



## LXI. A demonstration of Le Gendre's theorem for solving such spherical triangles as have their sides very small in proportion to the radius of the sphere

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means serious falls would be greatly avoided. On the presumption that the machine could thus be guided along any intended track, it might perhaps be practicable to change the men at several stages like coach horses. Indeed I should not despair of yet seeing some such method employed as the most expeditious for conveying the mail from one place to another.

This contrivance is no doubt very inferior to the organs of flight with which the feathered race are furnished, and which enable them to traverse the air with such admirable facility. But it is still a recommendation, that it is free from any reciprocating motion, the vanes obviously acting during every part of their revolution; which is a property entirely wanting in those unfortunate artificial wings contrived to act in imitation of the birds; since such unwieldy wings are not simply useless whilst returning to renew their stroke, but really retard and destroy the flight altogether, as the experiment has uniformly proved.

I have not yet attempted to compute the force to be exerted in supporting such a machine. This would be a task of some difficulty as well as uncertainty; since our best theories of the resistance of fluids are still something short of perfection. It might however, to a certain extent, be compared with the forces acting in the common windmill.

If the above scheme, which is perhaps as plausible as most of the kind that have been proposed, seem to deserve a place in the *Philosophical Magazine*, the insertion of it will oblige

Yours, &c.

Edinburgh, Sept. 29, 1821.

VOLATOR.

LXI. *A Demonstration of LE GENDRE'S Theorem for solving such spherical Triangles as have their Sides very small in Proportion to the Radius of the Sphere.* By JAMES IVORY, M.A. F.R.S.

THE theorem to be demonstrated is one of singular beauty, and of great usefulness in geodetical calculations. Although many demonstrations have already been given of it, yet the one which follows may merit attention on account of its simplicity.

The theorem is this :

“In a spherical triangle of which the sides are very small relatively to the radius of the sphere, if each of the three angles be diminished by one-third part of the excess of their sum above two right angles, the remainders will be the angles of a plane triangle that has its sides equal in length to those of the spherical triangle.”

Let  $r$  represent the radius of the sphere, and  $a, b, c$ , the three  
sides

sides of the triangle; then, these four quantities being measured in the same parts, as feet, yards, fathoms, &c. the sides of a similar triangle on the sphere whose radius is unit, will be  $\frac{a}{r}$ ,  $\frac{b}{r}$ ,  $\frac{c}{r}$ .

Suppose that  $A'$ ,  $B'$ ,  $C'$ , denote the angles opposite to  $a, b, c$  respectively; then, because the sines of the sides are proportional to the sines of the opposite angles, we shall have these equations,

$$\begin{aligned}\sin \frac{a}{r} \sin B' &= \sin \frac{b}{r} \sin A' \\ \sin \frac{a}{r} \sin C' &= \sin \frac{c}{r} \sin A'.\end{aligned}\quad (1)$$

Again, in the plane triangle that has its sides equal to  $a, b, c$ , let  $A, B, C$ , be the angles opposite to those sides; then, because, the sides are proportional to the sines of the opposite angles, we shall have

$$\begin{aligned}a \sin B &= b \sin A \\ a \sin C &= c \sin A.\end{aligned}$$

$$\begin{aligned}\text{Suppose} \quad A' &= A + \delta A \\ B' &= B + \delta B \\ C' &= C + \delta C;\end{aligned}\quad (2)$$

and, as the angles of one triangle are very little different from those of the other, we may neglect the squares of the small variations: then,

$$\sin A' = \sin A + \delta A \cos A = \sin A \left(1 + \frac{\delta A}{\tan A}\right)$$

$$\sin B' = \sin B \left(1 + \frac{\delta B}{\tan B}\right)$$

$$\sin C' = \sin C \left(1 + \frac{\delta C}{\tan C}\right).$$

Again,  $\frac{a}{r}$ ,  $\frac{b}{r}$ ,  $\frac{c}{r}$ , being small fractions, we may, with great exactness, suppose

$$\sin \frac{a}{r} = \frac{a}{r} - \frac{1}{6} \cdot \frac{a^3}{r^3} = \frac{a}{r} \left(1 - \frac{a^2}{6r^2}\right),$$

$$\sin \frac{b}{r} = \frac{b}{r} \left(1 - \frac{b^2}{6r^2}\right),$$

$$\sin \frac{c}{r} = \frac{c}{r} \left(1 - \frac{c^2}{6r^2}\right).$$

Now, let these different values be substituted in the equations (1); then,

$$\frac{a \sin B}{r} \left(1 - \frac{a^2}{6r^2}\right) \left(1 + \frac{\delta B}{\tan B}\right) = \frac{b \sin A}{r} \left(1 - \frac{b^2}{6r^2}\right) \left(1 + \frac{\delta A}{\tan A}\right),$$

$$\frac{a \sin C}{r} \left(1 - \frac{a^2}{6r^2}\right) \left(1 + \frac{\delta C}{\tan C}\right) = \frac{b \sin A}{r} \left(1 - \frac{c^2}{6r^2}\right) \left(1 + \frac{\delta A}{\tan A}\right):$$

and,

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and, omitting the equal factors on both sides of each equation;

$$\left(1 - \frac{a^2}{6r^2}\right) \left(1 + \frac{\delta B}{\tan B}\right) = \left(1 - \frac{b^2}{6r^2}\right) \left(1 + \frac{\delta A}{\tan A}\right),$$

$$\left(1 - \frac{a^2}{6r^2}\right) \left(1 + \frac{\delta C}{\tan C}\right) = \left(1 - \frac{c^2}{6r^2}\right) \left(1 + \frac{\delta A}{\tan A}\right):$$

and, by multiplying, and neglecting small quantities of the second order;

$$\begin{aligned} \frac{\delta B}{\tan A} - \frac{a^2}{6r^2} &= \frac{\delta A}{\tan A} - \frac{b^2}{6r^2}, \\ \frac{\delta C}{\tan C} - \frac{a^2}{6r^2} &= \frac{\delta A}{\tan A} - \frac{c^2}{6r^2}. \end{aligned} \quad (3)$$

Again, in the plane triangle, we have

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A, \\ b^2 &= a^2 + c^2 - 2ac \cos B, \\ c^2 &= a^2 + b^2 - 2ab \cos C: \end{aligned}$$

but, if  $s$  represent the area of the triangle, then  $2s = bc \sin A = ac \sin B = ab \sin C$ : and hence  $bc \cos A = \frac{2s}{\tan A}$ ,  $ac \cos B = \frac{2s}{\tan B}$ ;  $ab \cos C = \frac{2s}{\tan C}$ . The values of  $a^2$ ,  $b^2$ ,  $c^2$  may therefore be thus represented, viz.

$$\begin{aligned} a^2 &= \frac{a^2 + b^2 + c^2}{2} - \frac{2s}{\tan A}, \\ b^2 &= \frac{a^2 + b^2 + c^2}{2} - \frac{2s}{\tan B}, \\ c^2 &= \frac{a^2 + b^2 + c^2}{2} - \frac{2s}{\tan C}. \end{aligned}$$

Let these values be substituted in the equations (3); then

$$\frac{\delta B}{\tan B} + \frac{s}{3r^2 \tan A} = \frac{\delta A}{\tan A} + \frac{s}{3r^2 \tan B},$$

$$\frac{\delta C}{\tan C} + \frac{s}{3r^2 \tan A} = \frac{\delta A}{\tan A} + \frac{s}{3r^2 \tan C};$$

and hence,

$$\begin{aligned} \left(\delta B - \frac{s}{3r^2}\right) \tan A &= \left(\delta A - \frac{s}{3r^2}\right) \tan B, \\ \left(\delta C - \frac{s}{3r^2}\right) \tan A &= \left(\delta A - \frac{s}{3r^2}\right) \tan C. \end{aligned} \quad (4)$$

Take the sum of these two equations, and of this identical equation, viz.

$$\left(\delta A - \frac{s}{3r^2}\right) \tan A = \left(\delta A - \frac{s}{3r^2}\right) \tan A;$$

then,

$$\left(\delta A + \delta B + \delta C - \frac{s}{r^2}\right) \tan A = \left(\delta A - \frac{s}{3r^2}\right) (\tan A + \tan B + \tan C).$$

Now,

Now,  $\delta A + \delta B + \delta C$ , is the excess of the angles of the spherical triangle above those of the plane triangle, or above two right angles; and  $\frac{s}{r^2}$  is the area of the spherical triangle on the sphere whose radius is unit; and, by the well known theorem of Albert Girard, these quantities are equal. Wherefore

$$\delta A + \delta B + \delta C - \frac{s}{r^2} = 0;$$

consequently,

$$\left(\delta A - \frac{s}{3r^2}\right) (\tan A + \tan B + \tan C) = 0.$$

Because  $A + B + C = 180$ ,  $\tan C = -\tan(A + B)$ ; therefore,  $\tan A + \tan B + \tan C = \tan A + \tan B - \tan(A + B)$ , a quantity that in no circumstances can be equal to zero. Wherefore

$$\delta A - \frac{s}{3r^2} = 0;$$

and hence, by equat. (4),

$$\delta A = \frac{s}{3r^2},$$

$$\delta B = \frac{s}{3r^2},$$

$$\delta C = \frac{s}{3r^2}.$$

Consequently,

$$A = A' - \frac{s}{3r^2} = A' - \frac{\delta A + \delta B + \delta C}{3},$$

$$B = B' - \frac{s}{3r^2} = B' - \frac{\delta A + \delta B + \delta C}{3},$$

$$C = C' - \frac{s}{3r^2} = C' - \frac{\delta A + \delta B + \delta C}{3}.$$

J I V O R Y.

LXII. *On the Change of Colour in Blue vegetable Colours by metallic Salts.* By Mr. J. MURRAY.

WE had rested quietly in the belief that the relations of acids and alkalis to vegetable colours were uniform; that the first class of bodies turned vegetable blues to red, or restored the original tint obliterated by an alkali; and that the second class, or alkalis, restored the blue colour changed to red by acids, or deepened the yellow and red obtained from turmeric, Brasil wood, &c. into brown. It was at length discovered that boracic acid produced the same effect on turmeric as alkalis would do, and I further find that on tincture of cabbage, and syrup of violets, this peculiar characteristic is still maintained.

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