ADDENDUM TO A PAPER "ON THE INVERSION OF A REPEATED INFINITE INTEGRAL"*

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MR. G. H. HARDY has pointed out to me that the conditions (i.)–(v.) of § 5 of this paper are satisfied by the integral (*loc. cit.*, p. 192)

$$(1) \qquad \int_0^\infty x^{p-1} e^{-x} dx \int_0^\infty (e^{-y} - e^{-xy}) \frac{dy}{y},$$

provided that q (the real part of p) is *positive*; so that it is unnecessary to suppose $q > 1$ as I did in my original investigation. The following method is substantially the same as Mr. Hardy's.

It will be seen from my work on p. 193 that the condition $q > 1$ is only introduced in proving that the integral

$$(2) \qquad \int_0^\xi x^{p-1} e^{-x} dx \int_\eta^\infty (e^{-xy} - e^{-y}) \frac{dy}{y} \quad (0 < \xi \leqslant 1)$$

tends to zero as η tends to infinity.

Now the absolute value of (2) is less than

$$(3) \qquad \int_0^\xi x^{q-1} dx \int_\eta^\infty (e^{-xy} - e^{-y}) \frac{dy}{y}$$

and $$\int_\eta^\infty (e^{-xy} - e^{-y}) \frac{dy}{y} = \int_{x\eta}^\infty e^{-u} \frac{du}{u} - \int_\eta^\infty e^{-u} \frac{du}{u} = \int_{x\eta}^\eta e^{-u} \frac{du}{u}.$$

Thus $$\int_\eta^\infty (e^{-xy} - e^{-y}) \frac{dy}{y} < e^{-x\eta} \int_{x\eta}^\eta \frac{du}{u} = e^{-x\eta} \log\left(\frac{1}{x}\right),$$

and so (3) is less than †

$$(4) \qquad \int_0^\xi e^{-x\eta} \log\left(\frac{1}{x}\right) x^{q-1} dx \leqslant \int_0^1 e^{-x\eta} \log\left(\frac{1}{x}\right) x^{q-1} dx.$$

Now the last integral in (4) converges (at $x = 0$) *uniformly* for all

* *Proc. London Math. Soc.*, Ser. 2, Vol. 1, 1903, p. 176.

† This inequality constitutes the essential improvement introduced by Mr. Hardy; the method of my paper used $1/x$ instead of $\log(1/x)$.

positive values of η, since the integrand is less than that of the convergent integral

$$\int_0^1 \log\left(\frac{1}{x}\right) x^{q-1}\,dx \quad (\text{if } q > 0).$$

Thus, by a familiar result* relating to uniformly convergent integrals, we find

$$(5) \qquad \lim_{\eta\to\infty} \int_0^1 e^{-x\eta} \log\left(\frac{1}{x}\right) x^{q-1}\,dx = 0$$

because

$$\lim_{\eta\to\infty} e^{-x\eta} = 0 \quad (\text{if } x > 0).$$

From the results (3)–(5) we see that the integral (2) tends to zero, provided that q is *positive*: and so the conditions of my paper are then satisfied by the integral (1).

Now the condition $q > 0$ is certainly necessary† as well as sufficient for the inversion of the order of integration to be permissible in the integral (1). It is therefore suggested that the range of my conditions is not so limited as seemed probable from the original discussion of this example (see p. 194, top).

As a matter of fact, the conditions given in § 5 of my paper will be *necessary* as well as sufficient whenever the equation

$$(6) \qquad \int_\xi^\infty dx \int_\eta^\infty f(x, y)\,dy = \int_\eta^\infty dy \int_\xi^\infty f(x, y)\,dx$$

is true for *all* values of ξ, η. And this will be true in the ordinary applications of the conditions, although it would seem to be possible to build up examples in which (6) might hold for certain values of ξ, η, but not for others. Thus it appears that my conditions are necessary as well as sufficient whenever the difficulty is due *solely* to the presence of infinity in the upper limits of integration.

Dr. Hobson has remarked‡ that the condition (ii.) given on p. 185 of my paper really contains *both* conditions (i.) and (ii.) as given on p. 184; and therefore condition (i.) is rendered superfluous when the second form of condition (ii.) is used. This fact follows at once from the inequality (4) of p. 185: since, if ν is there allowed to tend to infinity, we find that

$$(7) \qquad |A - y_\mu| \leqslant \epsilon, \qquad |a - y_\mu| \leqslant \epsilon \quad (\text{if } \mu \geqslant m_0),$$

* See, for instance, Art. 172 of the Appendix to my book on *Infinite Series*.

† In fact, the integral (1) is not convergent unless $q > 0$.

‡ Hobson, *Theory of Functions of a Real Variable*, p. 446.

where A, a are the maximum and minimum limits of z_ν. Thus from (7) we have

$$A - a \leqslant 2\epsilon \quad (\text{if } \mu \geqslant m_0).$$

But A and a do not depend on the variable μ, and so we must have

$$A = a. \tag{8}$$

Thus from (7) and (8) we see that y_μ has a limit a whenever the inequality (4) of p. 185 is satisfied, and that z_ν has then the same limit a. Thus the sufficiency of condition (ii.) on p. 185 is completely established, and its necessity was proved by my former investigation ;* the condition is therefore both necessary and sufficient.

On the other hand, if we apply the method given here to the inequality (3) on p. 184, the existence of $a = \lim z_\nu$ can be at once deduced; but we can only prove that a is *one* of the limits of y_μ, and accordingly the existence of $\lim y_\mu$ is here an additional necessary condition.

* My proof shows that, if $\lim y_\mu$ and $\lim z_\nu$ exist and are equal, then the inequality (4) of p. 185 can be inferred. To prove the sufficiency, I contented myself with the remark that the inequality (4) is more stringent than (3) : the reason for this additional stringency is now evident, because the second inequality includes the first and *also* the condition (i.) of p. 184.