end{array}\right\}(5).$$

The order of the system is to be determined.

Generally, if $(\mu)$ denote a coefficient whose order is $\mu$, the system

$$(s)\,x^{2m}+(s+a)\,x^{2m-1}+(s+2a)\,x^{2m-2}+\ldots\ldots+[s+a]\,x+[s]=0$$
$$(t)\,x^{2n}+(t+a)\,x^{2n-1}+(t+2a)\,x^{2n-2}+\ldots\ldots+[t+a]\,x+[t]=0$$

may be written in the form

$$\left.\begin{array}{l}(s)(\cos\theta,\sin\theta)_m+(s+a)(\cos\theta,\sin\theta)_{m-1}+\ldots+(s+ma)=0\\ (t)(\cos\theta,\sin\theta)_n+(t+a)(\cos\theta,\sin\theta)_{n-1}+\ldots+(t+na)\ =0\end{array}\right\}\ldots(c),$$

where $(s)(\cos\theta,\sin\theta)_m$ means a homogeneous function of $\cos\theta$, $\sin\theta$ of the order $s$ as to variables in the coefficients, and the order $m$ as to $\cos\theta$, $\sin\theta$.

Combining this system with $\cos^2\theta+\sin^2\theta=1$, which is of the type

$$(0)(\cos\theta,\sin\theta)_2+(a)(\cos\theta,\sin\theta)_1+(2a)(\cos\theta,\sin\theta)_0=0$$

when $a$ is positive, we get for the order of the resultant ($\cos\theta$, $\sin\theta$ being eliminated)

$$2mn\left\{\frac{s+ma}{m}+\frac{t+na}{n}+a-2a\right\}=2ns+2mt+2mna.$$

If we multiply the second of equations $(c)$ by $\mathrm{A}x+\mathrm{B}$, A and B being constants, we change the form to

$$(t)\,x^{2n+1}+(t+a)\,x^{2n}+(t+2a)x^{2n-1}+\ldots\ldots+[t+a]\,x+[t]=0.$$

But the order of the resultant becomes

$$2ns+2mt+2mna+s+ma,$$

or

$$(2n+1)\,s+2mt+(2mn+m)a.$$

In like manner, for two equations of the degrees $2m+1$, $2n+1$, we get

$$(2n+1)\,s+(2m+1)\,t+(2mn+m+n)a.$$

Now the system (5) may be written, when $m$ is even, in the form

$$(0)(\cos\theta,\sin\theta)_m\ \ +(2)(\cos\theta,\sin\theta)_{m-2}+\ldots\ldots=0$$
$$(1)(\cos\theta,\sin\theta)_{m-1}+(3)(\cos\theta,\sin\theta)_{m-3}+\ldots\ldots=0,$$

or in the form

$$(0)\,\mathrm{X}^m\ \ +(2)\,\mathrm{X}^{m-1}+(4)\,\mathrm{X}^{m-2}+\ldots\ldots+[2]\,\mathrm{X}+[0]=0$$
$$(1)\,\mathrm{X}^{m-1}+(3)\,\mathrm{X}^{m-2}+(5)\,\mathrm{X}^{m-3}+\ldots\ldots+[3]\,\mathrm{X}+[1]=0,$$

and the order of the resultant is

$$m+m(m-1) = m^2.$$

Similarly, we get $m^2$ if $m$ is odd. Consequently, the degree of the locus is $m^2$.

12. The class can be determined from the consideration that a double point or cusp on the directrix gives $2m$ double points or cusps on the locus. If we take a system of $m$ right lines as the directrix, the locus consists of the lines themselves and $\frac{m(m-1)}{2}$ conics. The class is therefore $m(m-1)$. But, allowing for the effect of the double points, we obtain for the general class

$$m(m-1)+2m^2(m-1) = 2m^3-m^2-m;$$

and if there are $\delta$ double points and $\kappa$ cusps, the class is

$$2m^3-m^2-m-(4\delta+6\kappa)\,m, \quad \text{or} \quad 2mn+m(m-1),$$

where $n$ is the class of the directrix.

*Case of one directrix, $a=b$.*

13. Let us suppose that the directrices are identical, and that the describing point is on the perpendicular to the moving line at its middle point.

The system (1) becomes

$$\phi(x-a\cos\theta+c\sin\theta,\ \ y-a\sin\theta-c\cos\theta) = 0,$$
$$\phi\,\{x-(a-k)\cos\theta+c\sin\theta,\ \ y-(a-k)\sin\theta-c\cos\theta\} = 0,$$

$k$ being evanescent; and if, further, we now suppose $a$ to become evanescent, we have, by expansion, dividing by $k$, and then making $a$ and $k=0$,

$$\phi = \phi(x+c\sin\theta,\ \ y-c\cos\theta) = 0,$$
$$\frac{d\phi}{d\theta} = 0.$$

In fact, the moving line becomes infinitesimally small, the locus becomes an envelope, and is a parallel of the directrix.

Referring to the remarks of § 3, it is plain that, in the case under discussion, we have the moving radii equal and coincident, the describing curves coincide in one, and the locus is, in fact, the envelope of a curve moving parallel to itself about a circle. This envelope is, as I have elsewhere remarked, a parallel of the moving curve.

But the parallels are described only by points of the moving plane lying at any moment on a normal to the directrix; that is to say, on the perpendicular to the infinitesimal moving element. It is desirable to pay attention to the general locus, in connection with the special case in which the locus is a parallel, and has an exceptional interest of its own.

14. Imagine a straight line moving tangentially about a directrix $\phi(x, y) = 0$. We have to treat of the locus of the extremity of a perpendicular to the tangent, having its foot always at a constant distance from the point of contact, the length of the perpendicular being of course given.

The equations (1) become in this case

$$\phi = \phi(x - a\cos\theta + c\sin\theta,\ y - a\sin\theta - c\cos\theta) = 0 \ \dots\dots \ (a'),$$

$$\frac{d\phi}{dx}\cos\theta + \frac{d\phi}{dy}\sin\theta = 0 \ \dots\dots\dots\dots\dots\dots\dots\dots \ (b').$$

The directrix being free from singularities, the degree of the locus remains $2m^2$, and is the same as for parallels.

To find the class, we have, putting $\frac{dy}{dx} = k$,

$$\left\{\frac{d\phi}{dx}(c\cos\theta + a\sin\theta) + \frac{d\phi}{dy}(c\sin\theta - a\cos\theta)\right\}\frac{d\theta}{dx} + \frac{d\phi}{dx} + k\frac{d\phi}{dy} = 0,$$

or, by the second equation of condition,

$$a\left\{\frac{d\phi}{dx}\sin\theta - \frac{d\phi}{dy}\cos\theta\right\}\frac{d\theta}{dx} + \frac{d\phi}{dx} + k\frac{d\phi}{dy} = 0 \ \dots\dots\dots \ (c');$$

and differentiating the second equation, we get

$$\left\{\begin{array}{l} \left[\frac{d^2\phi}{dx^2}(c\cos\theta + a\sin\theta) + \frac{d^2\phi}{dx\,dy}(c\sin\theta - a\cos\theta)\right]\cos\theta \\ \quad + \left[\frac{d^2\phi}{dx\,dy}(c\cos\theta + a\sin\theta) + \frac{d^2\phi}{dy^2}(c\sin\theta - a\cos\theta)\right]\sin\theta \\ \qquad - a\left[\frac{d\phi}{dx}\sin\theta - \frac{d\phi}{dy}\cos\theta\right] \end{array}\right\}\frac{d\theta}{dx}$$

$$+ \frac{d^2\phi}{dx^2}\cos\theta + \frac{d^2\phi}{dx\,dy}\sin\theta + k\left\{\frac{d^2\phi}{dx\,dy}\cos\theta + \frac{d^2\phi}{dy^2}\sin\theta\right\} = 0 \ \dots \ (d').$$

If now we eliminate from $(b')$, $(c')$, $(d')$ the quantities $\frac{d\theta}{dx}$, $\cos\theta$, $\sin\theta$, so far as they are explicitly contained, we shall obtain an equation of the degree $8m(m-1)$ in $(x - a\cos\theta + c\sin\theta)$, $(y - a\sin\theta - c\cos\theta)$, say X, Y; and combining this with $(a')$, we have for the number of tangents parallel to a given line $8m(m-1)$. The two equations give, in fact, the points of the directrix corresponding to such parallel tangents.

Substituting X, Y in the expressions, we may write the equations

$$\left.\begin{array}{l} \qquad\qquad\qquad \phi(X, Y) = 0, \\ \left\{(a + ck)\frac{d\phi}{dY} + (c - ak)\frac{d\phi}{dX}\right\}^2 \\ \quad \times\left\{\left(\frac{d\phi}{dY}\right)^2\frac{d^2\phi}{dX^2} - 2\frac{d\phi}{dX}\frac{d\phi}{dY}\frac{d^2\phi}{dX\,dY} + \left(\frac{d\phi}{dX}\right)^2\frac{d^2\phi}{dY^2}\right\}^2 \\ \quad - a^2\left\{\left(\frac{d\phi}{dX}\right)^2 + \left(\frac{d\phi}{dY}\right)^2\right\}^3\left(\frac{d\phi}{dX} + k\frac{d\phi}{dY}\right)^2 = 0 \end{array}\right\} \dots (e').$$

If R be the radius of curvature at an intersection, the equation gives

$$\frac{(a+ck)\dfrac{d\phi}{dY}+(c-ak)\dfrac{d\phi}{dX}}{a\left(k\dfrac{d\phi}{dY}+\dfrac{d\phi}{dX}\right)}=\pm R,$$

an expression which can be geometrically interpreted.

If $a=0$, the factor dependent on $k$ is $\left(\dfrac{d\phi}{dX}+k\dfrac{d\phi}{dY}\right)^2=0$, giving $2m(m-1)$ for the class in the case of parallels. It is plain, in this case, that the points of the directrix with which we are concerned lie upon the first polar of the point at infinity which determines the direction of the tangents, and that each such point on the directrix corresponds to two tangents of the parallel.

15. It will be observed, that since

$$2m^2(2m^2-1)-8m(m-1)=4m^4-10m^2+8m,$$

and cusps do not enter the general locus, we may take the number of double points to be $2m^4-5m^2+4m$.

Now, in the case of parallels,

$$\begin{aligned} \text{number of double points} &= 2m^4-11m^2+10m,\\ \text{number of cusps} &= 6m^2-6m;\\ \text{so that the multiplicity} &= 2m^4-5m^2+4m, \end{aligned}$$

as in the general case above treated.

We may take it, then, that a certain number of double points, viz. $6m^2-6m$, are converted into cusps when the locus becomes an envelope relative to its two equations.

The foregoing result is confirmed by the consideration that the oblique parallels (as they may be called) and the parallels have a (1, 1) correspondence.*

---

* The result is accordant with a theorem mentioned by Dr. Salmon ("Cambridge and Dublin Mathematical Journal," Vol. III., p. 170, note), viz., the curve obtained by eliminating $t$ between

$$at^m+mbt^{m-1}+\frac{m(m-1)}{2}ct^{m-2}+\ldots=0,$$

$$a't^m+mb't^{m-1}+\frac{m(m-1)}{2}c't^{m-2}+\ldots=0,$$

where the coefficients are of the first order in the variables $x, y, z$, is of the $2m$th degree having the maximum number of double points; but if $a'=b$, $b'=c$, &c., then, the total number of double points remaining the same, the maximum number of these $[=3(m-1)]$ will become cusps. Under the limitations specified by Dr. Salmon, we have an envelope relative to $t$ as parameter. There seems to be some more general theorem on the subject.

The reduction in the degree of the locus on account of $\delta$ double points and $\kappa$ cusps on the directrix, is $4\delta+6\kappa$, as in the case of parallels. In both instances, a circle described about a double point or cusp fulfils the geometrical conditions, and by parity of reasoning the circle has to be taken twice for a double point, and three times for a cusp, in both cases. In fact, a double point evidently gives rise to a doubled circle. If we take a very small loop, and draw the corresponding locus by points, it immediately appears that another circle becomes part of the locus at the limit. The general degree therefore is $2(m+n)$, both for oblique and proper parallels.

The class for a conic is 16, and for two lines it is zero. The reduction for the double point is therefore 16; and since the effect is local, as in the case of proper parallels, the class remains $8n$.

The case of cusps requires special consideration. I do not find that a cusp on the directrix gives rise to cusps on the oblique parallel, and therefore conclude that, since the multiplicity (in this instance the deficiency also) is the same as in proper parallels, the class is $8n+2\kappa$.

In fact, reverting to the equation which determines by its intersections with the directrix the number of parallel tangents, let us take as the directrix the semicubical parabola $x^3-py^2=0$.

We have $$X^3-pY^2=0,$$

$$\{3(c-ak)X^2-2(a+ck)pY\}^2\{4p^2Y^2.\,6X-2p\,.\,9X^4\}^2$$
$$-a^2\{9X^4+4p^2Y^2\}^3\{3X^2-2kpY\}^2=0.$$

If we put in this $X^{\frac{3}{2}}$ for $\sqrt{p}\,Y$, it becomes divisible by $X^{11}$, or, when rationalized, by $X^{22}$. Hence the reduction on account of the cusp at the origin is 22, in conformity with the value $8n+2\kappa$.

More generally, we may take the curve $x^s-py^{s-1}=0$. The resulting equation contains the factor $X^{\left(8s-4-6\frac{s}{s-1}\right)}$, and, when rationalized, the factor $X^{8s^2-18s+4}$.

But the origin in this case being equivalent to $s-2$ cusps and $\frac{(s-2)(s-3)}{2}$ double points, we have

$$22(s-2)+16\frac{(s-2)(s-3)}{2}=8s^2-18s+4.$$

16. The principal conclusions up to this point are summed up in the following synoptic form. I denote the degree, the class, the number of double points, and the number of cusps of the directrix $C_m^n$, by $m$, $n$, $\delta$, $\kappa$, and the corresponding characteristics of the directrix $C_{m'}^{n'}$ by the same letters accented.

| Two Directrices. | | One Directrix. | | | |
|---|---|---|---|---|---|
| | | | $b=-a,\ c=0$ | $b=a$ | $b=a=0$ (Parallel) |
| Order | $2mm'$ | $2m^2$ | $m^2$ | $2(m+n)$ | $2(m+n)$ |
| Class | $2(mn'+m'n+mm')$ | $4mn+2m(m-1)$ | $2mn+m(m-1)$ | $8n+2\kappa$ | $2n$ |
| Double points | $mm'(2mm'-m-m')$ $+2(m\delta'+m'\delta)$ | $2m^3(m-1)+(4\delta+1)m$ | $\frac{m^4-2m^3+m}{2}+2\delta m$ | $2(m+n)^2-5(m+n)$ $+4m-\kappa$ | $2(m+n)^2-11(m+n)$ $+10m-3\kappa$ |
| Cusps | $2(m\kappa'+m'\kappa)$ | $4\kappa m$ | $2\kappa m$ | | $6n+2\kappa$ |

It is supposed that the line at infinity has no special relation to the directrices or directrix, as the case may be.

*Envelopes in the moving plane.*

17. I now pass to the question of envelopes. The envelope of a line which rests with a given interval on two directrices is of the class $4mm'$, where $m$, $m'$ denote the degrees of the directrices. In fact, Professor Cayley has pointed out, that if two points on two given curves $C_m$, $C_{m'}$ have an $(a, a')$ correspondence, the class of the envelope of the line joining those points is $ma'+m'a$. In the present case $a=2m$ and $a'=2m'$, since a circle of given radius about a point on the first directrix as centre meets the second directrix in $2m'$ points. The class is independent of double points or cusps on the directrices. The degree, however, is affected by these singularities. To each double point or cusp on the first directrix correspond $2m'$ double tangents, or stationary tangents on the envelope. Hence, if there are $\delta$ double points and $\kappa$ cusps on $C_m$, and $\delta'$ double points and $\kappa'$ cusps on $C_{m'}$, we shall have $2(m\delta'+m'\delta)$ double tangents, and $2(m\kappa'+m'\kappa)$ stationary tangents on the envelope. The degree of the envelope may now be determined, as follows:

Take two directrices consisting of $m$ and $m'$ right lines respectively. For a pair of lines the degree is 6 and the class 4. The degree of the envelope in the present case is therefore $6mm'$; and correcting this for the effect of the $\dfrac{m(m-1)}{2}$ double points on $C_m$, and the $\dfrac{m'(m'-1)}{2}$ double points on $C_{m'}$, we have

$$6mm'+2mm'(m'-1)+2m'm(m-1) = 2mm'(m+m'+1);$$

and if there are $\delta$ double points and $\kappa$ cusps on $C_m^n$, and $\delta'$ double points and $\kappa'$ cusps on $C_{m'}^{n'}$, we have to deduct $4(m\delta'+m'\delta)+6(m\kappa'+m'\kappa)$, and we may write the result $2(mn'+m'n+3mm')$. In the case of a pair of lines, there are 6 cusps and 4 double points, 3 double tangents, and no stationary tangents. Double points on the directrices do not introduce stationary tangents. Hence, for general directrices, there are

$2(m\kappa'+m'\kappa)$ stationary tangents (or inflexions),

$8m^2m'^2-mm'(m+m'+3)+2(m\delta'+m'\delta)$ double tangents,

$2(mn'+m'n+3mm')^2-(mn'+m'n+3mm')-2mm'-9mm'(m+m'-1)$
$-3(m\kappa'+m'\kappa)$ double points, and

$6mm'(m+m'-1)+2(m\kappa'+m'\kappa)$ cusps.

I am indebted to Mr. Cotterill for the following determination of the degree of the envelope when a right line and curve are directrices:—

Corresponding to each intersection of the line with the curve there will be two contacts of the envelope with the line. Also for each intersection of the line by the parallel of the curve (mod. length of chord) we have an intersection of the envelope and line. Hence the whole number of intersections, *i.e.* the degree of the envelope, is $4m+2(m+n)$, which agrees with the general formula.

The degree may be shown analytically as follows for two directrices without singularities:

The point of contact of the chord with its envelope is the foot of the perpendicular let fall on it from the intersections of the normals to the directrices at its extremities. If we write X, Y for the coordinates of the intersection of the normals, $x$, $y$ for those of the point of contact, and in other respects follow our former notation, we get the following conditions:

$$\phi(x_1, y_1) = 0, \quad \psi(x_2, y_2) = 0,$$
$$(\mathrm{X}-x_1)\frac{d\phi}{dy_1} - (\mathrm{Y}-y_1)\frac{d\phi}{dx_1} = 0,$$
$$(\mathrm{X}-x_2)\frac{d\psi}{dy_2} - (\mathrm{Y}-y_2)\frac{d\psi}{dx_2} = 0,$$
$$(x_1-x_2)^2 + (y_1-y_2)^2 = k^2,$$
$$(y-\mathrm{Y})(y_1-y_2) + (x-\mathrm{X})(x_1-x_2) = 0,$$
$$(y-y_1)(x_1-x_2) - (x-x_1)(y_1-y_2) = 0.$$

Since the axes are arbitrary, it is sufficient to find the degree of the resultant in $x$ or $y$, when $x_1$, $y_1$, $x_2$, $y_2$, X, Y are eliminated.

By eliminating X, Y and then $x$, $y$ successively, we get a system corresponding to

| Order in $y$ | Order in $x_1, y_1, x_2, y_2$ |
|---|---|
| 1 | $m+m'+1$ |
| 0 | $m+m'+1$ |
| 0 | $m$ |
| 0 | $m'$ |
| 0 | 2 |

and the order of the resultant in $y$ is therefore $2mm'(m+m+1)$.

The envelope will in general have $2m'(m+n)$ contacts with $\mathrm{C}_m^n$, and $2m(m'+n')$ contacts with $\mathrm{C}_{m'}^{n'}$.

These conclusions relate to the moving limited line itself. It remains to consider the envelope of a line rigidly connected with the primary moving line.

18. It is known that, if a limited line move about two linear directrices, the envelope of a parallel to the moving line is a parallel curve to the envelope of the moving line itself, and breaks up into two curves possessing characteristics like those of the envelope.

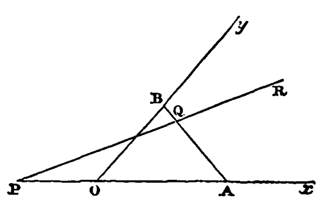
We may consider it, therefore, as a curve of the 12th degree with a multiplicity 56, one more than $\frac{11 \cdot 10}{2}$, the limit for a proper curve. Making use of the same reasoning as before, we may pass to the case of a limited line moving on two directrices $\mathrm{C}_m$, $\mathrm{C}_{m'}$. We have for the general degree and class of the envelope of a line parallel to the moving line $4mm'(m+m'+1)$ and $8mm'$ with $12mm'(m+m'-1)$ cusps, and

$8m^2m'^2(m+m'+1)^2-20mm'(m+m'+1)+32mm'$ double points. The multiplicity is therefore not always in excess of the limit for a proper curve, but we remark that it is precisely sufficient for two curves of the degree $2mm'(m+m'+1)$ and class $4mm'$ with $6mm'(m+m'-1)$ cusps and $2m^2m'^2(m+m'+1)^2-10mm'(m+m'+1)+16mm'$ double points. It is natural to conjecture that the parallel curve is compound in the general case. In fact, taking account only of the curve which corresponds to $+d$, $d$ being the distance of the generating line from the primary line, the whole reasoning appears applicable to the curve thus generated; and we have for a pair of lines the degree 6 and class 4; for two distinct pairs of lines as directrices, the degree 24 and class 16; and therefore, for two conics, the degree 40 and class 16, and so on.

I apprehend that when we are considering the envelope of a line in the moving plane parallel to the line joining the two points whose motions determine the motion of the plane, we must consider $d$, the distance between the lines, as persistently $+$ or $-$, and therefore that the degree and class of the corresponding envelope are $2mm'(m+m'+1)$ and $4mm'$.

The corrections for double points and cusps on the directrices will be similar to those for the preceding case. In fact, the general formulæ remain the same.

19. We shall now be able to deal with the case in which the enveloping line divides the moving chord in a given ratio, and makes with it a given angle.




Take, in the first instance, two linear directrices $OAx$, $OBy$, inclined at an angle $AOB=\omega$. Let $AB$ be a position of the moving bar; $PQR$ a line dividing $AB$ in $Q$ so that $AQ=a$, $BQ=b$; and let $\angle BAx=\alpha$, $\angle PQA=\phi$.

The coordinates of $Q$ are

$$\frac{b\sin(\alpha-\omega)}{\sin\omega},\quad \frac{a\sin\alpha}{\sin\omega};$$

and by the formula for the tangent of the angle between two lines referred to oblique coordinates, the equation of $PQR$ is of the form

$$\left(y-\frac{a\sin\alpha}{\sin\omega}\right)+K\left(x-\frac{b\sin(\alpha-\omega)}{\sin\omega}\right)=0,$$

where $$K=\frac{\sin\alpha\sin(\phi-\omega)-\sin\phi\sin(\alpha-\omega)}{\sin\alpha\sin\phi-\sin(\alpha-\omega)\sin(\phi+\omega)}.$$

The result is of the form

$$A\cos\alpha+B\sin\alpha+C\cos 2\alpha+D\sin 2\alpha+E=0,$$

where A, B are of the first order, and C, D, E are of the order zero relative to $x$ and $y$.

The envelope of the line is consequently of the 6th order, and has six cusps and four double points. This corresponds to the parallel when one sign of $d$ is taken.

A double point or cusp on the directrix $C_m$ appears to produce $2m'$ double tangents, or stationary tangents on the envelope; and, *mutatis mutandis*, the same is true for $C_{m'}$. Hence, if the two directrices are made up of $m$ lines and $m'$ lines respectively, the degree is $6mm'$; and adding on account of the double points $2mm'(m+m'-2)$, we have for the general degree $2mm'(m+m'+1)$. The class is $4mm'$. We get, therefore, the same general formulæ as in the particular cases just treated of.

If there is one directrix only, the class and degree of the envelope of a line require to be modified. Taking a system of $m$ lines, the class of the envelope is $4m^2-2m$, and is unaffected by the double points. This therefore is the general class. There will be, in general, $4\kappa m$ stationary tangents, and the deficiency appears to be the same as for a trajectory of a point connected with a constant chord. I infer, therefore, that the degree is $4m^3+2m^2-6m-8\delta m-12\kappa m$, a formula which, however, does not hold for the chord itself.*

20. The envelope of a curve traced on the moving plane is, in general, the same as the envelope of the trajectories of the points of the curve. ("Nouvelles Annales," April, 1869.)

For if an envelope with respect to $a, b, c$ as parameters is derived from

$$\chi(a, b, c) = 0,$$
$$\phi(x, y, a, b, c) = 0,$$
$$\psi(x, y, a, b, c) = 0,$$

it is immaterial as to nett result whether we eliminate $a, b$, and take the discriminant as to $c$; or eliminate $a, c$, and take the discriminant as to $b$.

But if $a, c$, in the system (1), are coordinates relative to the moving plane, the equation

$$\chi(a, c) = 0$$

represents a curve $\chi$ traced on it. If we combine this with (1), eliminate $\theta$, and take the discriminant as to $a$ or $c$ after eliminating the other letter, we get the envelope of the trajectories of points on $\chi(a, c) = 0$. If we eliminate $a, c$, we get the equation of $\chi$ referred to the fixed plane for a given value of $\theta$; and taking the discriminant as to $\theta$, we get the envelope of the curves $\chi$. The cases of simple sliding or simple rolling are exceptional. [See "Nouvelles Annales," t. vii. (Laisant), where the problem is treated geometrically.]

* For the chord, the class reduces to $2m^2-2m$, exclusive of $m$ points at infinity.

If there is a common point of the order $p$ on $C_{n'}$ and $q$ on $C_{m'}$, the class is $4mm'-2pq$, and the degree is $2\left\{mn'+m'n+3(mm'-pq)\right\}$.

In the case of the chord, there are still further reductions when there are common asymptotes.

21. If an envelope is derived from

$$\chi(a, b, c) = 0,$$
$$\xi(x, y, z;\ a, b, c) = 0,$$

the functions being homogeneous in $x$, $y$, $z$, and in $a$, $b$, $c$, the degree of the envelope in $x$, $y$, $z$ is reduced, in consequence of a double point or cusp on $\chi(a, b, c) = 0$ regarded as a point curve, by $2m_1$, or $3m_1$, where $m_1$ is the degree of the second equation in $x$, $y$, $z$. For if $a_1$, $b_1$, $c_1$ are simply double values of $a$, $b$, $c$ satisfying the first equation, the curve $\xi(x, y, z;\ a_1, b_1, c_1) = 0$ is consecutive to itself, and counts twice in the envelope which it enters as a factor. Hence the degree of the nett envelope is reduced by $2m''$. The result for a cusp follows similarly.

We are now supposing that the functions are general, except as expressly conditioned, at least in so far that $a$, $b$, $c$ enter into the second equation in a manner entirely independent of the form of $\chi$.

We have found that the envelope derived from $\mathrm{A}a + \mathrm{B}c + \mathrm{C} = 0$ and the system (1) is of the degree $2mm'(m+m'+1)$. Let us take a series of $m''$ lines on the moving plane, the degree of the envelope is then $2mm'm''(m+m'+1)$; and adding on account of the double points $2 \cdot 2mm'\dfrac{m''(m''-1)}{2}$, we get, for the general degree,

$$2mm'm''(m+m'+m'').$$

If the traced curve is a circle, its envelope will evidently be a parallel of the trajectory described by the centre. Hence its degree is $2\{2mm' + 2mm'(m + m' - 1)\} = 4mm'(m + m')$, and its class $4mm'(m+m'-1)$.

I obtain for the general class

$$2mm'(m+m'-1)\,m''(m''-1) + 4mm'm''.$$

If the degree is reduced by $2m_1$ or $3m_1$ for a double set $a_1$, $b_1$, $c_1$, it equally follows that the class is reduced by $2n_1$ or $3n_1$ under the same conditions. The general theorem is of wide application, and worth, I think, further investigation.*

---

* As a mere verification, take Prof. Henrici's result that the degree and class of the envelope of a unicursal series of curves $(\lambda)^r(x_1x_2x_3)_m = 0$ are respectively $2m(r-1)$ and $m\{3mr - 2(m+r) + 2\}$. If the parametric locus consists of $p$ linear factors, there are $p-1$ double points in excess of the number for a unicursal locus. The degree then for a unicursal locus is

$$2mp(r-1) + 2m(p-1) = 2m(pr-1).$$

The class will be

$$mp\{3mr - 2(m+r) + 2\} + 2m(m-1)(p-1) = m\{3mpr - 2(m+pr) + 2\};$$

these are the proper values, since the index of the system of curves is $pr$.

For a general parametric locus without double points or cusps, we have, in like manner,

$$\text{Degree} = 2mp(r-1) + mp(p-1) = mp\{2r + p - 3\},$$
$$\text{Class} = m(m-1)\{p^2 + 3p(r-1)\} + mpr.$$

For $\delta$ double points and $\kappa$ cusps on the parametric locus, the corrections are $-m(2\delta + 3\kappa)$ and $-m(m-1)(2\delta + 3\kappa)$. There are $3m^2p(p+r-3) - 6m^2\delta - 8m^2\kappa$ cusps. The above expression for the degree agrees with a known result relative to the Tact Invariant of two curves. (See Salmon's *Higher Algebra*, 2nd Ed., p. 154.)

With respect to the motion of a plane in plane space, however conditioned, the envelopes of parallel lines must form a system of involutes having a common evolute. We are therefore at liberty to select the line of the set which will give the simplest result.

In general, an envelope of a line, and a trajectory of a point, have a (1,1) correspondence. If two curves have a (1, 1) correspondence, their deficiency is the same. This correspondence may relate to points, or lines, or points of one and lines of the other. (Clebsch, Crelle, t. lxiv. p. 98, quoted by Cayley, Quarterly Journal, vol. xi. p. 185.)

We have seen that the double and stationary tangents of the envelope of a line are in number

$$8m^2m'^2-mm'(m+m'+3)+2\{m(\delta'+\kappa')+m'(\delta+\kappa)\},$$

and the class is $4mm'$. Therefore

Deficiency of Envelope $= mm'(m+m'-3)-2\{m(\delta'+\kappa')+m'(\delta+\kappa)\}+1$

$$=\frac{(2mm'-1)(2mm'-2)}{2}-\{mm'(2mm'-m-m')+2[m(\delta'+\kappa')+m'(\delta+\kappa)]\}$$

= Deficiency of Trajectory.

*Circular point directrices.*

22. When the directrices pass through the circular points at infinity, there are cases of peculiar difficulty. I shall endeavour to determine the formulæ when those points are points of simple multiplicity.

If the order of multiplicity is $p$ as to $C_m$ and $q$ as to $C_{m'}$, the degree of the trajectory seems to be $2mm'-2pq$. By the condition $\cos^2\theta+\sin^2\theta=1$, the equations (1) are, in the case supposed, reducible to the orders $m-p$, $m-q$, respectively, relative to $\cos\theta$ and $\sin\theta$, while they remain of the orders $m$, $m'$ in $x$ and $y$. They are of the types

$$(p)(\cos\theta,\sin\theta)_{m-p}+(p+1)(\cos\theta,\sin\theta)_{m-p-1}+\ldots=0,$$

$$(q)(\cos\theta,\sin\theta)_{m'-q}+(q+1)(\cos\theta,\sin\theta)_{m'-q-1}+\ldots=0;$$

where $(p)(\cos\theta,\sin\theta)_{m-p}$ means, as formerly, a homogenous function of $\cos\theta$, $\sin\theta$ of the order $m-p$, whose coefficients are of the order $p$ in $x$, $y$. Adding to these the equation $\cos^2\theta+\sin^2\theta-1=0$ of the type $(0)(\cos\theta,\sin\theta)_2+(1)(\cos\theta,\sin\theta)_1+(2)=0$,

we have, as before, for the order of the resultant relative to $\theta$,

$$2(m-p)(m'-q)\left(\frac{m}{m-p}+\frac{m'}{m'-q}+1-1-1\right)=2mm'-2pq.$$

This result may be verified by means of compound directrices; $C_m$ consisting, say, of $p$ circles and $m-2p$ lines, and $C_{m'}$ of $q$ circles and $m'-2q$ lines. The degree of the locus is

$$2(m-2p)(m-2q)+4p(m'-2q)+4q(m-2p)+6pq=2mm'-2pq;$$

for in the case of two circles as directrices the degree is 6, and in the case of a line and circle the degree is 4. The double points do not affect the degree.

23. The more difficult question of the class remains. When only one of the directrices (say $C_m$) passes through the circular points, the degree remains unaltered. But we cannot infer the same relative to the class. A double point or cusp on $C_m$ gives rise to $2m'$ double points or cusps on the locus. I apprehend, however, that a double point or cusp on $C_{m'}$ only gives rise to $2(m-1)$ double points or cusps on the locus, the circular points being extraneous since they correspond to every position of the extremity of the traversing chord which lies upon $C_{m'}$.*

If this be correct, we have for the class of the trajectory, by means of compound directrices consisting of lines,

$$2mm'+2(m-1)m'(m'-1)+2m'm(m-1),$$

or

$$2mm'(m+m'-1)-2m'(m'-1).$$

When $m'=1$, the class is the same, whether $C_m$ is circular or not; and this agrees with the result obtained for a circle and line as directrices, in which case the trajectory is a circular quartic with two double points.

Again, if we assume this last conclusion, and consider $C_m$ as made up of $p$ circles and a general curve of the degree $m-2p$, while $C_{m'}$ consists of $m'$ lines, we have for the class

$$2m'(m-2p)^2+8m'p,$$

and for the general case we must add the correction

$$2(m-p)m'(m'-1)+4m'\{2p(m-2p)+p(p-1)\},$$

giving for the class

$$2mm'(m+m'-1)-4m'p(p-1)-2pm'(m'-1).$$

If there are $\delta$ finite double points, and $\kappa$ finite cusps on $C_m$, $\delta'$ finite double points and $\kappa'$ finite cusps on $C_{m'}$, there is a reduction of

$$4\{(m-p)\delta'+m'\delta\}+6\{(m-p)\kappa'+m'\kappa\}.$$

The circular points are multiple in the degree $pm'$.

Taking now two sets of $m$ lines and $m'$ lines $L_m$, $L_{m'}$, the one containing $p$ pairs of circular asymptotes, the other containing $q$ such pairs. The trajectory consists of $mm'-2pq$ conics, and $2pq$ parallel lines. The class is therefore $2mm'-4pq$. But to obtain the general class, we

* In determining the number of normals which can be drawn to a curve from a given point, we meet with a similar case. For a point at infinity, there are in general $m+n$ normals, $n$ finite, and $m$ lying at infinity; but if the curve is circular, there are only $m-2$ normals at infinity. As M. Chasles has remarked (*Comptes Rendus*, April, 1871), the reduction is due, not simply to the tangents at the circular points being also normals, but to the fact that *every* line through a circular point is normal to the curve extraneously.

must correct for $\frac{m(m-1)}{2}-p(p-1)$ double points on $L_m$, and for $\frac{m'(m'-1)}{2}-q(q-1)$ on $L_{m'}$ (these being the numbers of the double points exclusive of the circular points). Hence we have

$$2mm'-4pq+4(m-p)\left\{\frac{m'(m'-1)}{2}-q(q-1)\right\}$$
$$+4(m'-q)\left\{\frac{m(m-1)}{2}-p(p-1)\right\}.$$

Or we may avoid the foregoing imaginary combinations as follows:—For two circles as directrices I find that the trajectory is of the degree 6, and has the circular points for triple points, and three other finite double points, giving the class 12.

Hence, if $C_m$ consists of $p$ circles and $m-2p$ lines, and $C_{m'}$ of $q$ circles and $m'-2q$ lines, the class is

$$12pq+8q(m-2p)+8p(m'-2q)+2(m-2p)(m'-2q).$$

For the general class we must add the correction for double points

$$4(m-p)\left\{\frac{(m'-2q)(m'-2q-1)}{2}+2q(m'-2q)+q(q-1)\right\}$$
$$+4(m'-q)\left\{\frac{(m-2p)(m-2p-1)}{2}+2p(m-2p)+p(p-1)\right\},$$

and the result is the same as before, viz.

$$2mm'(m+m'-1)-2pm'(m'-1)-2qm(m-1)$$
$$-4mq(q-1)-4m'p(p-1)+4pq(p+q-3).$$

If the double points, exclusive of circular points, on $C_m$, $C_{m'}$ are $\delta$, $\delta'$, and the cusps $\kappa$, $\kappa'$, there is a reduction of

$$4\{(m-p)\delta'+(m'-q)\delta\}+6\{(m-p)\kappa'+(m-q)\kappa\}.$$

The circular points at infinity are of the order $pm'+qm-pq$.

In the several cases, if there are no cusps on the directrices, there are none on the locus. Hence the number of cusps, when they exist, is known by their origin from cusps on the directrices, and Plücker's equations give the several characteristics.

24. Let us now consider the case in which the two circular directrices coincide.

Take $p$ circles and $m-2p$ lines, the degree of the locus is

$$2p+2.4p(m-2p)+6p(p-1)+2(m-2p)(m-2p-1)+2(m-2p).$$

Since a circle gives the degree 4, the locus being two concentric circles; a circle and line give 4; two circles give 6; and two lines give 2. Except for circles and lines taken singly, the numbers have to be doubled. Hence the degree is $2m^2-2p(p+1)$; and this is the general degree, because the double points do not affect it.

As to the class, we have, for a compound directrix consisting of $p$ circles and a general curve $C_{m-2p}$,

$$4p+12p(p-1)+\{4(m-2p)^3-2(m-2p)^2-2(m-2p)\}$$
$$+\ 2p\{2.2(m-2p)(m-2p+1)-2(m-2p)(m-2p-1)\},$$

and for the general correction

$$4.2(m-p)\{2p(m-2p)+p(p-1)\};$$

taking into account double points, not circular points.

The result is

$$4m^3-2m^2-2m-4pm(m-1)-8mp(p-1)+8p^3-12p^2-4p.$$

This formula, however, presents difficulties which I have not been able to remove, and it may require some modification. There seems to be a difference in respect to double points situate at the intersection of tangents at the circular points. The class for a circle as directrix being 4, if we now suppose it to break up into a pair of circular asymptotes, the locus consists of two imaginary circles, and the class remains 4. In fact, a circle of given radius about the double point as centre meets the circular asymptotes at points adjacent to the circular points at infinity; and all these intersections seem to be extraneous in some way.

Accordingly, if we take $m$ lines, containing a pair of circular asymptotes $p$ times over, we have for the class

$$4\left\{\frac{m(m-1)}{2}-p(p-1)\right\}+8(m-p)\left\{\frac{m(m-1)}{2}-p(p-1)\right\},$$

giving $4m^3-2m^2-2m-4pm(m-1)-8mp(p-1)+8p^3-12p^2+4p.$

But, in consequence of the $p$ real intersections of circular asymptotes, this result is in excess by $8p$, the last term being $+$ instead of $-$.

25. When $a=b$, and the sole directrix passes $p$ times through the circular points, we have, by taking the special case of $p$ circles and a curve of the degree $(m-2p)$,

$$\text{Order} = 2(m-2p)^2+4p.$$

But the $2p(m-2p)+p(p-1)$ double points not at the circular points have reduced the order by $4\{2p(m-2p)+p(p-1)\}$; the general expression therefore is $2m^2-4p^2$, or $2(m+n)-4p$ if $n$ is the class, an expression which holds good when there are double points and cusps besides the multiple circular points.

To obtain the class, we have in the first place, for the compound directrix, $\quad 8(m-2p)(m-2p-1)+4p,$

to which add the correction for the effect of double points not circular points, $\quad 16\{2p(m-2p)+p(p-1)\};$

and we have $\;8\{m(m-1)-2p(p-1)\}-12p = 8n-12p.$

In the corresponding case for parallel curves, we have $6n-12p$ cusps, and if these are changed into double points, the class is changed to $8n-12p$. If there are $\kappa$ cusps, the class will be $8n-12p+2\kappa$.

Whenever the line at infinity is specially related to the directrix, reductions similar to those in the corresponding cases for parallels may be looked for with respect to oblique parallels. Thus, if the directrix touches the line at infinity $p$ times, there is a reduction of the degree by $2p$. The degree being the same as that of the corresponding parallel, and the multiplicity being the same, we have, because there are in general no cusps, $8n+2k-6p$ for the class.

Thus, for the parabola $y^2-px=0$, the system of equations is

$$(y-a\sin\theta-c\cos\theta)^2-p\,(x-a\cos\theta+c\sin\theta)=0,$$
$$2\,(y-a\sin\theta-c\cos\theta)\sin\theta-p\cos\theta=0.$$

But the first of these may be written by means of the second,

$$(y-a\sin\theta-c\cos\theta)\cos\theta-2\sin\theta\,(x-a\cos\theta+c\sin\theta)=0\,;$$

and the elimination is reducible to that of $t$ from equations of the form

$$A_0t^4+B_1t^3+C_0t^2+D_1t+E_0=0,$$
$$A_0't^4+B_1't^3+C_0't^2+D_1't+E_0'=0,$$

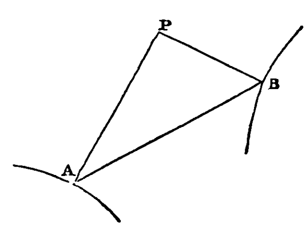
where the suffixes denote the orders of the coefficients in $x$, $y$. The resultant is therefore of the degree 6. Otherwise, writing for the first equation $\quad p\cos^2\theta-4\sin^2\theta\,(x-a\cos\theta+c\sin\theta)=0,$

we see that $y$ appears in the degree 6, and $x$ in the degree 4. But since $x=0$ meets the line at infinity at a point infinitely distant from the parabola, it appears that 6 is the degree of the locus. There are two adjacent double points where the directrix touches the line at infinity.

Again, substituting $p^{\frac{1}{4}}X^{\frac{1}{4}}$ for Y in the equation ($e'$) we obtain an equation of the 8th degree in X. But there are two tangents altogether at infinity. Hence the class is 10, and the oblique parallel is a unicursal sextic.

### *Locus of instantaneous centres relative to the fixed plane.*
### *Point directrices.*

26. The instantaneous centre for a given position of the moving line AB is the intersection of the normals PA, PB to the directrices $C_m$, $C_{m'}$ at the extremities A, B of the moving chord.




Hence if $\phi\,(x, y)=0$, $\psi\,(x, y)=0$ are the equations of the directrices, we have the conditions

$$(x_1-x_2)^2+(y_1-y_2)^2=k^2,$$
$$\phi\,(x_1, y_1)=0,\quad \psi\,(x_2, y_2)=0,$$

$$(x-x_1)\frac{d\phi\,(x_1, y_1)}{dy_1}-(y-y_1)\frac{d\phi\,(x_1, y_1)}{dx_1}=0,$$

$$(x-x_2)\frac{d\psi\,(x_2, y_2)}{dy_2}-(y-y_2)\frac{d\psi\,(x_2, y_2)}{dx_2}=0.$$

In order to obtain the equation to the locus of the instantaneous centres, we must eliminate $x_1, y_1, x_2, y_2$.

Since the axes are general and do not necessarily meet the locus at infinity, it will be sufficient to determine the highest power of $x$ or $y$ in the resultant.

But eliminating successively $x$ and $y$ from the last two equations, we have an equation $y$ in the first power without $x$, and another equation containing $x$ in the first power without $y$. The other variables enter in the degree $m+m'-1$ in both equations. The order of the resultant in $x$ or $y$ is $2mm'(m+m'-1)$, and this, therefore, is the degree of the locus of instantaneous centres relative to the fixed plane. This result may be verified by means of compound directrices. If there is a double point on $C_m$, every line through it may be regarded as a double normal in the same sense that every such line may also be regarded as a double tangent. It follows that $2m'$ normals to the directrix $C_{m'}$ are included in the locus, and are to be counted twice over. The double point, therefore, causes a reduction of $4m'$ in the degree of the locus. So also a double point on $C_{m'}$ causes a reduction of $4m$. When the double point becomes a cusp, the reductions in the corresponding cases are, by similar reasoning, $6m'$ and $6m$.

For a system of $m$ lines, and a system of $m'$ lines, the degree is $2mm'$, since two linear directrices give a circle. But this degree has suffered reduction on account of $\frac{m(m-1)}{2}$ double points on $C_m$ and $\frac{m'(m'-1)}{2}$ double points on $C_{m'}$. We must for the general degree add, therefore, $4m'\frac{m(m-1)}{2}+4m\frac{m'(m'-1)}{2}$, and the result is $2mm'(m+m'-1)$. The reduction for $\delta$ double points and $\kappa$ cusps on $C_m$, and $\delta'$ double points and $\kappa'$ cusps on $C_{m'}$ will be $4(m\delta'+m'\delta)+6(m\kappa'+m'\kappa)$. I infer also that the circular points will be multiple and of the order $mm'$. When the directrices are a right line and curve, it is evident that the intersections of the locus with the line will be those of the line and a parallel of the curve. Hence the degree of the locus is $2m+2n$, which agrees with the general expression $2(mn'+m'n+mm')$.

27. In order to obtain the class, I make use of the following considerations.

We have seen that the reduction of order on account of a double point or cusp on $C_m$ is $4m'$ or $6m'$, $m'$ being the degree of the other directrix. That is to say, the reduction is independent of the degree of

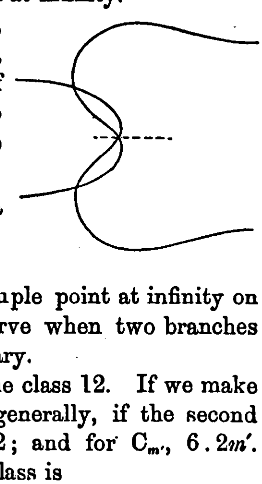
$C_m$ and also of its other double points or cusps; in fact, independent of its general form and characteristics, except at the singularity in question.

I apprehend we may, in the case before us, extend this conclusion to the reduction of class on account of a double point on a directrix, and consider that each corresponding double position of the moving chord gives rise to a reduction $-pm$ or $-pm'$ as the case may be, so that taking the class relative to two systems of $m$ right lines and $m'$ right lines, and adding the correction $p\left\{\frac{m(m-1)}{2}.2m'+\frac{m'(m'-1)}{2}.2m\right\}$, we have, for the general class

$$2mm'+p\,.\,mm'(m+m'-2),$$

where the numerical multipler $p$ has to be determined.

To determine $p$, we ought to take the case in which $C_{m-2}$ is a conic, and $C_{m'-1}$ a line; and if the class is $q$, a double point on $C_2$ reduces the class to 4, and we have an equation to determine $p$. But this case, simple as it is, gives an equation of the degree 8 in a form which renders it very difficult to determine the singularities with the necessary precision. I am led therefore to make use of the more general consideration, that the effect of a double point or cusp on $C_m$ in reducing the class, is independent of the form elsewhere of $C_m$, and remains the same when it passes through the circular points at infinity.



Now if we take as directrices a circle and line, the equation of the circle being $x^2+y^2-r^2=0$, and that of the line $x-a=0$, the equation of the locus of instantaneous centres relative to the fixed plane is without difficulty found to be

$$(x^2+y^2)(y^2-k^2+r^2+a^2)^2-4r^2(ax+y^2)^2=0.$$

This curve is of the 6th degree, having a double point at the origin, two double points where the lines $y^2-k^2+r^2+a^2=0$ meet the line $ax+k^2-r^2-a^2=0$; and, further, a quadruple point at infinity on $y=0$. The figure represents a form of the curve when two branches meeting the line at infinity on $y=0$ are imaginary.

The equivalent of double points is 9, and the class 12. If we make $r=0$, the class becomes zero. I infer that, generally, if the second directrix is of the order 1, the reduction is $6.2$; and for $C_{m'}$, $6.2m'$. It follows that for two general directrices the class is

$$2mm'+6mm'(m+m'-2)=2mm'(3m+3m'-5).$$

I suppose, from analogy, that a cusp will produce $\frac{3}{2}$ of the effect of a double point, and that for $\delta$ double points and $\kappa$ cusps on $C_m$, $\delta'$ double points and $\kappa'$ cusps on $C_{m'}$, the correction is

$$-12(\delta m'+\delta' m)-18(\kappa m'+\kappa' m).$$

It would seem that since, when there are no double points or cusps on the directrices, one point P of a trajectory corresponds to one point P′ of the locus of instantaneous centres relative to the fixed plane, and *vice versâ*, the curves have a (1, 1) correspondence, and consequently their deficiencies should be the same. The deficiency for a general trajectory is $mm'(m+m'-3)+1$, and this is the same as

$$\frac{\{2mm'(m+m'-1)-1\}\{2mm'(m+m'-1)-2\}}{2}$$
$$-\tfrac{1}{2}\{2mm'(m+m'-1)[2mm'(m+m'-1)-1]-2mm'(3m+3m'-5)\},$$

the deficiency for the locus in question.

28. We have next to determine the degree of the locus of instantaneous centres relative to the moving plane.

The quantities $a$, $c$, in (1), may now be taken as current coordinates; so that, following the same constructions and notation as before, we have the following equations of condition:

$$\begin{aligned} x-x_1 &\equiv a(x_2-x_1)-c(y_2-y_1), \\ y-y_1 &\equiv a(y_2-y_1)+c(x_2-x_1), \\ x-x_2 &\equiv (a-k)(x_2-x_1)-c(y_2-y_1), \\ y-y_2 &\equiv (a-k)(y_2-y_1)+c(x_2-x_1); \end{aligned}$$

and substituting these values in the equations of the normals at $x_1$, $y_1$, $x_2$, $y_2$, which pass through $x$, $y$, we have for our system of equations

$$\{a(x_2-x_1)-c(y_2-y_1)\}\frac{d\phi(x_1,y_1)}{dy_1}-\{a(y_2-y_1)+c(x_2-x_1)\}\frac{d\phi(x_1,y_1)}{dx_1}=0,$$

$$\{(a-k)(x_2-x_1)-c(y_2-y_1)\}\frac{d\psi(x_2,y_2)}{dy_2}$$
$$-\{(a-k)(y_2-y_1)+c(x_2-x_1)\}\frac{d\psi(x_2,y_2)}{dx_2}=0,$$

$$\phi(x_1,y_1)=0, \quad \psi(x_2,y_2)=0,$$
$$(x_2-x_1)^2+(y_2-y_1)^2=k^2.$$

We cannot, as in the previous case, reduce the order by the artifice of eliminating $a$ and $c$ successively. Whether we do this or not, the degree of the resultant is $2mm'(m+m')$ in $a$, $c$.

A difficulty was raised in my mind as to this conclusion, because, in the case of two right angles as directrices, the locus is generally taken to be a circle; AB then subtending a constant angle at P. But it will be remarked that the locus in this case is really a circle taken twice over, since to each position and form of the triangle APB corresponds another position and equal and similar form of the triangle.

As in the case of the locus relative to the fixed plane, a double point on $C_m$ causes the locus to contain $2m'$ lines doubled. Taking a system of $m$ lines and a system of $m'$ lines as directrices, the degree of the corresponding locus is $4mm'$. We obtain for the general degree, by adding the correction for double points

$$4\left(m.\frac{m'(m'-1)}{2}+m'.\frac{m(m-1)}{2}\right),$$

the expression, as before, $2mm'(m+m')$;

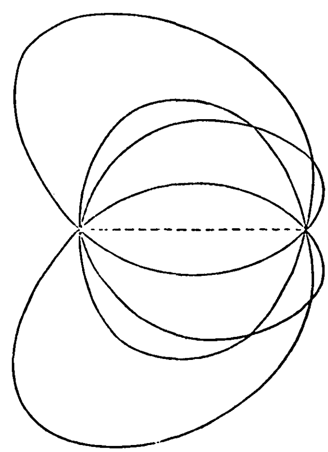
and using the former notation for double points and cusps on the directrices, we have a further correction

$$-4(m\delta'+m'\delta)-6(m\kappa'+m'\kappa).$$

29. To obtain the class, I follow the same course as in obtaining that of the locus relative to the fixed plane. Taking a circle and line for the directrices, the equations being $x^2+y^2-r^2=0$, $x-a=0$, I find for the locus now in question

$$\{[(x-k)^2+y^2][k^2x^2+a^2(x^2+y^2)]-r^2(x^2+y^2-kx)^2\}^2$$
$$-4a^2k^2(x^2+y^2)x^2[(x-k)^2+y^2]^2=0.$$

The curve has a quadruple point at the origin. The point $(x-k=0, y=0)$ is another quadruple point. Besides these, we have two pairs of adjacent or united double points at the circular points at infinity, and two double points where $x=0$ meets the line $(a^2-r^2)y^2+a^2k^2=0$. We have now accounted for the intersections of the curves represented by the squared factors equated to zero. But there are two more double points determined by $x^2+y^2=a^2=kx$. (See figure.) In all, then, there exists the equivalent of 20 double points, and the class is 16. If $r=0$, the class becomes zero. This reduction therefore is 16, and I infer that if $C_{m'}$ is general, the corresponding reduction on account of a double point on $C_m$ is $16m'$.



Hence, assuming, as in the previous case, that the class is of the form

$$4mm'+p\left\{m'.\frac{m(m-1)}{2}+m.\frac{m'(m'-1)}{2}\right\},$$

we have $p=16$, and the class is

$$2mm'(4m+4m'-6).$$

If we suppose, as I think is probably the case, that a cusp produces $\frac{3}{2}$ of the effect of a double point, we get the general correction, on account of singularities on the directrices,

$$-16(\delta m'+\delta'm)-24(\kappa m'+\kappa'm).$$

30. It is, unfortunately, difficult to verify these formulæ relating to instantaneous centres, on account of the high numbers which arise from the simplest cases. If we take as directrices the conic $Ax^2+By^2+C=0$ and the line $x-a=0$, I see no reason why the special relation of the line to the conic should reduce the class. But in this case I find the equation of the locus of instantaneous centres relative to the moving plane in the form

$$\begin{aligned}\{[AB(x^2+y^2-kx)^2+k^2A^2y^2][k^2x^2+a^2(x^2+y^2)]&\\ +CB(x^2+y^2-kx)^2(x^2+y^2)\}^2&\\ -4a^2k^2(x^2+y^2)x^2[AB(x^2+y^2-kx)^2+k^2A^2y^2]^2=0.&\end{aligned}$$

This is of the 12th degree. It appears to have an octuple point at the origin, and a quadruple point at $(x-k=0,\ y=0)$. We have also two double points where $x=0$ meets the lines $ABa^2y^2+A^2k^2a^2+CBy^2=0$, and at the circular points at infinity two quadruple points and two pairs of adjacent double points. There are, moreover, two double points at $x^2+y^2=a^2=kx$. In all, therefore, we have the equivalent of 54 double points, and the class is $12.11-2.54=24$, which is also the class given by the formula on putting $m=2$, $m'=1$.*

When $a=b=0$, the locus of instantaneous centres relative to the fixed plane is the evolute of the directrix, and the locus relative to the moving plane is the line constantly normal to the directrix. An involute, in fact, is generated by a point on an indefinite line rolling on the evolute. The circumstance that cusps of a parallel correspond to double points of an oblique parallel, is in harmony with this; for in the one case the generating point is on the rolling line, and gives rise to cusps when that point comes upon the evolute, while in the other case the generating point is off the rolling line.

31. If we eliminate $\theta$ from the system (1) as before, but consider $x$, $y$ as determined quantities, while $a$, $c$ are variables, we may regard the latter quantities as coordinates, relative to the moving plane, of a fixed point on the fixed plane. In other words, considering now the former moving plane as fixed, and the former fixed plane as moving, subject to the conditions of the construction, we have, in the present way of viewing the result of the elimination of $\theta$, the equation of the corresponding trajectory.

The degree of the equation is $4mm'$. The effect of a double point or

* There is obviously much which remains to be investigated. Circular directrices can be treated as in the case of a trajectory of a point. When there is only one directrix, the degrees of the loci seem to be $2m^3-m^2-m$ and $2m^3-2m$; but the class is difficult to determine.

I take the opportunity to correct an oversight at the end of § 10. The double point in question may be anywhere on the plane. It appears to follow that the double points of trajectories are distributed over the whole plane, on account of the manner in which the parameters $a$, $c$ enter their equations.

cusp on $C_m$ (now a moving directrix) is to give rise to $2m'$ double points or cusps on the locus, and similarly with respect to singularities on $C_{m'}$. Hence, taking compound directrices in the moving plane, consisting of $m$ lines and $m'$ lines, we have the class $4mm'$. Correct now for the effect of the double points, and we have in the general case $2mm'(m+m')$. If there are $\delta$ double points and $\kappa$ cusps on $C_m$, and $\delta'$ double points and $\kappa'$ cusps on $C_{m'}$, the further correction is $-4(m\delta'+m'\delta)-6(m\kappa'+m'\kappa)$. The number of cusps is $2mm'+2m\kappa'+2m'\kappa$, each of the circular points counting as $mm'$.

The deficiency is the same as for the original trajectory.

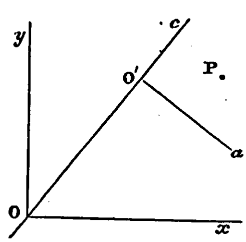
It is plain that, if lines AC, BC, containing a constant angle, move about the points A, B respectively, any point P, rigidly connected with them, will describe a Limaçon of Pascal of the degree and class 4, and having the circular points at infinity for cusps, and a finite double point. Any line rigidly connected with AC, BC envelopes a circle. In point of fact, it easily appears that if a constant angle circumscribes two circles, the locus of the apex, or of any point in the plane of the angle moving with it, is a Limaçon.

If there is only one directrix $C_m$, the corresponding degree of the trajectory of a point of its plane is $4m^2-2m$, and the class is

$$4m^3-4m-8m\delta-12m\kappa.$$

As to the envelope of a line in the moving plane, the class for directrices consisting of $m$ lines and $m'$ lines is $2mm'$; and since the double points and cusps on the directrices generally give rise to no double points or cusps on the envelope, I take the general class to be $2mm'$. But double points and cusps on the directrices give rise immediately to double tangents and stationary tangents on the envelope; so that we have, in general, $2mm'+2mm'(m+m'-2)=2mm'(m+m'-1)$ for the degree, with a reduction for double points and cusps, if any, $-4(m\delta'+m'\delta)-6(m\kappa'+m'\kappa)$, the number of stationary tangents being $2(m\kappa'+m'\kappa)$. The circular points are multiple in the degree $mm'$.

Taking the particular case when one of the directrices is a right line, we may proceed as follows:—




Let $Ox$, $Oy$ be rectilinear axes in the fixed plane. Let the linear moving directrix be $OO'c$; and let $O'a$, $O'c$ be rectilinear axes at $O'$ in the moving plane. Putting $\angle O'Ox=\theta$, $OO'=\rho$, we have for any point P of the moving plane

$$a=x\sin\theta-y\cos\theta,\quad c=x\cos\theta+y\sin\theta-\rho.$$

Hence, for a line $c-ma=0$, we have

$$(\cos\theta-m\sin\theta)\,x+(\sin\theta+m\cos\theta)\,y-\rho=0.$$

Put L for $\dfrac{\cos\theta-m\sin\theta}{\rho}$, M for $\dfrac{\sin\theta+m\cos\theta}{\rho}$, so that these may be taken as tangential coordinates of the line relative to the fixed plane.

Then $\sin\theta = \rho\,\dfrac{M-mL}{1+m^2}$, $\cos\theta = \rho\,\dfrac{mM+L}{1+m^2}$, $\rho^2 = \dfrac{1+m^2}{L^2+M^2}$.

But for a curve $\phi(a, c) = 0$ in the moving plane constantly passing through $(y=0, x=b)$, we have $\phi(b\sin\theta, b\cos\theta-\rho) = 0$, or

$$\phi\,\{b\,(M-mL),\; b\,(mM-L)-(1+m^2),\; \sqrt{(1+m^2)}\sqrt{(L^2+M^2)}\} = 0.$$

The class, therefore, of the envelope of $c-ma=0$ is $2m$, when $m$ is the degree of the curve. There are $m(m-1)$ double tangents, giving $2m^2$ for the degree.

32. Whatever may be the motion of a plane in plane space, it is clear that, if two trajectories of its points are given, any third may be described by means of a limited line moving between the given trajectories and rigidly connected with the describing point. But, in general, not only is the length of the limited line determined, but we cannot take all its positions. We can only conclude that the third trajectory in question is a part of the general locus. The length of the limited line must, of course, be the distance between the points which describe the given trajectories.

The kinematical properties of a plane involve much more than the determination of the degrees and characteristics of trajectories and envelopes. The correspondence extends to arcs, areas, and curvature. The particular study of these relations is, however, practically distinct from the subject of the present paper, though both lines of investigation would no doubt meet in a complete theory.

To facilitate the further study of the subject, it would be desirable to work out in detail the elementary cases. I am not aware that the simple case of two linear directrices has been elaborated; but probably a pretty complete account could be made up out of existing materials.

In conclusion, I may say that, while some of the methods which I have employed require great caution in their application, I believe them to be fundamentally sound. It is, of course, essential to solve accurately the different problems with reference to the elementary combinations. Any error in these particulars may lead to extravagantly wrong results; and I am not without apprehension that some of my conclusions may be faulty from this cause. But mistakes of this sort will not invalidate the principles used, which may, indeed, be probably carried much further. We may take systems of points instead of systems of lines, and *vice versâ*, or even employ combined systems of points and lines. I trust, however, that whatever may be thought of the means employed, some of my results possess interest.

The cases next in natural order seem to be the motion of a plane determined by the envelopes of two of its lines or by the locus of a point and the envelope of a line. I have obtained some general formulæ relating to these motions which may perhaps be worth more detailed discussion than I am able to give them at present. The motions of points and lines may be determined in an infinite number of ways, so that when the directrices are algebraical, the general motion shall be so too. Thus we may suppose two lines to be normal to two directrices, &c.* A more general case in which this holds is the motion determined by a curve sliding between given curves. The trajectories and envelopes will be algebraical when the given curves are so. This kind of motion is widely different in this respect from rolling, which under the simplest algebraical conditions generates transcendental curves.

Discussions took place on both communications.

Prof. Henrici exhibited card-board models of two ellipsoids, a hyperboloid of one sheet, and of an elliptic paraboloid; also stereograms of the models of surfaces exhibited at former meetings of the Society.

A brief discussion arose on the subject of Prof. Cayley's letter to the President (see Proceedings at the May meeting, p. 279); but it was determined that no definite action should be taken in the matter at the present time.

The following presents were then made to the Society, or subsequently received:—

Copies of the "Cambridge Mathematical Journal" for May, 1838; February and November, 1839; May and November, 1843; February, May, and November, 1844; February and May, 1845; "Quarterly Journal of Pure and Applied Mathematics," No. IV., March, 1855: from the President.

"Sulle trasformazioni razionali nello spazio," nota 1ª del Prof. L. Cremona, read May 4th, 1871: from the Author.

"Monatsbericht," April, June, and July, 1871.

"Proceedings of the Royal Society," Vol. XIX., Nos. 128, 129.

"Annali di Matematica," Serie IIª, Tom. IV., Fasc. 2, January, 1871.

"Journal of London Institution," Nos. 6, 7.

"Journal of Institute of Actuaries," July, 1871, No. LXXXIV.

"Annuaire de l'Académie Royale des Sciences, des Lettres, et des

---

* I have already alluded to the communication from M. Chasles to the *Comptes Rendus* for May 1871, from which I quote the following:—

"67. Le lieu d'un point d'où on peut mener à deux courbes $u^n$, $u^{n'}$, deux tangentes faisant un angle de grandeur donnée dans un sens de rotation déterminé est une courbe de l'ordre $2nn'$ qui a deux points multiples de l'ordre $nn'$ aux deux points circulaires à l'infini."

I had found that, for a point rigidly connected with the revolving angle, the trajectory is of the degree $4nn'$, and class $2(nm' + mn' + 2nn')$, where $m$, $m'$ are the degrees of the directrices. The circular points are cuspidal in the order $nn'$. But for the apex the order and class reduce to $2nn'$ and $nm' + n'm + 2nn'$, and the circular points are multiple in the order $nn'$.

Beaux Arts de Belgique," 1871; and "Bulletins de l'Académie Royale ..... de Belgique," 39me Année, 2me Série, Tomes 29, 30, 1870: from the Academy.

"Annual Report of the Board of Regents of the Smithsonian Institution for 1869": from the Smithsonian Institution.

"Crelle's Journal," 73 Band, drittes und viertes Heft.

"Appendix to Benjamin Anderson's Journey to Musada," New York, 1870; and "Transactions of the Connecticut Academy of Arts and Sciences," Vol. I., Part 2, 1867–71, New Haven: from the Academy.

"Verzeichniss der Abhandlungen der Königlich-Preussischen Akademie der Wissenschaften von 1710–1870, in Alphabetischer Folge der Verfasser," Berlin, 1871.

"Rapport sur les progres de Géométrie," par M. Chasles: from the President.

Model of the Surface called the "Cylindroid": from Prof. R. Stawell Ball.

[This is the model of a conoidal cubic surface which is presented in the Theory of the Geometrical Freedom of a Rigid Body. An abstract of a paper read before Section A of the British Association at Edinburgh, August, 1871, has been communicated by the Author to the "Philosophical Magazine and Journal of Science" for September, 1871. From this we learn that the name above given was suggested by Prof. Cayley, and that the equation to the surface is

$$z(x^2+y^2)-2axy=0.$$

In the model the parameter $a$ is 2·6 inches. The wires which correspond in the model with the generating lines of the surface represent the axes of the screws. The distribution of pitch upon the generating lines is shown by colouring a length of $2{\cdot}6\times\cos 2\theta$ inches upon each wire. The distinction between positive and negative pitches is indicated by colouring the former red and the latter black.]

# APPENDIX.

In a paper, read before the British Association Meeting, in August, 1871, Mr. Merrifield followed out the investigation of which he gave the first brief outline on pp. 222, 223. It is published also in "The Messenger of Mathematics" for October, 1871, pp. 81—88.

In connection with Mr. Todhunter's Historical Note (p. 232), the Secretaries have been favoured with the following references, pointed