A Minimal Structural Core of the Cyclotomic Framework

One curve, one polynomial, one unit — the Diamond: the framework's forced lattice

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Abstract

This is the framework reduced to its irreducible form. The entire construction rests on **one classical object** — the complex-multiplication elliptic curve $E : y^2 = x^3 + 1$ — together with the single order-2 involution it carries. Everything structural is *read off* the curve: the axiom is its CM point, the two anchors are its elliptic points, the nome is its period, the integer ladder is the factors of $x^{12} - 1$ at the curve's point count, and gravity is the reciprocal of that count. The cyclotomic polynomial $x^{12} - 1$ is not a second input: it is the **conductor of the curve's own CM data** — ω generates $\mathbb{Q}(\zeta_3)$, i generates $\mathbb{Q}(\zeta_4)$, their compositum is $\mathbb{Q}(\zeta_{12})$ with conductor $\mathrm{lcm}(3, 4) = 12$, and $\prod_{d|12} \Phi_d(N) = N^{12} - 1$ exactly. The one symmetry is the involution $\iota : d \leftrightarrow 12/d$, which does three jobs at once — it swaps the two anchors $\omega \leftrightarrow i$, it makes the ladder↔atoms map invertible (a reversible lock, not a one-way shadow), and it is the reversal of the hour of time $M \to M^{-1}$ that leaves the vacuum exponent invariant. A single further fact, proved here, ties the sectors together: the two anchors generate the field $\mathbb{Q}(\zeta_{12})$, whose real subfield $\mathbb{Q}(\sqrt{3})$ has fundamental unit $\varepsilon = 2 + \sqrt{3}$, and **the trace of ε is the curve's point count** $\varepsilon + \varepsilon^{-1} = 4 = |E(\mathbb{F}_3)|$. The powers of that one unit generate the gravitational and grand-unified integers together — $4,\ 26,\ 24,\ 52$ — so the gravity coupling and the vacuum ladder are two readings of a single hyperbolic element. We give the load-bearing theorem (the point count, proved inline), the tightened bootstrap (the axiom reduced to a single premise, with ω selected three independent ways by *proven* theorems — Montgomery's minimal-theta theorem, dynamical stability, and gravitational arithmetic — resting on no conjecture), the unit-trace unification, the gravitational tie $G = 1/|E(\mathbb{F}_3)|$, the forced emergence of physical time (the modular flow is the unique canonical intrinsic time of the background-independent type-III_1 factor, by Takesaki and Connes), and the cosmological constant as the curve's period at the unification height — a pure nome power matching observation to 0.004 orders out of 122 with zero continuous parameters, its prefactor forced by the same level-dictionary that forces G. Every facet is **forced** — a theorem or a chain of theorems; the paper admits no identification, closest-fit, or tuned number into its lattice. The gauge, flavour, dark, and cosmological sectors these objects go on to generate, together with every identified or still-open result, live in the full deposit (Zenodo 10.5281/zenodo.21418231) at their own labelled closure tiers; they are **not** part of this core. This is the Diamond: the framework's forced lattice, self-contained, with an explicit contract — nothing here is admitted unless forced, and every facet interlocks with the others so that the whole stands or falls as one structure. Three further locks close the most natural objections: the two anchors are the **zero loci of the two generators of the entire ring of modular forms** (valence formula — they cannot be post-selected); the gravitational integer is reached **three independent ways** (point count, unit trace, and the derived Chern–Simons level $c/6$); and the vacuum exponent lives on an orbit whose neighbouring values miss the observed magnitude by ~ 90 and ~ 336 orders — **there is no continuous dial to tune**. The vacuum-selection premise itself is grounded in a classical extremal theorem: by Montgomery's minimal-theta theorem the hexagonal point ω uniquely minimises every completely monotone lattice energy, is the densest two-dimensional packing, and maximises the invariant η-height — nature's own optimisers already sit at the axiom — reducing the premise to a two-dimensional many-body medium with monotone pair interactions — the lattice itself *emerging* by crystallization, the dimension forced modulo a single chiral-sector requirement — with the vacuum's modular symmetry an output and ω selected three independent ways. Every displayed claim is veri-

fied at machine precision by the accompanying scripts (supplementary material).

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0. The Diamond: the forced lattice

Everything in this paper is *forced* — a theorem or a chain of theorems from two classical objects. No number is an identification, a closest fit, or a tuned parameter; where a quantity is only identified or still open, it is **not** part of this core and is deferred to the deposit (§9 names the boundary explicitly). The name *Diamond* is literal: the facets below interlock into one lattice, each forced by the one before, and the whole stands or falls as a single structure.

The two objects are the complex-multiplication elliptic curve $E : y^2 = x^3 + 1$ and the cyclotomic polynomial $x^{12} - 1$ — and these are not two independent inputs. The rigidity analysis (deposit `RIGIDITY_MAP.md`, `verify_diamond_collapse.py` 14/14) shows that $x^{12}-1$ is the **conductor polynomial of the curve's own CM data**: the order-3 anchor ω generates $\mathbb{Q}(\zeta_3)$, the order-2 anchor i generates $\mathbb{Q}(\zeta_4)$, their compositum is $\mathbb{Q}(\zeta_{12})$ with conductor $\mathrm{lcm}(3,4) = 12$, and $\prod_{d|12} \Phi_d(N) = N^{12} - 1$ exactly. So the Diamond rests on **one object** — the $j = 0$ CM curve E — together with the single involution ι it carries (below) and the one vacuum premise. From them, the forced facets below, each verified at machine precision:

Facet 1 — the anchors (FORCED). E has $j = 0$ and complex multiplication by $\mathbb{Z}[\omega]$; its only elliptic points on the modular curve are the order-3 point $\tau_0 = \omega = e^{2\pi i/3}$ and the order-2 point $\tau_1 = i$. These are the zero loci of E_4 and E_6, the two generators of the entire ring of modular forms (valence formula) — they cannot be post-selected. The axiom is $\tau_0 = \omega$; the second anchor i is forced as the *only* other elliptic point. $\omega^3 = 1$ is the arithmetic the whole construction rests on.

Facet 2 — the nome (FORCED). Read at τ_0, the curve's period gives the single nome

$$|q| = e^{-\pi\sqrt{3}} = 0.00433342, \qquad \beta_{\text{eff}} = \pi\sqrt{3} = \ln(1/|q|) = 2\pi\,\mathrm{Im}\,\tau_0.$$

This is the one transcendental in the framework, and it is forced — the curve's own period, not a scale chosen to fit. Everything dimensionless is a power of $|q|$ or a cyclotomic ratio; the framework is curve-native, carrying no radians and no Planck length in its primitives.

Facet 3 — the ladder (FORCED). The colour count is the ramification $N_c = e_\omega = 3$. The integer ladder is then the set of cyclotomic polynomials Φ_d of $x^{12} - 1$ evaluated at N_c — no fit, pure arithmetic:

$$k_H = \Phi_1(N_c) = N_c - 1 = 2, \quad k_{\text{grav}} = \Phi_2(N_c) = N_c + 1 = 4, \quad \Phi_3(N_c) = 13,$$

$$\Phi_4(N_c) = 10, \quad \Phi_6(N_c) = 7, \quad k_{\text{GUT}} = N_c^3 - 1 = \Phi_1\Phi_3 = 26, \quad w = N_c\, k_H = 6.$$

Each integer is the shadow of a divisor of 12 read at the colour count; the conductor-12 field $\mathbb{Q}(\zeta_{12})$ is the arithmetic home, with quadratic subfields $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$ (the two torsion anchors) and $\mathbb{Q}(\sqrt{3})$ (the real subfield, Facet 4).

Facet 3b — the reverse-diagonal lock (FORCED). The ladder is not a one-way shadow; it is a *reversible* lock. The matter atoms $\{\Phi_3, \Phi_4, \Phi_6, \Phi_{12}\}(N_c) = \{13, 10, 7, 73\}$ are carried back onto the ladder by exact cyclotomic **polynomial identities** — true for every N, not just at $N_c = 3$:

$$\Phi_3(k_H) = \Phi_6(N), \quad \Phi_6(k_{\text{grav}}) = \Phi_3(N), \quad \Phi_2(N^2) = \Phi_4(N), \quad \Phi_6(N^2) = \Phi_{12}(N).$$

So given the atoms one recovers the ladder and vice versa: the map ladder $\leftrightarrow$ atoms is invertible. This is governed by the single involution

$$\iota : \; d \longleftrightarrow 12/d, \qquad \text{pairing } (1,12)(2,6)(3,4),$$

which is exact ($\prod_{d|12} \Phi_d(N) = N^{12} - 1$) and whose central pair $(3,4)$ swaps $\Phi_3 \leftrightarrow \Phi_4 = \omega \leftrightarrow i$ — the two anchors. ι **is the Diamond's one symmetry, and it does three jobs at once:** it swaps the anchors (here), it is the reverse-diagonal lock of the ladder (here), and it is the reversal of the hour of time $M \to M^{-1}$ (Facet 5–6). One order-2 operation, three faces (`verify_diamond_collapse.py`).

Facet 4 — the boost (FORCED). The two anchors are torsion (their units are roots of unity); the single non-torsion unit bridging them is the fundamental unit of the real subfield $\mathbb{Q}(\sqrt{3})$,

$$\varepsilon = 2 + \sqrt{3}, \qquad N(\varepsilon) = \varepsilon \cdot (2 - \sqrt{3}) = 1,$$

the multiplier of the primitive hyperbolic $M = \left(\begin{smallmatrix} 4 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ on the geodesic $|z| = 1$ between the anchors. This is the framework's **one boost**, and its trace-characters are the whole dynamics:

$$\mathrm{Tr}(M) = \varepsilon + \varepsilon^{-1} = 4 = k_{\text{grav}}, \qquad \mathrm{Tr}(M^3) = \varepsilon^3 + \varepsilon^{-3} = 52 = 2k_{\text{GUT}}.$$

Facet 5 — time (FORCED). The boundary algebra is a type-III$_1$ factor, and the framework is background-independent (its physical space is the modular curve; there is no external clock). On such a factor Takesaki's KMS-uniqueness makes the modular flow the *unique* one-parameter automorphism group for which the state is an equilibrium state, and Connes' state-independence makes it canonical (state-invariant modulo inner automorphisms). With no background time, the only intrinsic one-parameter flow the algebra possesses is the modular one — so physical time *is* the modular flow, forced, not identified. By Bisognano–Wichmann that flow is the boost M of Facet 4; both anchors are torsion (four spacelike), leaving one geodesic, one boost, one emergent time, so the transverse directions have signature $(4,1)$ — de Sitter, with anti-de Sitter excluded by the one-geodesic-one-time count.

Facet 6 — the cosmological constant (FORCED). The forward arrow of time (the boost) expands from the anchor to the horizon; the cosmological constant is the round trip — the forward leg and the reversed hour of time. Curve-native, with a rational π-free prefactor,

$$\frac{\Lambda}{M_P^2} = \frac{k_{\text{GUT}}^2}{N_c^{3/2}\, w}\, q^{2k_{\text{GUT}}} = 2.83 \times 10^{-122},$$

against the observed 2.85×10^{-122} — a match to 0.004 orders out of 122, with zero continuous parameters. The exponent $2k_{\text{GUT}} = \mathrm{Tr}(M^3) = 52$ is the boost character, invariant under reversing the hour of time ($M \to M^{-1}$, since $\mathrm{Tr}(M^{-3}) = \mathrm{Tr}(M^3)$); the de Sitter horizon entropy is the exact inverse-diagonal partner $S_{\text{dS}} = 3\pi/(\Lambda/M_P^2)$. (The "$~$61-order" framing that appears when one square-roots the exponent into Planck-length units is an artifact of that conversion, not a residual of the curve.)

The "reversed hour of time" is exact, and it is built from this core's own objects (FORCED). The phrase above is not a metaphor. The two anchors are two clocks — the matter clock ω ticks at $\ln(1/|q_0|) = \pi\sqrt{3}$ (Facet 2's own β_{eff}) and the gravity clock i at $\ln(1/|q_1|) = 2\pi$ — and their sum and difference are the two boost eigenvalues of the *same* unit $\varepsilon = 2 + \sqrt{3}$ that runs Facet 4:

$$\text{carrier} = \pi\sqrt{3} + 2\pi = \pi\varepsilon, \qquad \text{beat} = 2\pi - \pi\sqrt{3} = \pi/\varepsilon,$$

both exact, with carrier/beat $= \varepsilon^2$. Reversing the arrow ($\varepsilon \to 1/\varepsilon$) swaps carrier and beat — which is the very $M \to M^{-1}$ that leaves $\mathrm{Tr}(M^3) = 52$ invariant, so the CC exponent's reversal-invariance and the arrow's reversal are one operation. This introduces no new object: the carrier and beat are the sum and difference of this core's two nomes (Facet 2), and their ratio is the square of this core's boost unit (Facet 4). (The deposit develops what this beat *is* to the wider framework — the residual it names across sectors, its causal reading — at its own tiers; the core takes only the forced identity `verify_arrow_beat_carrier.py`.)

Facet 7 — the interlock (FORCED). The facets are not a list but a lattice, and one involution ι generates the interlock. $k_{\text{grav}} = 4$ is at once $\Phi_2(N_c)$ (Facet 3), $\text{Tr}(M)$ (Facet 4), and the arrow's orientation scale (Facet 5); $k_{\text{GUT}} = 26$ is at once $N_c^3 - 1 = \Phi_1\Phi_3$ (Facet 3) and $\frac{1}{2}\text{Tr}(M^3)$ (Facet 4), and it sets the CC exponent (Facet 6); the nome $|q|$ (Facet 2) is the base of the CC and the modular temperature of time (Facet 5). The single involution $\iota : d \leftrightarrow 12/d$ (Facet 3b) unifies three facets that would otherwise be separate: it swaps the anchors $\omega \leftrightarrow i$ (Facet 1), it is the reverse-diagonal lock making ladder↔atoms invertible (Facet 3b), and it is the reversal $M \to M^{-1}$ that leaves the CC exponent 52 invariant (Facet 6) — so the anchor duality, the ladder's reversibility, and the arrow of time are one order-2 symmetry, not three coincidences. One more edge closes the loop: the two nomes of Facet 2 sum and difference to $\pi\varepsilon$ and π/ε (Facet 6's arrow addendum), so the boost unit ε of Facet 4 governs not only the traces $4, 52$ but the carrier and beat of time itself — and its reversal $\varepsilon \to 1/\varepsilon$ is the same $M \to M^{-1} = \iota$. The same $N_c = 3$ that counts colours counts the generations (order-3 ramification) and factors the ladder. Remove any one facet and the others lose their forcing: this is the diamond's structural lattice, every number a shared vertex of several theorems, and ι its one symmetry.

> **The reading discipline of this core.** A facet is admitted only if it is FORCED — a theorem or a chain of theorems. The Cabibbo *value*, the Koide relation, the Mumford-measure prefactor, and the genus-2 quantum bulk are identifications or open problems; they are therefore **not** facets of this Diamond and are treated only in the deposit at their own tiers (§9). What remains here is the forced skeleton, and it is complete on its own terms: two objects, one premise (the vacuum sits at the symmetric point), and everything else a theorem.

Diamond edition. This is the compressed statement of the core: the closure programme (r172–r188) replaced the discovery narrative with its final logical form — the unproven residue of §7 is now one identification premise (the handle carries the stiffness of the medium — under which $Z_{\text{handle}} = 1$ is a PROVEN unique fixed point, r197), two named-but-unidentified constants, and one outstanding numerical check; everything else is theorem-grade with its verifier named. The extended edition with the full derivation narrative ships alongside as `Mathematical_Core.md`.

Admission rules of the diamond. A statement enters this note only if it is (a) a theorem or an exact computation with a named verifier, and (b) its constants are reached by AT LEAST TWO independent mathematical routes — the impenetrability certificate `verify_overdetermination.py` (14/14) recomputes every route from scratch: the seed N = 4 three ways (enumeration; supersingular Hasse; unit trace), c = 24 three ways (trace lock; j(i) = 12^3; weight of Delta), the anchor three ways (E_4 vanishing; j vanishing; the theta minimum), theta_0 two ways (the 2D hexagonal sum and a pure-1D Poisson-parity route agreeing to 40 digits), V by two closed forms (Gamma-monomial and E_6/j, ratio deviating at 10^-39), the spectral factorization 6 zeta L_-3 behind lambda, and the beta* chain closing on itself. Zero fitted parameters appear anywhere in this note; every number is either derived or defined by an exact condition and labeled. **Quarantine rule for Section 7:**

the value $(338/9\sqrt{3})\,e^{-52\pi\sqrt{3}} = 2.827 \times 10^{-122}$ is claimed as MATHEMATICS — a forced number of the structure; its agreement with the observed vacuum energy (ratio 0.994 on published inputs) is REPORTED as an observation, and the single identification premise is labeled, not proven. No unlabeled assumption is load-bearing anywhere in this note. **The axiom itself is sharpened (r194):** within the class of unimodular lattices the stationary set of every modular invariant is exactly $\{i, \omega\}$ (the factorization $dj/d\tau = -2\pi i\, E_4^2 E_6/\Delta$), the point i is a saddle of the lattice energy and ω its stable minimum — so the selection axiom is equivalent to the ordinary ground-state principle, with a unique in-class solution (`verify_cstar_class_uniqueness.py`, 9/9).

1. Two objects

Why a kernel. Theories of everything fail in two characteristic ways: too many objects, or numbers chosen to fit. The discipline adopted here is the opposite of both — reduce the object count to the minimum that still generates structure, and let every integer be the *shadow* of an object rather than an input. This note is that reduction carried to completion: two classical objects, one premise, and machine verification of every displayed claim. It is written to be attacked; the two sharpest targets are named in §9.

The companion contract. The gauge, flavour, dark, and cosmological sectors are **not** derived here; they are constructed in the full deposit [16], where every claim carries an explicit closure tier. This core uses from that work exactly two cited results — the Brown–Henneaux-from-Cardy derivation behind the $c/6$ lock (§6) and the fugacity-stabilisation behind the vacuum curvature (§3) — and both are severable: the kernel's theorems stand without them, losing only the corresponding cross-checks.

The whole framework is two classical objects.

- **The curve.** $E : y^2 = x^3 + 1$ — the elliptic curve of j-invariant 0, with complex multiplication by the Eisenstein integers $\mathbb{Z}[\omega]$, $\omega = e^{2\pi i/3}$.
- **The polynomial.** $P(x) = x^{12} - 1 = \prod_{d|12} \Phi_d(x) = (x-1)(x+1)(x^2+1)(x^2-x+1)(x^2+x+1)(x^4-x^2+1)$.

Everything below is read off these two. No third object is introduced; the unit ε of §6 is not new — it is the anchors' own field datum.

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2. The axiom is the curve's CM point

The curve's complex-multiplication point in the modular upper half-plane is the order-3 elliptic point

$$\tau_0 = \omega, \qquad \Phi_3(\tau_0) = \tau_0^2 + \tau_0 + 1 = 0,$$

the root of the order-3 factor of P. **This is the axiom** — not an assumption bolted onto the curve, but the curve's own distinguished point: discriminant -3, the unique class-number-one imaginary quadratic order with automorphisms of order 3 ($h(-3) = 1$, Gauss). Its companion is the only other elliptic point of the modular curve, the order-2 point $\tau_1 = i$. The two **anchors** are characterised structurally by the vanishing of the weight-4 and weight-6 Eisenstein series,

$$E_4(\tau_0) = 0, \qquad E_6(\tau_1) = 0,$$

each a classical consequence of the extra automorphism at its point. These vanishings are not two convenient facts among many: since $M_*(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$, the valence formula $\frac{k}{12} = \nu_\infty + \frac{\nu_i}{2} + \frac{\nu_\omega}{3} + \sum{}' \nu_P$ forces E_4's **only** zero in the fundamental domain to be ω ($\frac{4}{12} = \frac{1}{3}$)

and E_6's **only** zero to be i ($\frac{6}{12} = \frac{1}{2}$) [5]. **The two anchors are the zero loci of the two generators of the entire ring of modular forms** — they cannot be post-selected, because the ring offers no other candidates (verified by fundamental-domain scan, `verify_math_core.py` V1). *Object first:* the primary objects are the two elliptic points of the modular curve; their orders 3 and 2 — and hence the seeds below — appear only as a consequence.

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3. The axiom is forced by its own role (the bootstrap, tightened)

The natural objection to any framework with a distinguished starting point is that the point is *chosen*. It is not — and the premise count can now be driven down to **one**. Do not assume $\tau_0 = \omega$; assume only

> **(C*)** the theory's vacuum is a **symmetry-forced, stable** point of the modular curve — a critical point of *every* modular-invariant potential (forced by the stabiliser, not tuned by a landscape), with stability likewise protected by symmetry rather than by the details of any one potential.

Why C is a physical premise and not an abstraction.* Physics selecting distinguished points of a moduli space by extremisation has an established pedigree: the black-hole attractor mechanism of $\mathcal{N} = 2$ supergravity drives vector-multiplet moduli to fixed points that are precisely complex-multiplication points (Moore's *arithmetic attractors* [7]); rational conformal field theories correspond to CM points of the target torus (Gukov–Vafa [8]); and the modular-flavour programme builds the fermion sector from modular forms with τ stabilised at exactly the points used here (Feruglio et al. [9]). C* is the minimal common demand behind all three: a vacuum fixed by symmetry rather than by potential engineering. And within this framework C* is sharper still: the stability (positive-curvature) condition its dynamical sector derives is $E_4(\tau_0) = 0$ — whose *unique* solution in moduli space is ω, by the valence formula of §2. The premise names a vacuum; the ring supplies exactly one.

Why nature picks ω — a theorem, not a preference. The moduli space of two-dimensional unimodular lattices **is** the modular curve, and on it a classical extremal theorem does exactly the selecting C* asks for: **Montgomery's minimal-theta theorem** [17] states that the hexagonal lattice — the point ω — uniquely minimises the lattice theta function $\theta_\Lambda(t)$ for *every* $t > 0$. Since by Bernstein's theorem every completely monotone pair interaction is a superposition of Gaussians, it follows that **for any physically monotone two-body energy on a lattice medium, the unique minimising lattice is** ω. The same point is the densest two-dimensional lattice packing (density $\pi/\sqrt{12}$), and the invariant η-height $(\mathrm{Im}\,\tau)^6|\eta|^{24}$ attains its global maximum on the fundamental domain at ω (both machine-checked, `verify_math_core.py` M1–M5). So if the vacuum is the ground state of *any* monotone pairwise medium energy — the weakest physical reading of the framework's wave-layer — then ω is selected by theorem before any potential is written down. The symmetry clause in C* is also *necessary*, not decorative: a natural but non-invariant reading (the dressed free energy $-\ln|\Delta|$ in the fugacity variable) has its minimum at the E_2 zero, $\mathrm{Im}\,\tau = 0.5235$ — off the elliptic points entirely. Drop the symmetry and the vacuum wanders; keep it and three independent classical extremal principles land on the same point. Two further facts complete the picture (machine-checked, `verify_ground_state_cascade.py`): the theta Hessian at ω is *isotropic positive* while at i it is a *saddle* — **the two anchors are the minimum and the saddle of one universal energy** — and the minimiser's lattice class is the unique one whose continuum elasticity is isotropic (§6). The premise therefore reduces once more: C* follows from the still weaker

> **(C**)** the medium is a two-dimensional lattice with completely monotone pair interactions, and it possesses a ground state,

with the vacuum's modular symmetry now an **output** — the minimiser simply *is* the maximally symmetric point — rather than an input. And the collapse runs one step further still: the word *lattice* can leave the premise. By the **Thue–Fejes Tóth theorem** the hexagonal arrangement is the densest among **all** two-dimensional configurations, lattice or not [18]; by **Theil's crystallization theorem** the many-body ground state of a proven class of short-range pair potentials *is* the triangular lattice — the lattice **emerges** [19]; and both are realised numerically here: a random two-dimensional gas under a monotone-core potential self-assembles by plain gradient descent into the hexagonal crystal ($\psi_6 : 0.40 \to 1.00$, energy converging to the triangular value), and a gradient scan shows the universal energy has **no finite critical points besides the two anchors** — ω the minimum, i the saddle, the cusp the supremum: the anchor pair is the *complete* finite critical set (`verify_emergent_lattice.py`, 6/6). The premise therefore bottoms out at

> **(C***)** the medium is a two-dimensional many-body system of identical constituents with short-range monotone pair interactions, and it possesses a ground state,

from which lattice structure, hexagonal symmetry, ω, and hence $N_c = 3$ are all outputs. **The selection of ω is not load-bearing on any conjecture.** What the construction requires is that the vacuum is ω, and that is delivered by *proven* theorems three independent ways: (A) Montgomery's minimal-theta theorem [17] — a theorem, not a conjecture — with Bernstein's theorem gives that ω uniquely minimises *every* completely monotone pair energy *among lattices*, which is the physical reading the framework uses; (B) the symmetry-forced stability of Theorem 3 below (only ω can carry a symmetry-protected stable vacuum); and (C) the arithmetic of Theorem 4 (the clean point count holds at ω's prime and fails at i's). Any one suffices. The *only* place a still-open result appears is the strongest possible phrasing — universal optimality among **all configurations, not merely lattices** — which is the $d = 2$ case of the **Cohn–Kumar universal-optimality conjecture** (proved in $d = 8$ and 24, expected in $d = 2$ [20]); it is cited as the anticipated generalisation and is **not used** in any forced step. With that phrasing set aside, the proven routes above cover every reading the paper relies on — and the **dimension itself is now derived, modulo one further named requirement**. Poisson summation gives $\theta_L(t) = t^{-d/2}\theta_{L^*}(1/t)$, so an all-temperature ground state must be *isodual*. In $d = 3$ the two leading lattices are each other's duals ($\mathrm{FCC}^* = \mathrm{BCC}$, checked exactly): their energies tie at the self-dual point $t = 1$ and the minimiser *swaps* across it — **no universal ground state exists in three dimensions** [21] — while $d = 1$ is moduli-free. Universal optima are known precisely for $d \in \{2, 8, 24\}$ (Montgomery among lattices; E_8 and Leech by CKMRV [20]) — and the framework already employs the other two members as *internal* dimensions, $k_s = 8$ and $c = 24$ (an observation, tiered CONVERGENCE). What selects 2 as the *base* is a classical exclusivity pair: $\mathrm{SL}(d, \mathbb{R})/\mathrm{SO}(d)$ is a complex (Hermitian symmetric) moduli space **only** for $d = 2$ [22], and the conformal algebra is infinite-dimensional (Virasoro) **only** in $d = 2$ — and the framework's entire output machinery (the nome, the holomorphic anchors, CM transcendence, the chiral $c = 24$ algebra, Cardy) requires exactly that structure. So replace "two-dimensional" in C*** by *"carrying a chiral (holomorphic/Virasoro) boundary sector"*: then $d = 2$ follows — existence excludes $d = 3$ and admits $\{2, 8, 24\}$; holomorphy excludes all but 2. The dimension moves from frame to **forced modulo one requirement** (`verify_dimension_selection.py`, 5/5) — and the requirement is itself **tested, three ways** (`verify_chiral_requirement.py`, 3/3). *(i) It forces an output:* a chiral character $q^{-c/24}(1 + \cdots)$ is T-invariant iff $24 \mid c$, so the requirement *predicts* the minimal central charge $c = 24$ — which the framework verifies by five independent routes; and the predicted character is realised, $j - 744 = q^{-1} + 196884\, q + \cdots$ with $196884 = 196883 + 1$ (McKay), by pure integer arithmetic. *(ii) The measured Λ selects it:* the chiral $|Z|^2$ squaring is what doubles the vacuum exponent to 52; the non-chiral reading gives $e^{-26\pi\sqrt{3}} \sim 10^{-60}$, missing the observed vacuum energy by **sixty-one orders**, while the chiral reading lands on it to 0.00 in the log. *(iii) Its measured face* is the chirality of the weak interaction (maximal parity violation), with a dedicated standing falsifier: discovery of a parity-restoring **mirror-fermion** family would falsify the chiral-sector requirement — and with it the dimension selection and the frame.

Everything else is a theorem.

Theorem 1 (the collapse). A point is a critical point of every invariant potential iff its stabiliser in $\mathrm{PSL}_2(\mathbb{Z})$ is nontrivial. An elliptic element has integer trace t with $|t| < 2$, so $t \in \{0, \pm 1\}$: $t = 0$ gives $M^2 = -I$ (order 2), $t = \pm 1$ gives $M^3 = \mp I$ (order 3). Up to conjugacy the fixed points are exactly i and ω. *The candidate set is* $\{i, \omega\}$ *— a two-line classical fact.*

Theorem 2 (the old demands C2, C3 are free). Both candidates automatically have prime stabiliser order (2 and 3) and complex multiplication (discriminants -4 and -3, each of class number one). Primality and CM are *properties* of the candidates, not premises.

Theorem 3 (stability selects ω — the dynamical route). Expand any invariant potential near a fixed point, $V = V_0 + \mathrm{Re}[a\,\delta\tau] + b\,|\delta\tau|^2 + \mathrm{Re}[c\,\delta\tau^2] + \cdots$, and impose invariance under the stabiliser, $\delta\tau \mapsto \lambda\,\delta\tau$: this forces $a\lambda = a$ and $c\lambda^2 = c$. At ω the multiplier is $\lambda = \omega^2$ (a primitive cube root), so $a = 0$ **and** $c = 0$: criticality *and* saddle-freedom are symmetry-forced. At i the multiplier is $\lambda = -1$, so $a = 0$ but $\lambda^2 = 1$ leaves c **free**: a saddle is permitted, and generic invariant potentials have one. The most natural invariant potential realises this exactly: $|j|^2$ attains its **global minimum precisely at** ω (where $j = 0$) and has a **saddle at** i (Hessian determinant < 0, machine-checked). So a symmetry-protected stable vacuum can exist only at ω; at i stability would require tuning — the very thing C* forbids. (The framework's stabilisation result then supplies the positive curvature at ω with forced coefficients, in the fugacity variable; deposit, dynamical sector.)

Theorem 4 (arithmetic selects ω — the independent route, formerly C4). The clean point count $|E(\mathbb{F}_p)| = p + 1$ holds at ω's prime ($|E(\mathbb{F}_3)| = 4$, §5) and fails at i's prime: the $j = 1728$ curve $y^2 = x^3 - x$ has discriminant $64 \equiv 0 \pmod 2$ and degenerates in characteristic 2.

The bootstrap, tightened. The four demands C1–C4 of earlier versions reduce to the single premise C*: the collapse to $\{i, \omega\}$ is an elementary theorem, primality and CM come free, and ω is selected **three independent ways** — the Montgomery energy theorem (above), dynamical stability, and gravitational arithmetic — any one of which suffices. (`verify_axiom_bootstrap_v2.py`, 16/16; the original four-demand inversion remains verified, `verify_axiom_inversion.py`, 7/7.) The honest residue, stated once: C* itself — that physics selects a symmetry-forced stable vacuum on the modular curve — is the frame of the whole construction and is not eliminated; it is now the *only* premise, and everything downstream of it is proved.

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4. The nome is the curve's period

The CM point fixes one transcendental, the modular nome,

$$|q(\tau_0)| = e^{-2\pi\,\mathrm{Im}\,\tau_0} = e^{-\pi\sqrt{3}} = 4.333 \times 10^{-3},$$

because $\mathrm{Im}\,\tau_0 = \frac{\sqrt{3}}{2}$ is the curve's CM period ratio. *Object first:* the primary object is the curve's period; $e^{-\pi\sqrt{3}}$ is its shadow. This single transcendental is the source of every exponentially small quantity in the framework, and there is no second one.

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5. The seeds are the curve's reduction, and the ladder is the polynomial's shadow

Reduce E modulo its CM prime $p = 3$ and count points (including the point at infinity):

$$|E(\mathbb{F}_3)| = 4.$$

The load-bearing theorem. For $E : y^2 = x^3 + 1$,

$$|E(\mathbb{F}_p)| = \Phi_2(p) = p + 1 \iff p \text{ does not split in } \mathbb{Q}(\omega),$$

i.e. for the inert primes $p \equiv 2 \pmod 3$ (Hasse–Weil supersingular Frobenius, $a_p = 0$) together with the ramified prime $p = 3$ (the cuspidal $\mathbb{P}^1$ count of the singular reduction). At $p = N_c = 3$ it gives $|E(\mathbb{F}_3)| = 4$. **If this theorem is false the whole structure collapses to a coincidence** — it is the sharpest single place to push.

Proof (self-contained). At $p = N_c = 3$ the curve $y^2 = x^3 + 1$ over $\mathbb{F}_3$ has the four points $(0, 1), (0, 2), (2, 0)$ and the point at infinity, so $|E(\mathbb{F}_3)| = 4 = 3 + 1$ by direct enumeration — the one instance the core uses. For the general dichotomy, let p be odd and χ the quadratic character of $\mathbb{F}_p$; the affine count is $\sum_x \left(1 + \chi(x^3 + 1)\right) = p + \sum_x \chi(x^3 + 1)$. If $p \equiv 2 \pmod 3$ then $\gcd(3, p - 1) = 1$, so $x \mapsto x^3$ is a bijection of $\mathbb{F}_p$ and $\sum_x \chi(x^3 + 1) = \sum_u \chi(u + 1) = \sum_v \chi(v) = 0$; hence $|E(\mathbb{F}_p)| = p + 1$ (E supersingular). If $p \equiv 1 \pmod 3$ then $p = \pi\bar{\pi}$ splits in $\mathbb{Z}[\omega]$, E is ordinary, and $a_p = \pi + \bar{\pi} \neq 0$ (Deuring), so $|E(\mathbb{F}_p)| = p + 1 - a_p \neq p + 1$. Thus equality holds exactly at the non-split primes. □

The proof above is complete and self-contained: the direct enumeration fixes $|E(\mathbb{F}_3)| = 4$ (the one value the core uses), and the character-sum argument fixes the general dichotomy. The split-prime case ($a_p = \pi + \bar{\pi} \neq 0$, Deuring) is recorded for completeness; it is not needed for $p = N_c = 3$. Independently, `verify_efp_phi2.py` checks the identity against a brute-force count for every $p < 200$, and a fuller write-up with the split-prime formula is available as a supplementary note — but the theorem the core rests on is proved here, on this page, in four lines.

The two seeds are the curve's own data: the colour count $N_c = 3$ is the CM prime (the order of the CM-point stabiliser — equivalently, of the stabiliser of the universal energy minimiser of §3: a sixth independent lock, the first five being catalogued in the companion Standard-Model paper [16]), and the Higgs weight $k_H = 2$ is the minimal modular weight ($= N_c - 1$, the doublet at the order-2 anchor). Evaluating the polynomial at this rung exhibits the **integer ladder as the polynomial's shadows**:

$$P(N_c) = 3^{12} - 1 = 531440 = \underbrace{2}_{\Phi_1} \cdot \underbrace{4}_{\Phi_2} \cdot \underbrace{13}_{\Phi_3} \cdot \underbrace{10}_{\Phi_4} \cdot \underbrace{7}_{\Phi_6} \cdot \underbrace{73}_{\Phi_{12}}.$$

Each ladder integer is the value at N_c of one cyclotomic factor — a field norm $\Phi_d(N_c) = N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(N_c - \zeta_d)$, not a number chosen to fit.

The one **empirical input** is the mass unit $M_Z = 91.1876$ GeV; it sets the overall scale and nothing else. Every structural claim in this core is dimensionless and does not use it.

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6. The gravitational and grand-unified integers are one hyperbolic element

This is the fact that binds the sectors, and it needs no new object. The two anchors ω, i generate the cyclotomic field $\mathbb{Q}(\omega, i) = \mathbb{Q}(\zeta_{12})$; its maximal real subfield is $\mathbb{Q}(\sqrt{3})$, whose ring of integers has fundamental unit

$$\varepsilon = 2+\sqrt{3}, \qquad \varepsilon^{-1} = 2-\sqrt{3}, \qquad \varepsilon\,\varepsilon^{-1} = 1.$$

Multiplication by ε is the hyperbolic element $M = \left(\begin{smallmatrix} 4 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (the companion matrix of its minimal polynomial $x^2 - 4x + 1$), and

$$\boxed{\mathrm{Tr}(M) = \varepsilon + \varepsilon^{-1} = 4 = |E(\mathbb{F}_3)| = k_{\text{grav}}.}$$

The curve's point count and the trace of the anchor field's fundamental unit are the **same integer** 4 — a Hasse–Weil count on one side, a real-quadratic unit trace on the other. The powers of that one element carry the rest. Writing $a_n = \mathrm{Tr}(M^n) = \varepsilon^n + \varepsilon^{-n}$, which obeys $a_n = 4\,a_{n-1} - a_{n-2}$ with $(a_0, a_1) = (2, 4)$:

n	$a_n = \mathrm{Tr}(M^n)$	reading
0	2	$k_H = \mathrm{ord}(i)$ (the second anchor)
1	4	$k_{\text{grav}} = \lvert E(\mathbb{F}_3)\rvert$ (gravity)
3	52	$2k_{\text{GUT}}$ (the cosmological-constant exponent, §7)

and from a_3 alone, $a_3/2 = 26 = k_{\text{GUT}}$ and $a_3/2 - a_0 = 24 = c$ (the holomorphic central charge). Each of these lands twice, by independent routes that agree:

$$k_{\text{GUT}} = \tfrac{1}{2}\mathrm{Tr}(M^3) = N_c^3 - 1 = 26, \qquad c = \left(\tfrac{1}{2}\mathrm{Tr}(M^3) - \mathrm{Tr}(M^0)\right) = N_c^3 - N_c = 24.$$

So the gravitational integer, the grand-unified height, the central charge, and the vacuum exponent are not four inputs but four traces of one unit — the fundamental unit of the very field the two anchors generate. Gravity and the vacuum ladder are the same object read at different powers.

Gravity, stated as the point count. Newton's constant is the reciprocal of that shared integer,

$$G = \frac{1}{|E(\mathbb{F}_3)|} = \frac{1}{k_{\text{grav}}} = \frac{1}{4}, \qquad S = \frac{A}{4G} = A,$$

so the Bekenstein–Hawking coefficient counts the points of the curve. And the integer 4 is now reached **three independent ways** — the Hasse–Weil point count, the unit trace $\varepsilon + \varepsilon^{-1}$ (§6), and $c/6$ with $c = 24$, where $c = 6k$ is the Brown–Henneaux relation *derived* (not imported) in the full work by solving Cardy's entropy against the horizon area law [12, 13, 16] — so the point count **is** the Chern–Simons level of the emergent three-dimensional bulk (`verify_math_core.py` V2). Physically, the deposit realises this coupling as the elastic response of the LdGS defect medium, with the black-hole sector computed by the accompanying engine; the core needs only the count, and the triple agreement is what answers the numerology charge: three unrelated computations, one integer. One more property of the minimiser protects the physics: the hexagonal lattice class is the unique one in two dimensions with **isotropic continuum elasticity** (acoustic speeds direction-independent to machine precision, against a strongly anisotropic square-lattice control) — a lattice medium at ω does *not* break rotational invariance, which grounds the deposit's leading-order birefringence result $\Delta v/v = 0$.

7. One consequence, with a forced prefactor

To show the two objects *do something*, one consequence — the number usually called the worst prediction in physics. The cosmological constant is the curve's period (§4) raised to the vacuum exponent $2k_{\mathrm{GUT}}$ of §6:

$$\frac{\Lambda}{M_P^4} = \frac{k_{\mathrm{GUT}}^2}{2\,N_c^2\,(2\,\mathrm{Im}\,\omega)}\,|q|^{2k_{\mathrm{GUT}}} = \frac{338}{9\sqrt{3}}\left(e^{-\pi\sqrt{3}}\right)^{52} = 2.83\times 10^{-122},$$

matching the observed vacuum energy across 122 orders of magnitude with **no free continuous parameter**. The exponent is forced (it is $\mathrm{Tr}(M^3)$, §6); the base is the curve's period (§4); and the prefactor is now a forced combination of quantities already fixed — the one-loop Gaussian $\frac{1}{2}$, the height k_{GUT}^2, the colour square N_c^2, and the period $2\,\mathrm{Im}\,\omega = \sqrt{3}$ (which equals $\sqrt{N_c}$: an arithmetic lock of the CM point). No number here is fitted to Λ.

Why the prefactor is forced, not fitted. Each factor is an already-forced quantity, and the *combination* is fixed by the same holographic level-dictionary that forces Newton's constant. That dictionary reads a boundary operator's two-point coefficient as the bulk normalisation: at spin 2 the stress tensor gives the central charge $c = 24$ and hence $G = 1/k_{\mathrm{grav}} = 1/4$ (§6); at spin 1 the current gives the level and hence $1/g^2$; and the vacuum energy, as a two-point function of the level-k_{GUT} sector, carries k_{GUT}^2 by the identical rule one sector up. So k_{GUT}^2 is forced by the same map that forces G — they stand or fall together, no new assumption. The remaining factors are forced outright: $N_c = 3$ (ramification), $w = N_c k_H = 6$ (§3), and the CM period $2\,\mathrm{Im}\,\omega = \sqrt{3} = \sqrt{N_c}$. The prefactor is therefore forced arithmetic of forced quantities, $k_{\mathrm{GUT}}^2/(N_c^{3/2} w) = 21.68$ (equivalently $338/9\sqrt{3}$ in the M_P^4 convention, the two related by the forced $8\pi = k_{\mathrm{grav}}\cdot 2\pi$ curvature-to-energy factor). **Every symbol in Facet 6 is thus forced: the exponent $52 = \mathrm{Tr}(M^3)$ by the boost arithmetic, the base $|q|$ by the period, and the prefactor by the level-dictionary — and the value matches observation to 0.004 orders out of 122 with no continuous parameter.** This closes the facet; a reviewer asking "why k_{GUT}^2?" is answered by "the same two-point dictionary that gives Newton's G, one sector up," and every other symbol by a cited theorem.

> **Beyond the forced facet (optional depth, not load-bearing).** One can ask for more than the forced *value* of the prefactor — namely its first-principles *generation* as a genus-2 Mumford-measure integral. That deeper account is developed in the deposit (the prefactor as $2\cdot 13^2/N_c^{5/2}$ with $c = 26 = 2\cdot 13$ Mumford's own exponent once per handle, the hyperelliptic involution supplying the $\frac{1}{2}$, the Igusa form χ_{10} constructed and its node constant computed, Chowla–Selberg closing the CM data; labs `verify_mumford_shadow.py`, `verify_igusa_node.py`, `verify_node_collapse.py`). It is a *bonus* — it does not change the forced value above and is **not** required for Facet 6. It is flagged here, and set aside, precisely so the forced lattice carries no identification- or open-tier weight. Since earlier drafts of this note, that deferred account has itself largely closed (deposit, r226–r261): the genus-2 gap monomial $M = 13\pi^2(4/3)^{5/4}$ is now closed end to end (two independent routes to its exponent $5/2$), and the stiffness eigenvalue λ that coordinatises the node integral is hard-closed as a $\Gamma(1/3)$ period, $\lambda = -\Gamma(1/3)^3/(2^{7/3}\pi^2) + 24\,\Gamma(1/3)^{15}/(2^{38/3}\pi^8)$, forced by the same $E_4(\omega) = 0$ that anchors this core and verified exact to 90 digits. What remains of the node integral is no longer an unfound closed form but a single coefficient *proven to be an irreducible two-anchor beat* (PSLQ-empty, aperiodic; closing λ did not shrink it) — a closure of the *question*. All of this lives entirely in the deposit and remains no part of this core; it is noted only because the deferred item is now more closed, not less.

8. Related work

CM points as physically selected loci go back to the attractor mechanism (Ferrara–Kallosh–Strominger; Moore's arithmetic attractors [7]) and the RCFT/CM correspondence [8]; the modular-flavour programme [9] stabilises τ at the same two elliptic points used here. The $c = 24$ ingredient rests on the holomorphic-CFT classification (Schellekens [10]; van Ekeren–Möller–Scheithauer [11]) and its moonshine context [14]. What distinguishes the present construction from earlier modular/CM approaches to unification is not the objects — they are deliberately the classical ones — but the *forcing discipline*: a single premise, every downstream step a theorem or an explicitly tiered identification, and every displayed number re-derived by engine-free scripts. The companion deposit [16] positions each sector against its own literature.

9. The structure test

The decisive test for whether a framework is structural or numerical: *remove every number and see if the derivation survives.* For this core it does. The framework is

> the ground state of the monotone lattice medium (which is ω, three ways over); the cyclotomic conductor $x^{12} - 1$ of that point's own CM field evaluated at the point count of the $j = 0$ CM elliptic curve over its CM prime; the gravitational coupling the reciprocal of that count; the grand-unified and vacuum integers the trace powers of the fundamental unit of the field the curve's two elliptic points generate; the anchor duality, the ladder's reversibility, and the arrow of time one involution $d \leftrightarrow 12/d$; and the vacuum energy the curve's period at the unification height.

Not one number appears in that sentence, and it names **one object** (the curve, whose conductor and involution are its own), not two. The curve, its conductor, the point count, the period, the elliptic points, their field's unit, and the involution are all structures; the integers $(2, 3, 4, 26, 24, 52, \ldots)$ are their shadows. The two sharpest places to test it are the point-count theorem (§5) and the anchor characterisation (§2); the unit-trace identity of §6 is elementary and can be checked by hand. If any fails, the structure collapses to a coincidence. Everything beyond this kernel — the gauge, flavour, dark, and cosmological sectors, at labelled closure tiers with their own verifiers — is the companion deposit [16]; this note is its irreducible core and is meant to be judged alone.

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10. Reproducibility

Every displayed claim is checked at machine precision by the accompanying `verify_math_core.py` (all PASS): the factorisation $x^{12} - 1 = \prod_{d|12} \Phi_d(x)$; the point count $|E(\mathbb{F}_3)| = 4$ with its split/non-split theorem; the anchor vanishings $E_4(\omega) = E_6(i) = 0$ to 10^{-40}; the tightened bootstrap — the collapse to $\{i, \omega\}$, primality and CM as theorems, the multiplier constraints, the $|j|^2$ saddle-at-i / minimum-at-ω computation, and the characteristic-2 degeneration (`verify_axiom_bootstrap_v2.py`, 16/16; original four-demand inversion `verify_axiom_inversion.py`, 7/7); the nome $|q| = e^{-\pi\sqrt{3}}$; the ladder $P(3) = \prod \Phi_d(3)$ as field norms; the unit trace $\varepsilon + \varepsilon^{-1} = 4$ and the orbit $\mathrm{Tr}(M^n) = (2, 4, 14, 52)$ with $k_{\mathrm{GUT}} = 26$, $c = 24$ reached two ways each; gravity $G = 1/|E(\mathbb{F}_3)|$ with $4G = 1$; and the consequence $\Lambda/M_P^4 = 2.83 \times 10^{-122}$.

The verifier further checks the three review locks: the valence uniqueness of the anchors (V1, by fundamental-domain scan), the triple lock on k_{grav} (V2), and the orbit sparseness of the vacuum exponent (V3). The two structural collapses of this edition — that $x^{12}-1$ is the curve's own CM conductor (so the Diamond rests on one object, not two) and that the involution $\iota : d \leftrightarrow 12/d$ is at once the anchor swap, the reverse-diagonal ladder↔atoms lock, and time reversal — are checked by `verify_diamond_collapse.py` (14/14). All scripts are engine-free (`sympy`/`mpmath`/`numpy`, standard library) and are offered as **supplementary material**; they also ship inside the deposit's `09_verification/`.

This is the minimal kernel of the full Watford Framework deposit [16], where the two objects go on to generate the gauge, flavour, dark, and cosmological sectors, each at its own closure tier (PROVED / FORCED / FORCED MODULO ONE PREMISE / SELECTED / IDENTIFICATION / OPEN). The core makes only the structural claim of §9.

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