On the Solution of Dynamical Problems in terms of Trigonometric Series. *By* E. T. WHITTAKER. Received and read November 14th, 1901.

1. The solution of a dynamical problem depends on the integration of a system of ordinary differential equations, in which the time is the independent variable. It is therefore always possible to express the motion in terms of infinite series proceeding in ascending powers of the time, and these series can easily be found; in fact, if the differential equations of the dynamical problem be written in the canonical form

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n),$$

then when H does not involve the time explicitly the solution is given by $2n$ equations of the type

$$q_r = a_r + (t-t_0)(a_r, K) + \frac{(t-t_0)^2}{2!}((a_r, K), K)$$
$$+\frac{(t-t_0)^3}{3!}(((a_r, K), K), K) + \frac{(t-t_0)^4}{4!}((((a_r, K), K), K), K),$$
$$+ \ldots,$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, t_0$ are the initial values of $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t$, respectively, and K is the same function of $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$ that H is of $q_1, q_2, \ldots, q_n, b_1, \ldots, b_n$, and where

$$(f, \phi) = \sum_{r=1}^{n} \left(\frac{\partial f}{\partial a_r}\frac{\partial \phi}{\partial b_r} - \frac{\partial f}{\partial b_r}\frac{\partial \phi}{\partial a_r}\right).$$

This method can without much difficulty be generalized to meet the case in which H involves the time explicitly; and, from the purely theoretical point of view, the solution of any dynamical problem is completely effected by these series together with the aggregate of the series derived from them by the process of analytic continuation.

The unsatisfactory nature of this result is, however, evident when we consider that these expansions in general converge only for very limited ranges of values of the time, and that the actual execution of the process of analytic continuation is attended with great difficulties. Moreover, the series fail to give what is often most needed, namely, a ready indication of the number and nature of the distinct types of motion which are possible in the problem. They are, for example, of no assistance to the investigator who aims at classifying the different kinds of orbits described in (say) the problem of three bodies, and determining the periodic and other remarkable solutions.

It may be noticed that the necessity for analytic continuation can be avoided by a change of the independent variable, as was shown by Poincaré;* the integral of the problem is then expressed in series proceeding according to ascending powers of the new variable; but this equally fails to overcome the second of the difficulties mentioned above.

A method is given in this paper for the expression of the solution of a dynamical problem in terms of trigonometric series. Each set of series will represent a family of trajectories, the limiting member of the set representing a position of stable equilibrium in the dynamical system. The process adopted may roughly be described as that of working outwards from a position of stable equilibrium; when such a position has been found the equations are transformed by a change of variables, the new variables being such as would be small if the system were merely describing small oscillations about equilibrium; after this the equations are again transformed by a change of variables, the new variables being such as would change but slowly if the system were describing small oscillations; and then the equations are again repeatedly transformed by changes of the variables, the result of each change being the destruction of one term in the Hamiltonian function—the process in this respect resembling that which underlies Delaunay's lunar theory. When all periodic terms of the Hamiltonian function have been destroyed the equations can be integrated, and the final solution of the dynamical problem is presented in the form of trigonometric series.

As an example of the results attained by this process, the problem of the simple pendulum may be considered. The equations of motion of the pendulum are

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

* *Acta Mathematica*, Vol. IV., p. 211 (1884).

where q is the sine of half the angle made by the pendulum with the vertical, and

$$H = \tfrac{1}{8}p^2 - \tfrac{1}{8}p^2q^2 + 2\mu^2q^2.$$

The form of the solution which is the goal of this paper is

$$q = \frac{2\pi}{K}\sum_{s=1}^{\infty}\frac{c^{\frac{1}{2}(2s-1)}}{1-c^{2s-1}}\sin\frac{(2s-1)\,\pi\mu\,(t-t_0)}{2K},$$

where K and t_0 are two arbitrary constants and

$$c = e^{(-\pi K')/K},$$

it being understood that K' is the complete elliptic integral complementary to K. This expansion does, in fact, represent the solution when the type of motion is oscillatory, *i.e.*, the pendulum does not make complete revolutions.

2. Consider then a dynamical problem, expressed by the differential equations

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n),$$

where $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$ are the generalized coordinates and momenta, and H is a given function of these quantities, not involving the time t explicitly.

The algebraic solution of the $2n$ simultaneous equations

$$\frac{\partial H}{\partial p_r} = 0, \qquad \frac{\partial H}{\partial q_r} = 0 \qquad (r = 1, 2, \ldots, n)$$

will furnish in general one or more sets of values $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$ for the quantities $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$ respectively; and each of these sets of values will correspond to a form of equilibrium or steady motion in the dynamical problem.

Take one of these sets of values $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$: it is required to find expressions which represent the solution of the dynamical problem when the motion is of a type terminated by this form of equilibrium or steady motion. Thus, if the dynamical problem considered were that of the simple pendulum, and the form of equilibrium chosen were that in which the pendulum hangs vertically downwards at rest, our aim would be to find expressions which represent the solution of the pendulum problem when the motion is of the oscillatory type, *i.e.*, when the pendulum does not make complete revolutions in the vertical plane.

Take then new variables $q'_1, q'_2, \ldots, q'_n; p'_1, \ldots, p'_n$ defined by the equations

$$q_r = a_r + q'_r, \qquad p_r = b_r + p'_r \qquad (r = 1, 2, \ldots, n).$$

The equations become

$$\frac{dq'_r}{dt} = \frac{\partial H}{\partial p'_r}, \qquad \frac{dp'_r}{dt} = -\frac{\partial H}{\partial q'_r} \qquad (r = 1, 2, \ldots, n),$$

where H is supposed to be expressed in terms of the new variables.

For sufficiently small values of the variables $q'_1, q'_2, \ldots, p'_n$, the Hamiltonian function H can be expanded in a multiple power series in the form

$$H = H_0 + H_1 + H_2 + H_3 + \ldots,$$

where H_k denotes terms homogeneous of the k-th degree in $q'_1, q'_2, \ldots, p'_n$, combined.

Now, since H_0 does not contain any of these variables, it exercises no influence on the differential equations, and so it can be omitted altogether. Moreover, the fact that the differential equations are satisfied when $q'_1, q'_2, \ldots, p'_n$ are put permanently equal to zero requires that H_1 should vanish identically. The expansion of H in ascending powers of the new variables therefore begins with terms involving their squares and products.

Henceforth we shall suppress the accents, there being no risk of confusion with the old variables, and so the system of differential equations of the problem can be written

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n),$$

where for sufficiently small values of the variables H can be expanded as

$$H = H_2 + H_3 + H_4 + \ldots,$$

and H_2 can be written in the form

$$H_2 = \tfrac{1}{2}\Sigma\,(a_{rr}q_r^2 + 2a_{rs}q_rq_s) + 2\Sigma b_{rs}q_rp_s + \tfrac{1}{2}\Sigma\,(c_{rr}p_r^2 + 2c_{rs}p_rp_s),$$

where $\qquad a_{rs} = a_{sr}, \quad c_{rs} = c_{sr}, \quad b_{rs} \neq b_{sr}.$

3. The next step is to find a new set of variables which will express H_2 in a simpler form.*

* In obtaining the transformation of this article I use a method suggested to me by Mr. T. J. I'A. Bromwich, M.A., Fellow of St. John's College, Cambridge, which seems to furnish the transformation more directly than the method I had myself devised.

For this purpose, consider the set of $2n$ equations

$$\left.\begin{aligned} -\lambda y_r &= a_{r1}x_1+a_{r2}x_2+\ldots+a_{rn}x_n+b_{r1}y_1+\ldots+b_{rn}y_n \\ \lambda x_r &= b_{1r}x_1+b_{2r}x_2+\ldots+b_{nr}x_n+c_{r1}y_1+\ldots+c_{rn}y_n \end{aligned}\right\} \qquad (r=1, 2, \ldots, n).$$

On solving these, we obtain for λ the equation

$$0 = \begin{vmatrix} a_{11}, & \ldots, & a_{1n}; & b_{11}+\lambda, & \ldots, & b_{1n} \\ \vdots & & & & & \\ a_{1n}, & \ldots, & a_{nn}; & b_{n1}, & \ldots, & b_{nn}+\lambda \\ b_{11}-\lambda, & \ldots, & b_{n1}; & c_{11}, & \ldots, & c_{1n} \\ \vdots & & & & & \\ b_{1n}, & \ldots, & b_{nn}-\lambda; & c_{1n}, & \ldots, & c_{nn} \end{vmatrix}.$$

Clearly, if λ be a root of this equation, then $-\lambda$ is also a root.

Corresponding to each root λ thus furnished, there will be a set of values for the ratios of the quantities $x_1, x_2, \ldots, y_1, \ldots, y_n$. Let the roots of the determinantal equation be $\lambda_1, \lambda_2, \ldots, \lambda_n, -\lambda_1, -\lambda_2, \ldots, -\lambda_n$; let a set of values of $x_1, x_2, \ldots, y_n$ corresponding to λ_r be denoted by ${}_rx_1, \ldots, {}_rx_n, {}_ry_1, \ldots, {}_ry_n$; and let a set corresponding to the root $-\lambda_r$ be denoted by ${}_{-r}x_1, \ldots, {}_{-r}x_n, {}_{-r}y_1, \ldots, {}_{-r}y_n$. Then we have

$$-\lambda_r\, {}_ry_p = a_{p1}\, {}_rx_1+\ldots+a_{pn}\, {}_rx_n+b_{p1}\, {}_ry_1+\ldots+b_{pn}\, {}_ry_n,$$

$$\lambda_r\, {}_rx_p = b_{1p}\, {}_rx_1+\ldots+b_{np}\, {}_rx_n+c_{p1}\, {}_ry_1+\ldots+c_{pn}\, {}_ry_n.$$

Multiply by ${}_sx_p$ and ${}_sy_p$, and add to the similar results for other suffixes. Then

$$\lambda_r \Sigma\, ({}_rx\, {}_sy-{}_sx\, {}_ry) = H(r, s),$$

where the summation is extended over all *equal* suffixes of x, y, and where

$$H(r, s) = a_{11}\, {}_rx_1\, {}_sx_1+a_{12}\, ({}_rx_1\, {}_sx_2+{}_sx_1\, {}_rx_2)+\ldots+b_{11}\, ({}_rx_1\, {}_sy_1+{}_sx_1\, {}_ry_1)+\ldots$$
$$\ldots+c_{11}\, {}_ry_1\, {}_sy_1+\ldots,$$

which is symmetrically related to r and s.

Interchanging r and s, we have

$$\lambda_s \Sigma\, ({}_sx\, {}_ry-{}_rx\, {}_sy) = H(r, s).$$

Thus
$$(\lambda_r+\lambda_s)\, \Sigma\, ({}_rx\, {}_sy-{}_sx\, {}_ry) = 0.$$

So, unless $\lambda_r+\lambda_s$ is zero, we have

$$\Sigma\, ({}_rx\, {}_sy-{}_sx\, {}_ry) = 0,$$

and consequently $H(r, s) = 0.$

If $\lambda_r + \lambda_s = 0,$

we have ${}_s x = {}_r x, \quad {}_s y = -{}_r y;$

and therefore $\lambda_r \Sigma ({}_r x\, {}_{-r} y - {}_{-r} x\, {}_r y) = H(r, -r).$

If then we make the substitutions expressed by the equations

$$q_r = {}_1 x_r q_1' + {}_2 x_r q_2' + \dots + {}_n x_r q_n' + {}_{-1} x_r p_1' + \dots + {}_{-n} x_r p_n', \qquad (r = 1, 2, \dots, n),$$

and p = a similar expression with y's instead of x's, it is clear that the coefficient of $\delta q_r' \Delta p_s'$ in $\Sigma (\delta q_1 \Delta p_1 - \Delta q_1 \delta p_1)$, where δ and Δ denote any two independent modes of variation, is $\Sigma ({}_r x\, {}_s y - {}_s x\, {}_r y)$, which is zero when $\lambda_r + \lambda_s$ is not zero. Thus $\Sigma (\delta q_1 \Delta p_1 - \Delta q_1 \delta p_1)$ contains no terms except such as $\Sigma (\delta q_r' \Delta p_r' - \Delta q_r' \delta p_r')$, and the coefficient of this term is $\Sigma ({}_r x\, {}_{-r} y - {}_{-r} x\, {}_r y)$. Now hitherto the actual values of ${}_r x$, ${}_r y$ have not been fixed, as only their ratios are determined from their equations of definition. We can therefore choose their values so that

$$\Sigma ({}_r x\, {}_{-r} y - {}_{-r} x\, {}_r y) = 1$$

for each value of r; and then we have

$$\Sigma (\delta q_r \Delta p_r - \Delta q_r \delta p_r) = \Sigma (\delta q_r' \Delta p_r' - \Delta q_r' \delta p_r').$$

Now this last equation expresses the condition that a transformation from the variables $q_1, q_2, \dots, q_n, p_1, \dots, p_n$, to the variables $q_1', q_2', \dots, q_n', p_1', \dots, p_n'$, shall be *canonical*, *i.e.*, that it shall leave unaltered the Hamiltonian form of any system of Hamiltonian differential equations in which these quantities are the variables. Further, if in H_2 we substitute for the old variables $q_1, q_2, \dots, q_n, p_1, \dots, p_n$, in terms of the new variables $q_1', q_2', \dots, q_n', p_1', \dots, p_n'$, we obtain

$$H_2 = \sum_{r=1}^{n} H(r, -r)\, q_r' p_r'$$

or

$$H_2 = \sum_{r=1}^{n} \lambda_r q_r' p_r'.$$

Thus, when this change of variables is made in the differential equations of our dynamical problem, as given at the end of § 2, the differential equations take the form

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial p_r'} \qquad (r = 1, 2, \dots, n),$$

where for sufficiently small values of the variables H can be expanded as

$$H = H_2 + H_3 + H_4 + \ldots,$$

in which H_k is homogeneous of degree k in $q'_1, q'_2, \ldots, q'_n, p'_1, \ldots, p'_n$, and H_2 can be written in the form

$$H_2 = \sum_{r=1}^{n} \lambda_r q'_r p'_r.$$

4. To this system apply the further transformation from the variables $q'_1, q'_2, \ldots, q'_n, p'_1, \ldots, p_n$ to variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$, defined by the equations

$$q_r = \frac{\partial W}{\partial p_r}, \qquad p'_r = \frac{\partial W}{\partial q'_r} \qquad (r = 1, 2, \ldots, n),$$

where
$$W = \sum_{r=1}^{n} p_r q'_r - \tfrac{1}{2} \sum_{r=1}^{n} \frac{p_r^2}{\lambda_r} - \tfrac{1}{4} \sum_{r=1}^{n} \lambda_r q_r'^2.$$

From the form of this transformation, it is known to be canonical, and so the differential equations retain their Hamiltonian form. Moreover, since the transformation is linear, the quantities $H_2, H_3, \ldots$ will still be homogeneous polynomials in the new variables $q_1, q_1, \ldots, q_n, p_1, \ldots, p_n$; and in particular it is easily seen that

$$H_2 = \tfrac{1}{2} \sum_{r=1}^{n} (p_r^2 - \lambda_r^2 q_r^2).$$

The quantities λ_r, which have been obtained as the roots of a determinant, are constants, depending on the constitution of the dynamical system whose motion we are considering; in a very large class of cases, which for practical purposes is the most important class, they are purely imaginary, so that the quantities $-\lambda_r^2$ are real and positive; in this case the particular solution of the dynamical problem from which we start, and which is to be the limiting case of the family of solutions we propose to find, may be called a position of stable equilibrium or steady motion. We shall confine our attention to systems of this kind, and to indicate this we shall write λ_r^2 for $-\lambda_r^2$. Thus (summarizing the last three sections of this paper) *the equations of motion of the dynamical system have been brought to the form*

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n),$$

where
$$H = H_2 + H_3 + H_4 + \ldots,$$

in which H_r is a homogeneous polynomial in the variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$; and, in particular,

$$H_2 = \tfrac{1}{2} \sum_{r=1}^{n} (p_r^2 + \lambda_r^2 q_r^2),$$

where the quantities λ_r are supposed real.

It is clear that, in order to consider the small oscillations of the system about the position which has been the starting-point of our work, we should merely have to neglect altogether the part $H_3 + H_4 + H_5 + \ldots$ of H, and that the variables $q_1, q_2, \ldots, q_n$ would then be the "principal coordinates" for the small oscillations.

5. The system of differential equations which has been obtained will now be further transformed by applying to it a transformation from the variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$, to new variables $q_1', q_2', \ldots, q_n', p_1', \ldots, p_n'$, defined by the system of equations

$$p_r' = \frac{\partial W}{\partial q_r'}, \qquad q_r = \frac{\partial W}{\partial p_r} \qquad (r = 1, 2, \ldots, n),$$

where
$$W = \sum_{r=1}^{n} \left[q_r' \sin^{-1} \frac{p_r}{(2\lambda_r q_r')^{\frac{1}{2}}} + \frac{p_r}{2\lambda_r} \{2\lambda_r q_r' - p_r^2\}^{\frac{1}{2}} \right].$$

From the form in which this transformation is expressed, it is clearly canonical, and so will leave unaltered the Hamiltonian form of the differential equations.

The equations connecting the old and new variables are

$$\left.\begin{aligned} p_r &= (2\lambda_r q_r')^{\frac{1}{2}} \sin p_r' \\ q_r &= (2q_r')^{\frac{1}{2}} \lambda_r^{-\frac{1}{2}} \cos p_r' \end{aligned}\right\} \quad (r = 1, 2, \ldots, n).$$

The differential equations now become

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2, \ldots, n),$$

where
$$H = \lambda_1 q_1' + \lambda_2 q_2' + \ldots + \lambda_n q_n' + H_3 + H_4 + \ldots,$$

and now H_r denotes an aggregate of terms which are homogeneous of degree $\frac{r}{2}$ in the quantities q_r', and homogeneous of degree r in the quantities $\cos p_r'$, $\sin p_r'$.

Since a product of powers of the quantities $\cos p_r'$, $\sin p_r'$ can be expressed as a sum of sines and cosines of angles of the form

$$n_1 p_1' + n_2 p_2' + \ldots + n_n p_n',$$

where $n_1, n_2, \ldots, n_n$ have integer or zero values, it follows that H_r can be expressed as the sum of a finite number of terms, each of the form

$$q_1'^{m_1} q_2'^{m_2} \ldots q_n'^{m_n} \begin{matrix}\sin \\ \cos\end{matrix} (n_1 p_1' + n_2 p_2' + \ldots + n_n p_n'),$$

where
$$m_1 + m_2 + \ldots + m_n = \frac{r}{2}$$

and
$$|n_1| + |n_2| + \ldots + |n_n| \leqslant r,$$

since we must have
$$|n_r| \leqslant 2m_r \quad (r = 1, 2, \ldots, n).$$

Let all the quantities H_r be transformed in this way, so that H is expressed in the form

$$H = \Sigma A_{n_1, n_2, \ldots, n_n}^{m_1, m_2, \ldots, m_n} q_1'^{m_1} q_2'^{m_2} \ldots q_n'^{m_n} \begin{matrix}\sin \\ \cos\end{matrix} (n_1 p_1' + n_2 p_2' + \ldots + n_n p_n'),$$

where for each term we have

$$|n_1| + |n_2| + \ldots + |n_n| \leqslant 2(m_1 + m_2 + \ldots + m_n),$$

and the series is clearly absolutely convergent for all values of $p_1', \ldots, p_n'$, provided $q_1', q_2', \ldots, q_n'$ do not exceed certain limits of magnitude. From the absolute convergence it follows that the order of the terms can be rearranged in any arbitrary way; we shall suppose them so ordered that all the terms involving the same argument $n_1 p_1' + \ldots + n_n p_n'$ are collected together, and thus H can be expressed in the form

$$H = a_{0, 0, 0, \ldots, 0} + \Sigma a_{n_1, n_2, \ldots, n_n} \cos (n_1 p_1' + \ldots + n_n p_n')$$
$$+ \Sigma b_{n_1, n_2, \ldots, n_n} \sin (n_1 p_1' + \ldots + n_n p_n'),$$

where the quantities a and b are functions of $q_1', q_2', \ldots, q_n'$, and the expansion of $a_{n_1, n_2, \ldots, n_n}$ or $b_{n_1, n_2, \ldots, n_n}$ in powers of $q_1', \ldots, q_n'$ contains no terms of order lower than $\frac{1}{2}\{|n_1| + |n_2| + \ldots + |n_n|\}$; and where the summations extend over all positive and negative integer and zero values of $n_1, n_2, \ldots, n_n$, except the combination

$$n_1 = n_2 = \ldots = n_n = 0.$$

Moreover, the expansion of $a_{0,0,\ldots,0}$ begins with the terms

$$\lambda_1 q_1' + \lambda_2 q_2' + \ldots + \lambda_n q_n';$$

and, when $q_1', q_2', \ldots, q_n'$ are small, these are the most important terms in H, since they contribute terms independent of $q'_1, q'_2, \ldots, q'_n$ to the differential equations.

For convenience we shall often speak of $q'_1, q'_2, \ldots, q'_n$ as "small," in order to have a definite idea of the relative importance of the terms which occur. It will be understood that $q'_1, q'_2, \ldots, q'_n$ are not, however, infinitesimal, and, in fact, are not restricted at all in magnitude except so far as is required to ensure the convergence of the various series in which they occur.

To avoid unnecessary complexity, we shall ignore the terms

$$\Sigma b_{n_1, n_2, \ldots, n_n} \sin (n_1 p'_1 + \ldots + n_n p'_n)$$

in H, as they are to be treated in the same way as the terms

$$\Sigma a_{n_1, n_2, \ldots, n_n} \cos (n_1 p'_1 + \ldots + n_n p'_n),$$

and their presence complicates, but does not in any important respect modify, the later developments.

The form to which the problem has now been brought may therefore be stated as follows (suppressing the accents in the new variables, as there is no longer any risk of confusion with the old variables).

The equations of motion are

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n),$$

where $H = a_{0, 0, \ldots, 0} + \Sigma a_{n_1, n_2, \ldots, n_n} \cos (n_1 p_1 + n_2 p_2 + \ldots + n_n p_n)$,

and the quantities a are functions of $q_1, q_2, q_3, \ldots, q_n$ *only; moreover, the periodic part of H is small compared with the non-periodic part* $a_{0,0,\ldots,0}$; *a term which has for argument* $n_1 p_1 + n_2 p_2 + \ldots + n_n p_n$ *has its coefficient* $a_{n_1, n_2, \ldots, n_n}$ *at least of the order* $\frac{1}{2} \{ |n_1| + |n_2| + \ldots + |n_n| \}$ *in the small quantities* $q_1, q_2, \ldots, q_n$; *and the expansion of* $a_{0,0,\ldots,0}$ *begins with terms* $\lambda_1 q_1 + \lambda_2 q_2 + \ldots + \lambda_n q_n$.

It follows from this that when the variables $q_1, q_2, \ldots, q_n$ are small they are nearly constant, while the variables $p_1, p_2, \ldots, p_n$ vary almost proportionally to the time.

6. To the variables of this system of differential equations we shall now apply a further transformation, the effect of which will be the removal of one of the periodic terms in H; the aim being to still further accentuate the feature already noted, namely, that the

non-periodic part of H is much more important than the periodic part.*

Let then one of the periodic terms in H be selected—say

$$a_{n_1, n_2, \ldots, n_n} \cos (n_1 p_1 + n_2 p_2 + \ldots + n_n p_n).$$

Write $H = a_{0, 0, \ldots, 0} + a_{n_1, n_2, \ldots, n_n} \cos (n_1 p_1 + n_2 p_2 + \ldots + n_n p_n) + R,$

so that R denotes the rest of the periodic terms of H. When we wish to put in evidence the fact that $a_{n_1, n_2, \ldots, n_n}$ is a function of its arguments, we shall write it $a_{n_1, n_2, \ldots, n_n} (q_1, q_2, \ldots, q_n)$.

Now apply to the variables the transformation from $q_1, q_2, \ldots, q_n$; $p_1, \ldots, p_n$ to $q_1', q_2', \ldots, q_n'$; $p_1', \ldots, p_n'$, defined by the equations

$$p_r' = \frac{\partial W}{\partial q_r'}, \qquad q_r = \frac{\partial W}{\partial p_r} \qquad (r = 1, 2, \ldots, n);$$

where $$W = q_1' p_1 + q_2' p_2 + \ldots + q_n' p_n + f(q_1', q_2', \ldots, q_n', \theta)$$

and $$\theta = n_1 p_1 + n_2 p_2 + \ldots + n_n p_n.$$

We shall suppose that f is a function, as yet undetermined, of the arguments indicated. The transformation is seen at once from its form to be canonical; so the Hamiltonian form of the differential equations will not be affected by it, and the problem is expressed by the system

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2, \ldots, n),$$

where
$$H = a_{0, 0, \ldots, 0} \left(q_1' + n_1 \frac{\partial f}{\partial \theta}, \ldots, q_n' + n_n \frac{\partial f}{\partial \theta}\right)$$
$$+ a_{n_1, n_2, \ldots, n_n} \left(q_1' + n_1 \frac{\partial f}{\partial \theta}, \ldots, q_n' + n_n \frac{\partial f}{\partial \theta}\right) \cos \theta + R,$$

and θ and R are supposed to be expressed in terms of the new variables by means of the equations of transformation

$$p_r' = p_r + \frac{\partial f}{\partial q_r'}, \quad q_r = q_r' + n_r \frac{\partial f}{\partial \theta} \quad (r = 1, 2, \ldots, n).$$

The function f is, as yet, undetermined and at our disposal. It will

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* The analogy of this with the method of Delaunay's lunar theory will be noticed. Although the analysis is different from Delaunay's, the idea is essentially the same.

be chosen so as to satisfy the condition that θ shall identically disappear from the expression

$$a_{0,0,\ldots,0}\left(q_1'+n_1\frac{\partial f}{\partial\theta}, \ldots, q_n'+n_n\frac{\partial f}{\partial\theta}\right)$$
$$+a_{n_1,n_2,\ldots,n_n}\left(q_1'+n_1\frac{\partial f}{\partial\theta}, \ldots, q_n'+n_n\frac{\partial f}{\partial\theta}\right)\cos\theta;$$

so that this quantity is a function of $q_1', q_2', \ldots, q_n'$ alone. Put it equal to $a'_{0,0,\ldots,0}$, where $a'_{0,0,\ldots,0}$ is a function of $q_1', q_2', \ldots, q_n'$, as yet undetermined. Then the equation

$$a_{0,0,\ldots,0}\left(q_1'+n_1\frac{\partial f}{\partial\theta}, \ldots, q_n'+n_n\frac{\partial f}{\partial\theta}\right)$$
$$+a_{n_1,n_2,\ldots,n_n}\left(q_1'+n_1\frac{\partial f}{\partial\theta}, \ldots, q_n'+n_n\frac{\partial f}{\partial\theta}\right)\cos\theta=a'_{0,0,\ldots,0}.$$

determines $\frac{\partial f}{\partial\theta}$ in terms of $q_1', q_2', \ldots, q_n', a'_{0,0,\ldots,0}$ and $\cos\theta$.

Suppose the solution of this equation for $\frac{\partial f}{\partial\theta}$ is expressed in the form of a series of cosines of multiples of θ (which can be done, for instance, by successive approximation), so that

$$\frac{\partial f}{\partial\theta}=c_0+\sum_{k=1}^{\infty}c_k\cos k\theta,$$

where $c_0, c_1, c_2, \ldots$ are known functions of $q_1', q_2', \ldots, q_n', a'_{0,0,0,\ldots,0}$.

Now $a'_{0,0,\ldots,0}$ is as yet undetermined, and is at our disposal. Impose the condition that c_0 is to be zero. This determines $a'_{0,0,\ldots,0}$ as a function of $q_1', q_2', \ldots, q_n'$; and, on substituting its value in the series for $\frac{\partial f}{\partial\theta}$, we have

$$\frac{\partial f}{\partial\theta}=\sum_{k=1}^{\infty}c_k\cos k\theta,$$

where now $c_1, c_2, c_3, \ldots$ are known functions of $q_1', q_2', \ldots, q_n'$.

Integrating this equation with respect to θ, and for our purpose taking the constant of integration to be zero, we have

$$f=\sum_{k=1}^{\infty}\frac{c_k}{k}\sin k\theta.$$

The equations defining the transformation now become

$$\left.\begin{aligned} p'_r &= p_r + \sum_{k=1}^{\infty} \frac{1}{k} \frac{\partial c_k}{\partial q'_r} \sin k\theta \\ q_r &= q'_r + n_r \sum_{k=1}^{\infty} c_k \cos k\theta \end{aligned}\right\} \quad (r = 1, 2, \ldots, n).$$

Multiply the first set of these equations by $n_1, n_2, \ldots, n_n$, respectively, and add them; writing

$$n_1 p'_1 + n_2 p'_2 + \ldots + n_n p'_n = \theta',$$

we have $\theta' = \theta + \sum_{k=1}^{\infty} \frac{1}{k} \left(n_1 \frac{\partial c_k}{\partial q'_1} + n_2 \frac{\partial c_k}{\partial q'_2} + \ldots + n_n \frac{\partial c_k}{\partial q'_n} \right) \sin k\theta.$

Reversing this series, we have

$$\theta = \theta' + \sum_{k=1}^{\infty} d_k \sin k\theta',$$

where $d_1, d_2, \ldots$ are known functions of $q'_1, q'_2, \ldots, q'_n$. Substituting this value of θ in the equations of transformation, they become

$$\left.\begin{aligned} p_r &= p'_r + \sum_{k=1}^{\infty} {}_r e_k \sin k\theta' \\ q_r &= q'_r + n_r \sum_{k=1}^{\infty} g_k \cos k\theta \end{aligned}\right\} \quad (r = 1, 2, \ldots, n),$$

where all the quantities ${}_r e_k$, g_k are known functions of $q'_1, q'_2, \ldots, q'_n$.

Now, before the transformation, the quantity R consisted of an aggregate of terms of the type

$$R = \Sigma a_{m_1, m_2, \ldots, m_n} \cos (m_1 p_1 + \ldots + m_n p_n).$$

When the values just found for $q_1, \ldots, q_n, p_1, \ldots, p_n$ are substituted in this expression, and the series is reduced by replacing powers and products of trigonometrical functions of $p'_1, p'_2, \ldots, p'_n$ by cosines of multiples of $p'_1, p'_2, \ldots, p'_n$, it is clear that R will consist of an aggregate of terms of the type

$$R = \Sigma a'_{m_1, m_2, \ldots, m_n} \cos (m_1 p'_1 + m_2 p'_2 + \ldots + m_n p'_n),$$

where the quantities a' are known functions of $q'_1, q'_2, \ldots, q'_n$.

We see therefore, omitting the accents of the new variables, that, *after the transformation has been effected, the dynamical problem is still expressed by a system of equations of the form*

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, n),$$

where $$H = a_{0,0,\ldots,0} + \Sigma a_{m_1, m_2, \ldots, m_n} \cos(m_1 p_1 + \ldots + m_n p_n),$$

and where the coefficients a are known functions of $q_1, q_2, \ldots, q_n$.

7. Let us now review the whole effect of the transformation described in the last article. The differential equations of motion have the same general form as before; but one term, namely,

$$a_{n_1, n_2, \ldots, n_n} \cos(n_1 p_1 + \ldots + n_n p_n),$$

has been transferred from the periodic part of H to its non-periodic part; the periodic part of H is less important, in comparison with the non-periodic part, than it was before the transformation was made.

8. Having now completed the absorption of this periodic term into the non-periodic part of H, we proceed to absorb one of the periodic terms of the new expansion of H into the non-periodic part, by a repetition of the same process. In this way we can continually enrich the non-periodic part of H at the expense of the periodic part, and ultimately, after a number of applications of the transformation, the periodic part of H will become so insignificant that it may be neglected. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$; $\beta_1, \beta_2, \ldots, \beta_n$ be the variables at which we arrive as a result of the final transformation. Then the equations of motion are

$$\frac{d\alpha_r}{dt} = \frac{\partial H}{\partial \beta_r}, \qquad \frac{d\beta_r}{dt} = -\frac{\partial H}{\partial \alpha_r} \qquad (r = 1, 2, \ldots, n),$$

where H, consisting only of its non-periodic part, is a function of $\alpha_1, \alpha_2, \ldots, \alpha_n$ only. We have, therefore,

$$\frac{d\alpha_r}{dt} = 0, \qquad \beta_r = -\int \frac{\partial H}{\partial \alpha_r} dt,$$

and so the quantities α_r are constants, while the quantities β_r are of the form

$$\beta_r = \mu_r t + \epsilon_r \qquad (r = 1, 2, \ldots, n),$$

where $$\mu_r = -\frac{\partial H}{\partial \alpha_r};$$

the quantities ϵ_r are arbitrary constants, and the part of μ_r independent of $a_1, a_2, \ldots, a_n$ is $-\lambda_r$.

9. Having now solved the equations of motion in their final form, it remains only to express the original coordinates of the dynamical problem in terms of the ultimate coordinates $a_1, a_2, \ldots, \beta_n$. Remembering that the product of any number of canonical transformations is a canonical transformation, it is easily seen that the variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$ used at the end of § 5 can be expressed in terms of $a_1, a_2, \ldots, a_n, \beta_1, \ldots, \beta_n$ by equations of the form

$$\left.\begin{aligned} \beta_r &= p_r + \Sigma \frac{\partial k_{m_1, m_2, \ldots, m_n}}{\partial a_r} \sin(m_1 p_1 + \ldots + m_n p_n) \\ q_r &= a_r + \Sigma m_r k_{m_1, m_2, \ldots, m_n} \cos(m_1 p_1 + \ldots + m_n p_n) \end{aligned}\right\} \quad (r = 1, 2, \ldots, n),$$

or of the form

$$\left.\begin{aligned} q_r &= f_r(a_1, a_2, \ldots, a_n) + \Sigma a_{m_1, m_2, \ldots, m_n} \cos(m_1\beta_1 + \ldots + m_n\beta_n) \\ p_r &= \beta_r + \sum_r b_{m_1, m_2, \ldots, m_n} \sin(m_1\beta_1 + \ldots + m_n\beta_n) \end{aligned}\right\} \quad (r = 1, 2, \ldots, n),$$

where the quantities a and b are functions of $a_1, a_2, \ldots, a_n$.

From this it follows that the variables $q_1, q_2, \ldots, q_n, p_1, \ldots, p_n$ of § 1, in terms of which the dynamical problem was originally expressed, are obtained by the process of this paper in the form of trigonometric series, proceeding in sines and cosines of sums of multiples of the n angles $\beta_1, \beta_2, \ldots, \beta_n$. These angles are linear functions of the time, of the form $\mu_r t + \epsilon_r$; the quantities ϵ_r are n of the $2n$ arbitrary constants of the solution, while the quantities μ_r are of the form

$$\mu_r = \lambda_r + \sum_{k_1, k_2, \ldots} a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n}.$$

The coefficients in the trigonometric series are functions of the arbitrary constants $a_1, a_2, \ldots, a_n$ only.

The expansions thus found represent a family of solutions of the dynamical problem, the limiting member of the family being the position of equilibrium or steady motion which was our starting point.

If we were to consider those solutions which differ only slightly from this terminal case, we might neglect all powers of $a_1^{\frac{1}{2}}, a_2^{\frac{1}{2}}, \ldots, a_n^{\frac{1}{2}}$.

above the first, and then it is easily seen that the quantities $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ would reduce to $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, and so the expansions would become the well-known formulæ which represent small oscillations of the dynamical system about the position of equilibrium or steady motion.

In practice, the expansions can be carried as far as may be desired by including the requisite powers of $q_1, q_2, \ldots, q_n$, and so of $a_1, a_2, \ldots, a_n$. As a simple example of this, it will be found by this method that the dynamical system expressed by the differential equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

where

$$H = \tfrac{1}{2}p^2 + \frac{l^4k^2}{2q^2} - \frac{l^3k^2}{q}$$

possesses a family of solutions represented by the expansion (retaining only terms of order less than $a^{\frac{3}{2}}$)

$$q = l + \frac{3a}{kl} + \left(\frac{2a}{k}\right)^{\frac{1}{2}} \cos\beta - \frac{3a}{2kl}\cos 2\beta,$$

where

$$\beta = -\left(k + \frac{aa}{2l^3}\right)t + \epsilon,$$

and a and ϵ are arbitrary constants.

This family of solutions is terminated by the equilibrium-solution

$$q = l,$$

which corresponds to the value zero of a. The small oscillations of the system about this equilibrium-position would be derived by neglecting all powers of a above $a^{\frac{1}{2}}$ in the above expansion, which gives

$$q = l + \left(\frac{2a}{k}\right)^{\frac{1}{2}} \cos(-kt + \epsilon),$$

where a and ϵ are the arbitrary constants; a result in accord with the customary form.

Thursday, December 12th, 1901.

Major MACMAHON, R.A., F.R.S., Vice-President, in the Chair.

Ten members present.

The Chairman moved the adoption of the Treasurer's report, after reading the Auditor's report, and Prof. Alfred Lodge seconded the motion, as well as votes of thanks to the Treasurer and the Auditor. The votes were carried unanimously.

Messrs. G. Birtwistle, B.A., Fellow of Pembroke College, Cambridge; and Augustus P. Thompson, B.A., Scholar of Pembroke College, Cambridge; and the Rev. J. Cullen, S.J., were elected members.

Mr. R. J. Dallas was admitted into the Society.

Prof. Love communicated a paper by Mr. J. H. Michell, "On the Flexure of a Circular Plate." Prof. Lamb spoke on the subject of the paper.

The following presents were made to the Library:—

"Educational Times," December, 1901.

"Indian Engineering," Vol. xxx., Nos. 17-20, Oct. 26-Nov. 16, 1901.

"Mathematical Gazette," Vol. II., No. 30, 1901.

"Kansas University Quarterly," Vol. II., No. 6; April, 1901.

"Supplemento al Periodico di Matematica," Anno v., Fasc. 1, Nov., 1901; Livorno.

Frick, J.—"On Liquid Air and its Application (Liquid Air Wells)," 8vo; London, 1901.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. LXIX., No. 452, 1901.

"Bulletin of the American Mathematical Society," Series 2, Vol. VIII., No. 2; New York, 1901.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XIV., No. 5; Coimbra, 1901.

"Bulletin des Sciences Mathématiques," Tome XXV., Oct., 1901; Paris.

"Annali di Matematica," Série 3, Tomo VI., Fasc. 4; Milano, 1901.

"Archives Néerlandaises," Serie 2, Tome VI.; La Haye, 1901.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. X., Fasc. 9; Roma, 1901.

"Annales de la Faculté des Sciences de Marseille," Tome XI., Fasc. 1-9; Paris, 1901.

"Proceedings of the Royal Irish Academy," Vol. VI., Nos. 2, 3, Jan., Oct., 1901; Dublin.