

## NEWTON'S THEOREMS ON THE ATTRACTION OF SPHERES.

IT seems quite certain that Newton was stopped at first from pursuing the problem of the planetary motions because he very soon arrived at the idea of universal gravitation, and then thought that the calculation of the whole resultant attraction of a large piece of matter, such as the earth or the sun, is such a complicated problem that it could not at that time be solved with any approach to accuracy. Thus the description of the celestial motions which was founded on the consideration of sun and planets as points apparently could only be regarded as a very rough approximation. In 1685, however, Newton found that the problem of the attraction of a sphere on an external point admits of an unexpectedly simple solution, and it was then, under the influence of the vivid interest that would naturally be caused by the discovery that the nearly spherical planets could be treated, with a very close approximation, as mathematical points, that he returned with renewed enthusiasm to his task. We may, it seems, also conclude that the problem solved in the seventy-first Proposition of Section XII of the first Book of the *Principia*, in which the attraction of a thin spherical shell on a particle outside it was found, preceded the discovery of the seventeenth Proposition, in which the attraction of a thin spherical shell on a particle *inside* it was found. To the well-known letter to Halley of June 20, 1686, which fixes the date of the

former discovery as 1685, we may add Newton's words in the eighth Proposition of the third Book of the *Principia*:

"After I had found that the force of gravity toward a whole planet did arise from, and was compounded of, the forces of gravity toward all its parts, and toward each part was in the inverse proportion of the squares of the distances from the part; I was yet in doubt whether that inverse duplicate proportion accurately held, or only nearly so, in the total force compounded of so many partial ones. For it might be that the proportion which accurately enough held for greater distances should be wide of the truth near the surface of the planet, where the distances of the particles are unequal and their situation dissimilar. But by the help of the seventy-fifth and the seventy-sixth Propositions of the first Book and their Corollaries, I was at last satisfied of the truth of this proposition."

The methods that Newton gave in his *Principia* for finding the attractions of spherical shells and bodies are the same in all editions. He found the attraction of a thin spherical shell of uniform density on an external point, which was the first step to determining the attraction of a solid sphere on such a point, in his seventy-first Proposition. In all that follows,  $S$  will denote the center of the shell which is of radius  $r$ , and  $P$  will denote the external point. Newton first found the attraction on  $P$  of a thin zone on the shell which is the surface cut out by an infinitely small arc of a great circle in the plane of the drawing we would naturally make, when this great circle revolves round  $SP$  as diameter and therefore perpendicularly to the plane of the drawing. Each element of the surface of this zone exerts an attraction on  $P$  which is proportional directly to the area of the element and inversely to the square of its distance from  $P$ . The first step is, then, to find the area of such a zone, which is the first step in the process, usual since the time of Archimedes, for finding the

area of the surface of a sphere. By the use of mathematical artifices which are not of a very complicated character, Newton found that the component of the attraction of the zone along  $SP$ —the total attraction of the zone on  $P$  being, by symmetry, along  $SP$ —is inversely proportional to the square of  $SP$ . The same is true of any other zone whose plane is parallel to that of the zone just considered. Therefore, by composition of ratios, the whole attraction of the spherical surface is in the same duplicate ratio.

From this we can proceed to the case of a solid sphere, by summing together various homogeneous spherical shells of center  $S$ , and we thus find that the mass of the whole set of shells is as the cube of the sphere's diameter, and the attraction on  $P$  is directly as this mass and inversely as the square of  $SP$ . This is the result of the seventy-fourth Proposition together with part of the seventy-second Proposition.

In the *Principia*, the seventieth Proposition concerns the attraction exerted by a spherical shell on a particle  $P$  within it. If we imagine an infinitely small double cone drawn so that  $P$  is the vertex and the two ends of the cone cut the sphere in small circles, the attractions of these small circles on  $P$  are opposite and proportional to the squares of their diameters while they are inversely proportional to the squares of their distances from  $P$ . Imagination of the geometrical figure made by  $P$  and any section of the shell passing through  $Q$  and the axis of the cone, and use of the thirty-fifth Proposition of Euclid's third Book show that these attractions on  $P$  are equal and opposite, and that they therefore destroy one another.

An application of the seventieth and seventy-second Propositions is contained in the seventy-third Proposition, in which is determined the attraction on a particle which is situated inside a solid sphere. If another concentric sphere is imagined which passes through the particle in question,

the solid sphere is divided into two parts: the part outside P, by the seventieth Proposition, exerts no attraction on P; the other part, being a sphere attracting an external particle whose distance from the center is equal to the radius, exerts, by the seventy-second Proposition, a force of attraction to its center which is proportional to  $r^3/r^2$ , where  $r$  is the radius SP. The fact that the attraction on a particle within the surface of the earth is proportional to the distance from the center instead of being inversely proportional to the square of this distance was touched upon by Newton in his correspondence with Halley of 1686 on the subject of the experiment suggested to Hooke in 1679. In his letters to Hooke, Newton carelessly said that the falling body considered described a spiral path, which, as results from the fourth Section of the second Book of the *Principia*, is true for a resisting medium such as our atmosphere. But Newton, in the figure that he drew for Hooke, made this spiral continue to the center of the earth, though he seems to have acknowledged to Hooke that such a speculation was of "no use." Newton was obviously correct in supposing that Hooke did not know the law of attraction underneath the earth's surface.

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