Notes on the Normals of Conics. By SAMUEL ROBERTS, M.A.

[*Read December* 13, 1877.]

On a subject so well explored I do not venture to assert that my results are new. They are obtained by simple methods, and it may be of some use if I bring together in a more or less consecutive way theorems scattered in various repertories. At the same time, I am inclined to think that a few of my conclusions are, if not altogether novel, put in a somewhat new light.

I.

1. The poles of a rectangular pair of conjugate straight lines, with respect to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, are connected by the equations

$$x_1 x_2 = \frac{a^4}{c^2}, \quad y_1 y_2 = -\frac{b^4}{c^2}.$$

If, therefore, one pole describes the curve $F(x, y) = 0$, the contra-pole describes the curve

$$F\left(\frac{a^4}{c^2 x}, -\frac{b^4}{c^2 y}\right) = 0.$$

In this and similar cases it seems unnecessary to specify characteristics which may be derived from the theory of inversion and substitution according to known rules.

If the lines $\alpha x+\beta y+\gamma=0 \ldots (1)$, $\alpha' x+\beta' y+\gamma'=0 \ldots (2)$ are rectangular conjugates with respect to the same conic, we have

$$\frac{\alpha\alpha'}{\gamma\gamma'} = \frac{1}{c^2}, \quad \frac{\beta\beta'}{\gamma\gamma'} = -\frac{1}{c^2}.$$

When the line (1) envelopes the class curve $F(\alpha, \beta, \gamma) = 0$, the line (2) envelopes the class curve $F\left(\frac{1}{\alpha}, -\frac{1}{\beta}, \frac{c^2}{\gamma}\right) = 0$.

It follows from the forms of the expressions that confocal conics have the same rectangular conjugates.

If (x_1, y_1), the pole of (1), describes the curve $F(x_1, y_1, 1) = 0$, the normal conjugate (2) envelopes the class curve

$$F\left(\frac{a^2}{\alpha}, -\frac{b^2}{\beta}, -\frac{c^2}{\gamma}\right) = 0.$$

All the polars of a point (x_1, y_1) with respect to a system of concentric, coaxial, and similar conics have the same normal conjugate.

In virtue of the relation of pole and polar, if (x_1, y_1) describes the curve $F(x, y, 1) = 0$, the line (1) will envelope the class curve $F(a^2\alpha, b^2\beta, -\gamma) = 0$.

2. In particular, if the line (1) envelope the point $A\alpha+B\beta+C\gamma=0$, the normal conjugate (2) will envelope the parabola

$$\frac{A}{\alpha}-\frac{B}{\beta}+\frac{c^2C}{\gamma}=0$$

(Chasles' "Coniques," p. 145), which remains the same for a confocal system of conics.

Considering c^2 as a variable parameter, we have a system of conics with three common tangents and a common point of contact, or four common tangents, two of which are consecutive. The point-pair $\alpha\beta=0$ consists of the poles of the axes. The other point-pair $\gamma(A\beta-B\alpha)=0$ consists of the common centre and the point at infinity of the parabolas.

For a given confocal system, the common tangents of the parabola and the conics of the system determine, at the points of contact with a particular curve, the feet of normals concurrent at the given point.

The curve touches the polars of the point with respect to the system, that is to say, the equation is satisfied by $\left(\frac{A}{a^2+k}, \frac{B}{b^2+k}, -C\right)$.

This envelope, which is well known to be a parabola, has therefore for its Cartesian equation

$$(xx_1+yy_1-a^2-b^2)^2+4(a^2yy_1+b^2xx_1-a^2b^2)=0,$$

obtained in the usual way from

$$\frac{x_1x}{a^2+k}+\frac{y_1y}{b^2+}-1=0.$$

The equation may be written

$$(x_1y+y_1x)^2+(xx_1-yy_1-c^2)^2-(xy_1-x_1y)^2=0,$$

showing the focus and the directrix which passes through the centre.

The parabola can readily be drawn by points, for it touches the axis of x, where the polar of the given point (x_1, y_1) with respect to the concentric circle through the real foci meets the axis; and it meets the axis of y in a point similarly determined by the polar of the point $(-x_1, -y_1)$ relative to the same circle. This conic seems to be as interesting in its geometrical relations as the perhaps more familiar hyperbola

$$-\frac{Aa^2}{x}+\frac{Bb^2}{y}+Cc^2=0,$$

which remains the same for a similar system of conics, and passes through the given point. It is the locus of the pole of the normal conjugate (2), that is to say, the reciprocal of the parabola relative to the given conic.

3. In like manner, if (x_1, y_1), the pole of (1), describes the line $Ax+By+C=0$, the normal conjugate (2) envelopes the parabola

$\frac{Aa^2}{a}-\frac{Bb^2}{\beta}-\frac{Cc^2}{\gamma}=0$, which remains the same for a system of concentric, coaxial, and similar conics.

This parabola and the corresponding hyperbola, $\frac{Aa^4}{c^2x}-\frac{Bb^4}{c^2y}+C=0$, are the same as before, but arranged in different systems.

4. Two rectangular conjugates, with the polar of their intersection or apex, constitute a right-angled self-conjugate triangle. The coordinates of the apex are

$$X=\frac{b^4+c^2y_1^2}{a^4y_1^2+b^4x_1^2}\,a^2x_1=\frac{b^4+c^2y_2^2}{a^4y_2^2+b^4x_2^2}\,a^2x_2,$$

$$Y=\frac{a^4-c^2x_1^2}{a^4y_1^2+b^4x_1^2}\,b^2y_1=\frac{a^4-c^2x_2^2}{a^4y_2^2+b^4x_2^2}\,b^2y_2.$$

We can, therefore, write down the equation of the locus of the poles when that of (X, Y) is given.

If, as before, $\alpha x+\beta y+\gamma=0\ \ldots(1)$, $\alpha'x+\beta'y+\gamma'=0\ \ldots(2)$ are the equations of the rectangular conjugates, the coordinates of the apex are

$$X=\frac{\alpha\,(\gamma^2+c^2\beta^2)}{-\gamma\,(\alpha^2+\beta^2)}=\frac{\alpha'\,(\gamma'^2+c^2\beta'^2)}{-\gamma'(\alpha'^2+\beta'^2)},$$

$$Y=\frac{\beta\,(\gamma^2-c^2\alpha^2)}{-\gamma\,(\alpha^2+\beta^2)}=\frac{\beta'\,(\gamma'^2-c^2\alpha'^2)}{-\gamma'(\alpha'^2+\beta'^2)}.$$

Thus, if $Y=0$, the conjugates envelope the foci $\gamma\pm c\alpha=0$, *i.e.*, we have the theorem of De la Hire. The locus of each of the poles is a directrix. The point $\beta=0$ is the pole of the major axis.

The equation of the hypothenuse of the triangle, being the polar of the apex, is

$$X\frac{\alpha}{a^2}(\gamma^2+c^2\beta^2)+Y\frac{\beta}{b^2}(\gamma^2-c^2\alpha^2)+\gamma\,(\alpha^2+\beta^2)=0\,;$$

so that in general, if the conjugates envelope a curve of class n, the hypothenuse envelopes a curve of class $3n$. This class is, however, reduced for special forms. If a conjugate envelopes a point $\beta=k\gamma$ on the minor axis, the envelope of the hypothenuse is a conic. The other conjugate also envelopes a point on the minor axis, and the apex describes a circle.

In fact, if the coordinates of the apex (X, Y) are given, we get, to determine (x_1, y_1), (x_2, y_2), the equations

$$x^2-\frac{a^2x}{c^2X}(X^2+Y^2+c^2)+\frac{a^4}{c^2}=0,$$

$$y^2+\frac{b^2y}{c^2Y}(X^2+Y^2-c^2)-\frac{b^4}{c^2}=0.$$

We can therefore write down the equation of the locus of the apex, when the locus of the middle point of the hypothenuse is given.

Thus, if $\frac{1}{2}(y_1+y_2)=k$, which also implies that (x_1, y_1), (x_2, y_2) move on lines parallel to the major axis, the locus of X, Y is a circle, viz., we have

$$b^2(X^2+Y^2-c^2)+2c^2kY=0.$$

The two normal conjugates envelope the extremities of the diameter coincident with the minor axis. In general, if the locus of the middle point is a line, the locus of (X, Y) is a circular cubic. The locus of the apex is, from the nature of the case, the locus of the apex of a right angle whose sides envelope certain curves. If one of the conjugates envelopes a point, the locus of the apex is a pedal of the parabola enveloped by the normal conjugate.

5. When the pole (x_1, y_1) of (1) moves on a concentric and coaxial conic,

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}-1=0,$$

the normal conjugate (2) envelopes the curve

$$\frac{a^4}{a_1^2\,\alpha^2}+\frac{b^4}{b_1^2\,\beta^2}-\frac{c^4}{\gamma^2}=0.$$

This is the tangential equation of the evolute of

$$\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}-1=0,$$

provided $$a_2^2=\frac{a_1^2\,b_1^4\,a^4\,c^4}{(a^4b_1^2-b^4a_1^2)^2},\quad b_2^2=\frac{a_1^4\,b_1\,b^4\,c^4}{(a^4b_1^2-b^4a_1^2)^2}.$$

Hence, if the pole of a line moves on a concentric and coaxial conic, the normal conjugates are normals to a third concentric and coaxial conic.

We have also $$a_1^2=\frac{c_2^4\,a^4}{c^4a_2^2},\quad b_1^2=\frac{c_2^4\,b^4}{c^4b_2^2}.$$

Hence the poles of the normal conjugates of the normals of a conic

$$\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}-1=0,$$

with respect to the conic $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

lie on the conic $$\frac{c^4\,a_2^2}{c_2^4\,a^4}x^2+\frac{c^4\,b_2^2}{c_2^4\,b^4}y^2-1=0;$$

and the poles of the normals themselves with respect to the original conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ lie on the curve

$$\frac{a_2^2\,a^4}{c_2^4\,x^2}+\frac{b_2^2\,b^4}{c_2^4\,y^2}-1=0.$$

6. In like manner, if $\alpha x+\beta y+\gamma=0 \ldots(1)$ envelopes the conic whose tangential equation is

$$a_1^2\alpha^2+b_1^2\beta^2-\gamma^2=0,$$

the corresponding normal conjugates will envelope

$$\frac{a_1^2}{\alpha^2}+\frac{b_1^2}{\beta^2}-\frac{c^4}{\gamma^2}=0,$$

which is the evolute of $\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}-1=0,$

provided $a_2^2=\frac{c^4}{c_1^4}a_1^2, \quad b_2^2=\frac{c^4}{c_1^4}b_1^2;$

the normal conjugates of (1) are therefore normals of a third concentric and coaxial conic.

Without departing from the relation of special quadric inversion, we might take the polar of a point (x_1, y_1) with regard to

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}-1=0,$$

and its normal conjugate with regard to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

The poles of the lines with respect to the first conic are connected by

$$x_1x_2=\frac{a_1^4}{c^2}, \qquad y_1y_2=-\frac{b_1^4}{c^2}.\ *$$

II.

7. If p is the perpendicular from the centre to the normal of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

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* I have been referred to a paper "On Normals to Conics: a New Treatment of the Subject," by Professor Cremona ("Messenger of Mathematics," Vol. III., pp. 88–91, 1864). The author, in his usual luminous way, applies the quadric transformation in its geometrical form, when lines correspond to lines and points to conics. The general correspondence and its special case are thus stated:—"Let aa', bb', cc' be the vertices of a quadrilateral whose diagonals aa', bb', cc' form a triangle $\alpha\beta\gamma$. Any line R intersects the diagonals in three points whose harmonic conjugates relative to the couples aa', bb', cc' respectively lie on another line R' which may be said to correspond to R." "When R turns round a fixed point p, R' envelopes a conic P inscribed in $\alpha\beta\gamma$." And then, "if the points c, c' coincide with the imaginary circular points at infinity, the inscribed conics will form a system of confocal conics: a, a' and b, b' being their common foci (real and imaginary), and γ their common centre." "Corresponding lines R, R' are now perpendicular to each other, and divide harmonically the focal segments aa', bb'."

What I have given is merely an analytical representation of this theory, and, as such, is perhaps worth retaining.

we have $$p^2 = \frac{c^4 \sin^2\theta \cos^2\theta}{a^2\cos^2\theta + b^2\sin^2\theta},$$

where θ is the angle which the normal makes with the axis of x. The equation, in fact, represents the pedal of the evolute with regard to the centre. If, then, $xM - yL = 0$ is a normal referred to axes parallel to the principal axes of the conic, and X, Y are the coordinates of the centre, $$(XM - YL)^2 = \frac{c^4L^2M^2}{a^2L^2 + b^2M^2}.$$

We may interpret this result in several ways. If X, Y are considered as current coordinates, and the origin is supposed to be at the centre, the equation represents two parallel normals. If X, Y are variable coordinates of the centre, two parallel lines are represented along which the centre of the conic must move, while the given line $xM - yL = 0$ remains normal, and the conic is simply translated. If we take L, M as variable parameters, the equation represents (1) the four normals which can be drawn to the conic for a given position of its centre; (2) the four normals which can be drawn to the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ through a point (X, Y).

The expanded form is

$$a^2Y^2L^4 - 2a^2XYL^3M + (b^2Y^2 + a^2X^2 - c^4)L^2M^2 - 2b^2XYLM^3 + b^2X^2M^4 = 0.$$

For the parabola $y^2 = px$ we get a corresponding equation,

$$M^3p + 2(p - 2X)L^2M + 4L^3Y = 0.$$

8. If m_1, m_2, m_3, m_4, μ are the tangents of the angles which the four normals through (X, Y) and the line joining (X, Y) to the centre make with the axis of x,

$$m_1 + m_2 + m_3 + m_4 = 2\mu, \qquad \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4} = \frac{2}{\mu}.$$

The quadratic and cubic invariants are

$$S = \tfrac{1}{12}(a^2x^2 + b^2y^2 - c^4)^2,$$
$$T = -\tfrac{1}{216}\{(a^2x^2 + b^2y^2 - c^4)^3 + 54a^2b^2c^4x^2y^2z^2\}.$$

The conic $a^2x^2 + b^2y^2 - c^4 = 0$ is the locus of points normals from which form an equi-anharmonic pencil, *i.e.*, the value of the anharmonic ratio is one of the imaginary cube roots of negative unity. The conic is the reciprocal of the given conic with respect to either concentric circle through two foci. In like manner, $T = 0$ (J. J. Walker, "Educational Times," April, 1871) is the locus of points from which the normals constitute an harmonic pencil, and generally for a constant anharmonic ratio $T^2 - k^2S^3 = 0$, or $T \pm kS^{\frac{3}{2}} = 0$, a locus consisting of two curves of the sixth degree. In the case of the discriminant, these become the evolute and the axes and line at infinity twice over. For shortness, I sometimes call the conic $S^4=0$ the conic of the cusps. In the case of the evolute, it is touched by the eight tangents at the four double points. Several relations noticed by M. Laguerre ("Comptes Rendus," Janvier, 1877) may be immediately obtained from the foregoing equation of normals.

9. Consider $ax+\beta y+\gamma=0$ as a chord of the conic

$$F(x, y) = a'x^2+b'y^2+c'+2f'y+2g'x+2h'xy = 0.$$

To obtain the equation of the normal conjugates relative to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ of the polars of the intersections of the chord with respect to the latter conic, we have

$$F(x_1, y_1) = 0,$$

$$ax_1+\beta y_1+\gamma = 0,$$

$$\frac{a^2X}{c^2x_1}-\frac{b^2Y}{c^2y_1}-1 = 0.$$

Let the elimination of y_1 from the first and second of these equations give

$$Lx_1^2+Mx_1+N = 0.$$

By the second and third of the equations,

$$c^2ax_1^2-Px_1-a^2\gamma X = 0,$$

where P is written for $a^2aX+b^2\beta Y-c^2\gamma$.

The required equation therefore is

$$U = (c^2aN+a^2\gamma LX)^2+(PL+c^2aM)(PN-a^2\gamma MX) = 0.$$

We can get the coordinates of the intersection of these most readily by making $\frac{dU}{dY}=0$, $\frac{dU}{dX}=0$.

Thus we find

$$2NLP-a^2\gamma XML+c^2aMN = 0,$$

$$-MLP+2a^2\gamma L^2X+c^2a(2LN-M^2) = 0,$$

giving

$$a^2\gamma LX+c^2aN = 0.$$

The Y coordinate is obtained by symmetry, and we have

$$X = -\frac{c^2a(b'\gamma^2+c'\beta^2-2f'\beta\gamma)}{a^2\gamma(a'\beta^2+b'a^2-2h'a\beta)},$$

$$Y = \frac{c^2\beta(a'\gamma^2+c'a^2-2g'a\gamma)}{a^2\gamma(a'\beta^2+b'a^2-2h'a\beta)}.$$

Thus, when the locus of (X, Y) is given, we can write down the tangential equation of the envelope of the chord.

10. In particular, if the conic $F(x, y)$ is identical with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

we have
$$X=-\frac{c^2\alpha\left(\frac{\gamma^2}{b^2}-\beta^2\right)}{a^2\gamma\left(\frac{\beta^2}{a^2}+\frac{\alpha^2}{b^2}\right)},\quad Y=\frac{c^2\beta\left(\frac{\gamma^2}{a^2}-\alpha^2\right)}{b^2\gamma\left(\frac{\beta^2}{a^2}+\frac{\alpha^2}{b^2}\right)}.$$

If the intersection of the normals at the extremities of the chord moves on a line, $Ax+By+C=0,$
the chord envelopes the class cubic

$$-c^2\alpha\,(\gamma^2-\beta^2b^2)\,A+c^2\beta\,(\gamma^2-\alpha^2a^2)\,B+\gamma\,(\beta^2b^2+\alpha^2a^2)\,C=0.$$

It is easy to see, from the form of the equation and *à priori*, the principal specialities of the curve, which has been found geometrically and described at some length by Dr. Emil Weyr ("Schlömilch," Z. xvi., 440-4). He has remarked that, if the given line is a normal, the above curve breaks up into a parabola and the foot of the normal (*A*). In fact, by comparing the equation with

$$(p\alpha+q\beta+r\gamma)\,(t\beta\gamma+u\alpha\gamma+v\alpha\beta),$$

the equation of the parabola is obtained in the form

$$\frac{1}{c^2A\alpha}-\frac{1}{c^2B\beta}+\frac{1}{C\gamma}=0,$$

with the linear factor
$$-\frac{a^2\alpha}{A}+\frac{b^2\beta}{B}+\frac{c^2\gamma}{C}=0,$$

the tangential equation of the foot of the fixed normal.

The Cartesian equation of the parabola in one form is

$$\left(\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+1\right)^2-\frac{4x_1y_1xy}{a^2b^2}=0,$$

where (x_1, y_1) is the foot of the fixed normal. The focus is on the tangent at $(-x_1, -y_1)$. The directrix passes through the centre, and is parallel to the normal at $(-y_1, x_1)$. The curve can be readily constructed by points. The locus of foci of the parabolas is the central pedal of the conic (*B*).*

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* In a paper "On Normals to Conics and Quadrics," ("Quarterly Journal of Mathematics," Vol. viii., p. 66, 1866,) Mr. Purser has given a number of interesting theorems, depending, so far as conics are concerned, on an identity which in its homogeneous form is

$$Ax^2+By^2+Cz^2+k\,(axy+byz+czx)=(\alpha x+\beta y+\gamma z)\,(\alpha'x+\beta'y+\gamma'z),$$

giving
$$\alpha\alpha'=A,\quad \beta\beta'=B,\quad \gamma\gamma'=C.$$

This the Author applies to the chords joining the feet of concurrent normals. Mr. Purser's results include (*A*) and (*B*).

11. Taking now, as previously, (x_1, y_1) for the pole of the chord, we have

$$\alpha : \beta : \gamma :: \frac{x_1}{a^2} : \frac{y_1}{b^2} : -1.$$

Consequently, if the intersection of the normals moves on the line $Ax+By+C=0$, the pole of the chord describes the cubic

$$-\frac{c^2x}{a^2}\left(1-\frac{y^2}{b^2}\right)A+\frac{c^2y}{b^2}\left(1-\frac{x^2}{a^2}\right)B-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)C=0,$$

or, if $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=U$,

$$\frac{c^2x^3}{a^4}A-\frac{c^2y^3}{b^4}B+C-U\left(\frac{c^2A}{a^2}x-\frac{c^2B}{b^2}y-C\right)=0.$$

We know, by simple substitution of the coordinates of a centre of curvature, that $\frac{c^2x^3}{a^4}A-\frac{c^2y^3}{b^4}B+C=0$ gives, by its intersections with the conic, six points whose centres of curvature range on the line $Ax+By+C=0$.

12. If this directrix is a normal, we have, as before, a compound locus consisting of the hyperbola

$$\frac{b^2x_1}{B}-\frac{a^2y_1}{A}+\frac{c^2x_1y_1}{C}=0,$$

and the line factor

$$-\frac{x_1}{A}+\frac{y_1}{B}-\frac{c^2}{C},$$

representing the tangent at the foot of the fixed normal.

The intersections, then, of the above hyperbola with the conic determine points whose centres of curvature lie on a normal, *i.e.*, they are the four simple intersections of a normal with the evolute.

When we write $\frac{x}{x_2}+\frac{y}{y_2}+1$ for $Ax+By+C=0$, the hyperbola in question becomes identical with that which determines the feet of normals concurrent at (x_2, y_2).

Since the directrix line is also a normal, we have the further condition,

$$a^2x_2^2+b^2y_2^2-c^4=0;$$

that is to say, the four simple intersections of a normal with the evolute are the centres of curvature of points the normals at which meet on the conic of the cusps.

The point (x_2, y_2) is formed by taking the intercepts of the directrix normal on the axes from the centre in a sense of opposite sign for the coordinates of the point. We thus have a construction by which, for a given normal, we can find the two real intersections with the evolute.

We have to draw normals from (x_2, y_2). Their intersections with the given normal are the points required, which are at the same time the centres of curvature of the curve at their feet. There are some interesting details with regard to this construction which I pass over.

13. In the case of the parabola, the conic of the cusps degenerates into the principal ordinate through the centre of curvature of the vertex and the line at infinity. To adapt the construction to this case, we may make use of the angles with the principal axis formed by the given normal and a right line joining (x_2, y_2) and the intersection of the given normal with the axis. Let ϕ be the first angle, and θ the second; then $\tan\phi = -2\tan\theta$. Suppose that a normal p is drawn to a parabola. Another normal q, having its angle θ with the axis determined by the foregoing equation (exhibiting a simple geometrical relation), will intersect the first normal p at the centre of curvature belonging to the foot of the secondary normal q. This construction corresponds to an infinitely distant point (x_2, y_2). But a normal will intersect the principal ordinate through the centre of curvature of the vertex (being the other line of the degenerate conic of the cusps) in a finite point. In connection with this intersection, the intercept on the normal between the ordinate and the axis is one-half of the intercept between the axis and the centre of curvature. It will be seen now that this is the construction given immediately by the equation of normals. For a parabola we have $m_1+m_2+m_3=0$, and, making $m_2=m_3$, we get $m_1=-2m_2$.

In the case of a central conic, we have, making $m_3=m_4$,

$$\tan\chi = \frac{m_1+m_2}{1-m_1m_2} = 2\,\frac{\mu-m_3}{1+m_3\mu} = 2\tan\psi,$$

which is useful in some constructions. It thus appears that a normal to the conic intercepts the conic of the cusps in two points, whose coordinates, with their signs changed, determine the two normals which can be drawn from the centre of curvature of the foot of the first normal. We are thus led back to a tangential equation of the evolute,

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}-c^4=0,$$

where x, y are now considered as tangential coordinates.

14. The four centres of curvature on normals meeting at a point of the conic of the cusps lie on a linear transversal, and form an equi-anharmonic range, since the four normals or tangents of the evolute form an equi-anharmonic pencil.

Again, with regard to the reciprocal quartic, if four tangents be drawn from any point on the curve, they form an equi-anharmonic pencil. Moreover, they meet the quartic again in points on a line which form an equi-anharmonic range. It is a speciality of the reciprocal unicursal quartic that the tangents at the nodes are harmonic conjugates to the adjacent nodal chords, a property which carries with it other special singularities, and cannot be altered by projection. We are, however, entitled to say that a unicursal quartic possessing those singularities is such that tangents from a point on it form a pencil whose anharmonic ratio is constant, and this is of some interest in connection with the well-known corresponding property of cubic curves.

February 14th, 1878.

Lord RAYLEIGH, F.R.S., President, in the Chair, and subsequently C. W. MERRIFIELD, Esq., F.R.S., Vice-President.

The following communications were made to the Society:—

"On a General Method of Solving Partial Differential Equations:" Prof. H. W. Lloyd Tanner.

"On the Conditions for Steady Motion of a Fluid:" Prof. H. Lamb, Adelaide.

"On a property of the Four-piece Linkage," and "On a curious Locus in Linkages:" Mr. A. B. Kempe.

"On Robert Flower's new Mode of Computing Logarithms:" Mr. S. M. Drach.

"On the Plückerian Characteristics of the Modular Equations:" Prof. H. J. S. Smith, Vice-President.

Mr. Drach also exhibited a large collection of drawings of Tricircloids made many years since for Mr. Perigal.

The following presents were received:—

"Nouveau Mode de Représentation plane de Classes de Surfaces régleés" (29 Oct. 1877).

"Applications d'un Mode de Représentation plane de Classes de Surfaces régleés" (5 Nov. 1877).

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On a General Method of Solving Partial Differential Equations. *By* H. W. Lloyd Tanner, M.A.

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It is well known that the general method of solving a partial differential equation of the first order consists in forming n equations, including the given one, such as to render

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0 \quad \ldots\ldots\ldots\ldots (a)$$

an integrable equation. These equations are found by solving certain partial differential equations usually written, for short,

$$[F_i, F_j] = 0 \quad \ldots\ldots\ldots\ldots (b).$$

The object of the following paper is to deduce a system equivalent to (b) directly from the conditions of integrability of (a). Such a system is, in fact, obtained, and precisely the same form of equation serves to integrate equations of the higher orders.

It is shown that only one system of the kind just mentioned has to be integrated in order to get a final integral of an equation or system of equations, although some of them may be of an order higher than the first.

When we have to deal with equations of the second or higher orders, we find it necessary to solve a second set of auxiliary equations which have no analogue in the theory of equations of the first order. One system of this kind must be solved before passing to the first integral, another before we can get a second integral, and so on. No such system has to be solved before passing from the penultimate integral to the solution.