Hyperbolic Quaternions. By Alexander Macfarlane, Lehigh University, South Bethlehem, Pennsylvania. (With a Plate.)

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It is well known that quaternions are intimately connected with spherical trigonometry, and in fact they reduce that subject to a branch of algebra. The question is suggested whether there is not a system of quaternions complementary to that of Hamilton, which is capable of expressing trigonometry on the surface of the equilateral hyperboloids. The rules of vector-analysts are approximately complementary to those of quaternions. In this paper I propose to show how they can be made completely complementary, and that, when so rectified, they yield the hyperbolic counterpart of the spherical quaternions.

The celebrated rules discovered by Hamilton are :—

$$\begin{array}{lll} i^2 = -1 & j^2 = -1 & k^2 = -1 \\ ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j. \end{array}$$

This is the statement of the rules as enunciated by Hamilton ; it supposes an order of the symbols from right to left. When the order is changed to that from left to right, they become :—

$$\begin{array}{lll} i^2 = -1 & j^2 = -1 & k^2 = -1 \\ ij = -k & jk = -i & ki = -j \\ ji = k & kj = i & ik = k. \end{array}$$

The rules used by vector-analysts are :—

$$\begin{array}{lll} i^2 = +1 & j^2 = +1 & k^2 = +1 \\ ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j, \end{array}$$

and they suppose an order from left to right. They lead to products in which the manner of associating the factors is essential, in this respect differing from the rules of quaternions. Can they be modified so that the order of the factors will be preserved,

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while the products become associative? I find that the desired modification is accomplished by introducing $\sqrt{-1}$ before the second and third sets. The rules then become

$$i^2=+1 \qquad j^2=+1 \qquad k^2=+1$$
$$ij=\sqrt{-1}k \qquad jk=\sqrt{-1}i \qquad ki=\sqrt{-1}j$$
$$ji=-\sqrt{-1}k \qquad kj=-\sqrt{-1}i \qquad ik=-\sqrt{-1}j.$$

As the quaternion $i\,j\,k$ are quadrantal unit-vectors, they can be analysed into $\sqrt{-1}i_0$, $\sqrt{-1}j_0$, $\sqrt{-1}k_0$, where $i_0 j_0 k_0$ are unit-vectors.

The quaternion rules, modified for order, then become

$$(\sqrt{-1}i_0)(\sqrt{-1}i_0)=-1 \qquad (\sqrt{-1}j_0)(\sqrt{-1}j_0)=-1$$
$$(\sqrt{-1}k_0)(\sqrt{-1}k_0=-1$$
$$(\sqrt{-1}i_0)(\sqrt{-1}j_0=-\sqrt{-1}k_0 \qquad (\sqrt{-1}j_0)(\sqrt{-1}k_0)=-\sqrt{-1}i_0$$
$$(\sqrt{-1}k_0)(\sqrt{-1}i_0)=\sqrt{-1}j_0$$
$$(\sqrt{-1}j_0)(\sqrt{-1}i_0)=\sqrt{-1}k_0 \qquad (\sqrt{-1}k_0)(\sqrt{-1}j_0)=\sqrt{-1}i_0$$
$$(\sqrt{-1}i_0)(\sqrt{-1}k_0)=\sqrt{-1}j_0.$$

These rules are in perfect harmony with the vector rules when made associative as above; for, on dividing the left hand by $\sqrt{-1}$ $\sqrt{-1}$, and the right hand side by the equivalent $-$, they yield

$$i_0{}^2=1 \qquad j_0{}^2=1 \qquad k_0{}^2=1$$
$$i_0j_0=\sqrt{-1}k_0 \qquad j_0k_0=\sqrt{-1}i_0 \qquad k_0i_0=\sqrt{-1}j_0$$
$$j_0i_0=-\sqrt{-1}k_0 \qquad k_0j_0=-\sqrt{-1}i_0 \qquad i_0k_0=-\sqrt{-1}j_0.$$

Let ρ denote any real unit axis; then $\rho^2=1$. Similarly for any imaginary unit axis $(\sqrt{-1}\rho)^2=-1$. It is evident that $\rho^2=1$ is in nature a principle of reduction. But there is also the principle of reduction $\rho/\rho=1$ or $\sqrt{-1}\rho/\sqrt{-1}\rho=1$. This latter is a more absolute principle, and the reduction specified can be made at any time; whereas the former is legitimate only under certain conditions. The rules of the form $ij=\sqrt{-1}k$ are also principles of reduction of a relative nature.

A more general statement of these rules is as follows:—For any two real unit axes β and γ.

$$\beta\gamma=\cos\beta\gamma+\sin\beta\gamma\sqrt{-1}\,\overline{\beta\gamma}$$

where $\overline{\beta\gamma}$ denotes in the simplest case the axis perpendicular to β and γ, but more correctly the axis conjugate to the plane of

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β and γ. Similarly for any two imaginary axes $\sqrt{-1}\beta$ and $\sqrt{-1}\gamma$
$(\sqrt{-1}\beta)(\sqrt{-1}\gamma) = -\cos\beta\gamma - \sin\beta\gamma\ \sqrt{-1}\,\overline{\beta\gamma}$.

I proceed now to apply these principles to the investigation of the fundamental theorems of hyperboloidal trigonometry. I shall consider only the hyperboloid of equal axes, but the results can easily be extended to the general hyperboloid.

On account of the symmetry of the sphere with respect to its centre, spherical quaternions are independent of rectangular axes. It is otherwise with hyperboloidal quaternions, for the equilateral hyperboloid has an axis of revolution. In order to treat of trigonometry on the hyperboloid, it is necessary first to treat the trigonometry of the sphere with reference to the same axis of revolution. In the figure (fig. 1) OA is the axis of revolution, and the surfaces considered are those generated by the circle and by the equilateral hyperbolas. From this point of view the circle appears as consisting of a real part PQ corresponding to the real hyperbola P′Q′, and an imaginary part QR corresponding to the imaginary hyperbola Q′R′. Consequently the sphere appears broken up into a double sheet traced out by PQ and RS, and a single sheet traced out by QR.

The algebraic expression for a circular angle is $e^{b\sqrt{-1}}$. As the axis of the plane is not specified, the denotation of the expression is necessarily limited to angles in a constant plane. Let β be introduced to denote the axis, then $e^{b\sqrt{-1}\beta}$ is the proper expression for an angle in any plane. We have

$$e^{b\sqrt{-1}\beta} = 1 + b\sqrt{-1}\beta + \frac{(b\sqrt{-1}\beta)^2}{2\,!} + \frac{(b\sqrt{-1}\beta)^3}{3\,!} + .$$

Let the principle of reduction be introduced, which reduces $(\sqrt{-1}\beta)^2 = -1$; then the right hand member becomes

$$\begin{aligned} &1 + b\sqrt{-1}\beta - \frac{b^2}{2\,!} - \frac{b^3}{3\,!}\sqrt{-1}\beta + \text{etc.} \\ &= 1 - \frac{b^2}{2\,!} + \frac{b^4}{3\,!} - \\ &\quad + \left(b - \frac{b^3}{3\,!} + \frac{b^5}{5\,!} - \right)\sqrt{-1}\beta \\ &= \mathrm{SU}q + \mathrm{VU}q \\ &= \cos b + \sin b\,(\sqrt{-1}\beta). \end{aligned}$$

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Note that the expression SUq + VUq is not the complete equivalent of Uq; the binomial is a reduced equivalent. For, if β is variable, the result of differentiating $e^{b\sqrt{-1}\beta}$ will be different from the result of differentiating $\cos b + \sin b\ (\sqrt{-1}\beta)$.

If we enquire for the analogous expression for a hyperbolic angle, we find that there is none furnished by Algebra. It is not e^b, for

$$e^b = 1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} +$$

and there is here no ground for breaking up the series into two components; all the terms are real, and so add directly. For the same reason it cannot be e^{-b}. But we know that

$$\cosh b = 1 + \frac{b^2}{2!} + \frac{b^4}{4!} + ,$$

$$\sinh b = b + \frac{b^3}{3!} + \frac{b^5}{5!} + ;$$

there must therefore be some proper way of expressing a hyperbolic angle by means of an exponential function. Try the effect of dropping $\sqrt{-1}$ from the circular expression $e^{b\sqrt{-1}\beta}$. We get

$$e^{b\beta} = 1 + b\beta + \frac{(b\beta)^2}{2!} + \frac{(b\beta)^3}{3!} + .$$

Now introduce the corresponding principle of reduction, namely, $\beta^2 = +1$; then

$$\begin{aligned} e^{b\beta} &= 1 + b\beta + \frac{b^2}{2!} + \frac{b^3}{3!}\beta + \\ &= 1 + \frac{b^2}{2!} + \frac{b^3}{3!} + \\ &\quad + (b + \frac{b^3}{3!} + \frac{b^5}{5!} +)\beta \\ &= \mathrm{SU}q' + \mathrm{VU}q' \end{aligned}$$

if q' denotes a hyperbolic quaternion. Hence it appears that $e^{b\beta}$ is the proper expression for the angle of an equilateral hyperbola.

It follows that the expression for the spherical quaternion is $re^{b\sqrt{-1}\beta}$, which, after expansion and reduction, gives the spherical complex quantity of the form $x + y\sqrt{-1}\beta$. Similarly the expression for the equilateral hyperbolic quaternion is $re^{b\beta}$, which, after expansion and reduction, gives the hyperbolic complex quantity of the form $x + y\beta$. In the former case we have $r = \sqrt{x^2 + y^2}$; in the latter, $r = \sqrt{x^2 - y^2}$. Suppose the objection

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made, x may be equal to y, what then becomes of the modulus? The answer is, the cosine is then $\frac{x}{o}$, which shows that the angle is infinitely great, and this is the geometrical truth. Suppose that the objection is made, x may be less than y, what then becomes of the modulus? The modulus then takes on a form appropriate to the conjugate hyperbola, and by the hypothesis the angle lies in the conjugate hyperbola.

The above expression for a spherical quaternion has a resemblance to the *Drehstreckung* of Professor Klein. But r does not mean an expansion and $e^{b\sqrt{-1}\beta}$ a rotation; the former is a multiplier simply, and the latter a circular angle. The existence of the analogous expression $re^{b\beta}$, and the application of these expressions to develop the trigonometry of surfaces of the second order show that his theory of quaternions is inadequate, and the sphere of applicability which he assigns them too narrow. According to his idea, quaternions will be in place when we wish to have a convenient algorithm for the combination of rotations and dilatations; the true idea is that quaternions contains the elements of the algebra of space.

In investigating the fundamental principles of hyperboloidal trigonometry, the first problem is to find the general expression for a spherical versor, when reference is made to the axis of revolution.

Let OA (fig. 2) represent the axis of revolution, and let it be denoted by α. Any versor, POA, passing through the axis of revolution, may be denoted by $e^{b\sqrt{-1}\beta}$, where β denotes a unit axis perpendicular to α. Similarly AOQ, another versor, passing through the axis of revolution, may be denoted by $e^{c\sqrt{-1}\gamma}$, where γ denotes a unit axis perpendicular to α. The product versor POQ is circular, but it will not in general pass through OA; let it be denoted by $e^{a\sqrt{-1}\xi}$.

Now $e^{a\sqrt{-1}\xi} = e^{b\sqrt{-1}\beta}e^{c\sqrt{-1}\gamma}$

$$= (S+V)(S'+V')$$
$$= SS + SV' + S'V + VV'$$
$$= \cos b \cos c + \cos c \sin b\sqrt{-1}\beta + \cos b \sin c \sqrt{-1}\gamma + \sin b \sin c \sqrt{-1}\beta\sqrt{-1}\gamma;$$
$$= \cos b \cos c - \sin b \sin c \cos \beta\gamma$$
$$+ \sqrt{-1}\{\cos c \sin b \cdot \beta + \cos b \sin c . \gamma - \sin b \sin c \sin \beta\gamma . \overline{\beta\gamma}\}.$$

We observe that the directed sine may be broken up into two components—namely, $\cos c \sin b.\beta + \cos b \sin c\cdot\gamma$, which is perpendicular to the axis of revolution, and $-\sin b \sin c \sin\beta\gamma.\overline{\beta\gamma}$, which has the direction of the negative of the axis of revolution, for $\overline{\beta\gamma}$ is identical with α.

Draw OS to represent the first component $\cos c \sin b\cdot\beta$, OT to represent the second component $\cos b \sin c\cdot\gamma$, and OU to represent the third component $-\cos b \cos c \sin\beta\gamma\cdot\alpha$ Draw OV, the resultant of the first two, and OR, the resultant of all three; then

$$\cos a = \cos b \cos c - \sin b \sin c \cos\beta\gamma$$

$$\text{and } \xi = \frac{\mathrm{OR}}{\sin a} = \frac{\cos c \sin b\cdot\beta + \cos b \sin c\cdot\gamma - \sin b \sin c \sin\beta\gamma\cdot\alpha}{\sqrt{1-(\cos b \cos c - \sin b \sin c \cos\beta\gamma)^2}}.$$

The plane of OA and OV passes through OR, which is normal to the plane POQ; hence these planes cut orthogonally in a line OX, and the angle between OA and OX is equal to that between OV and OR, for OV is perpendicular to OA and OR to OX. Let θ denote the angle AOX; then

$$\sin\theta = \frac{\sin b \sin c \sin\beta\gamma}{\sqrt{1-(\cos b \cos c - \sin b \sin c \sin\beta\gamma)^2}}.$$

The figure (fig. 3) represents a section through the plane of OA and OV; MX represents $\sin\theta$. Hence the axis ξ can be put in the form $\cos\theta\cdot\epsilon - \sin\theta\cdot\alpha$, where ϵ denotes a unit axis perpendicular to α. The unit axis ϵ may be expressed in terms of two axes j and k, forming an orthogonal system with the axis of revolution, which may be denoted by i. Hence a perfectly general expression for any spherical versor is $e^{a\sqrt{-1}\xi}$, where

$$\xi = \sqrt{-1}\{\cos\theta\cdot(\cos\phi\cdot j + \sin\phi\cdot k) - \sin\theta\cdot i\}.$$

We observe that if $e^{a\sqrt{-1}\xi}$ is an angle in the double sheet, $\sqrt{-1}\xi$ is a vector to the surface of the single sheet.

It is now easy to find the solution of the analogous problem, namely, the product of two diplanar hyperbolic versors when the plane of each passes through the axis of revolution.

The axis of the versor is perpendicular to the plane of the versor when the latter passes through the axis of revolution; and we shall assume that it is of unit length, an assumption which is afterwards

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completely justified. Let the two versors POA and AOQ (fig. 4) be denoted by $e^{b\beta}$ and $e^{c\gamma}$, the axes β and γ being both perpendicular to the axis of revolution α, and of unit length.

Then $$e^{b\beta}\ e^{c\gamma} = (S+V)(S'+V')$$
$$= SS' + S'V + SV' + VV'$$
$$= \cosh b \cosh c + \cosh c \sinh b \cdot \beta + \cosh b \sinh c \cdot \gamma + \sinh b \sinh c \cdot \beta\gamma.$$

Now $$\beta\gamma = \cos \beta\gamma + \sqrt{-1} \sin \beta\gamma \cdot \beta\gamma$$
$$= \cos \beta\gamma + \sqrt{-1} \sin \beta\gamma \cdot \alpha.$$

Hence $$e^{b\beta}e^{c\gamma} = \cosh b \cosh c + \sinh b \sinh c \cos \beta\gamma$$
$$+ \cosh c \sinh b \cdot \beta + \cosh b \sinh c \cdot \gamma + \sqrt{-1} \sinh b \sinh c \sin \beta\gamma \cdot \alpha.$$

Hence $$\cosh e^{b\beta}e^{c\gamma} = \cosh b \cosh c + \sinh b \sinh c \cos \beta\gamma$$

and $$\text{Sinh}\ e^{b\beta}e^{c\gamma} = \cosh c \sinh b \cdot \beta + \cosh b \sinh c \cdot \gamma + \sqrt{-1} \sinh b \sinh c \sin \beta\gamma \cdot \alpha.$$

The first and second components of the directed sinh (denoted by Sinh) are perpendicular to the axis of revolution, hence their resultant $\cosh c \sinh b \cdot \beta + \cosh b \sinh c \cdot \gamma$ is also perpendicular to the principal axis. Let it be represented by OV in the figure. The difficulty consists in finding the true direction of the third component $\sqrt{-1} \sinh b \sinh c \sin \beta\gamma \cdot \alpha$ on account of the presence of $\sqrt{-1}$. It will be found that $\sqrt{-1}$ has here nothing to do with the direction; and as the term is otherwise in the positive direction of α, we represent it by OU in the figure. In the case of the sphere OU is drawn in the direction opposite to α. Let OR be the resultant of OU and OV; it represents the directed Sinh both in magnitude and direction.

The square of the length of OR is

$$\cosh^2 c \sinh^2 b + \cosh^2 b \sinh^2 c + 2 \cosh c \cosh b \sinh c \sinh b \cos \beta\gamma + \sinh^2 b \sinh^2 c \sin^2 \beta\gamma.$$

But the square of the modulus of OR is the same with a negative sign before the last term; added to the square of $\cosh e^{b\beta}e^{c\gamma}$ it yields 1.

The directed sinh OR is not normal to the plane POQ; how is it related to that plane? If we draw $OU' = -OU$ and find OR'

the resultant, it is OR′ and not OR which is normal to the plane of OP and OQ. The expressions for the three vectors OR′, OP, OQ are

$$\mathrm{OR}' = \cosh c \sinh b \cdot \beta + \cosh b \sinh c \cdot \gamma - \sinh b \sinh c \sin \beta\gamma \cdot a$$

$$\mathrm{OP} = -\sinh b \frac{\cos \beta\gamma}{\sin \beta\gamma} \cdot \beta + \sinh b \frac{1}{\sin \beta\gamma} \cdot \gamma + \cosh b \cdot a$$

$$\mathrm{OQ} = -\sinh c \frac{1}{\sin \beta\gamma} \cdot \beta - \sinh c \frac{\cos \beta\gamma}{\sin \beta\gamma} \cdot \gamma + \cosh c \cdot \gamma$$

from which it follows that S(OR′)(OP) = 0 and S(OR′)(OQ) = 0. Hence OR′ is normal to the plane of POQ. How is the direction of OR related to that plane? The plane of OA and OV (fig. 5) cuts the equilateral hyperboloid in an equilateral hyperbola; and as it passes through the normal OR′, it must cut the plane POQ orthogonally.

Let OX be the line of intersection. Draw XM perpendicular to OA, draw XD a tangent to the equilateral hyperbola at X (fig. 5), and XA′ parallel to OA. Let θ denote the hyperbolic angle AOX. As OR′ is normal to the plane POQ, it is perpendicular to OX; but OV is perpendicular to OA, therefore the angle AOX is equal to the angle VOR′. Now the angle AOR is the complement of ROV, and A′XD the complement of AOX; therefore the line OR is parallel to the tangent XD. Thus the direction of the directed sinh is that of the conjugate axis to the plane of OP and OQ. This idea of *conjugate* instead of *normal* also applies to the spherical case, from which it follows that $ij = \sqrt{-1}k$ means that k is the axis conjugate to i and j.

$$\text{Now } \sinh \theta = \frac{\mathrm{MX}}{\mathrm{OA}} = \frac{\mathrm{VR}}{\sqrt{\mathrm{OV}^2 - \mathrm{VR}^2}}$$

$$= \frac{\sinh b \sinh c \sin \beta\gamma}{\sqrt{(\cosh b \cosh c + \sinh b \sinh c \cos \beta\gamma)^2 - 1}}$$

The above analysis shows that the product versor POQ may be specified by the following three elements:—*First*, ϵ, a unit axis drawn perpendicular to OA in the plane of OA and the normal to the plane POQ; *second*, θ, the hyperbolic angle determined by OA and OX, which is drawn at right angles to the normal in the plane of OA and the normal; *third*, a, the angle of the hyperbolic sector

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OPXQ, which is a sector of the hyperbola having OX for semi-major axis, and for semi-minor axis OB which is equal to OA and perpendicular to OA and OV. This hyperbola is not an equilateral hyperbola; PXQ is the curve of intersection of the hyperboloid with a plane through the points O, P, Q. An angle of this hyperbola is specified by the ratio of the sector to half of the rectangle formed by OX and OB. Thus a is the ratio of the sector POQ to half of the rectangle formed by OX and OB.

Hence the product versor may be expressed by means of a hyberbolic angle a and a hyberbolic axis of the form

$$\cosh \theta \cdot \epsilon + \sqrt{-1} \sinh \theta \cdot \alpha,$$

where, as before, ϵ denotes a unit axis normal to α, the axis of revolution. Let ξ denote the above axis; the actual components from which it is constructed are $\cosh \theta \cdot \epsilon$ and $\sinh \theta \cdot \alpha$ It is not of unit length, but it has a unit modulus. The former is $\sqrt{\cosh^2\theta + \sinh^2\theta}$, the latter is $\sqrt{\cosh^2\theta - \sinh^2\theta}$.

Hence the product versor may be expressed by

$$e^{a\xi} = e^{a\,(\cosh \theta \cdot \epsilon + \sinh \theta \cdot \alpha)}.$$

And to determine these quantities we have the three analogous equations

$$\cosh a = \cosh b \cosh c + \sinh b \sinh c \cos \beta\gamma \quad (1)$$

$$\cosh \theta = \frac{\sinh b \sinh c \sin \beta\gamma}{\sinh a}$$

$$\epsilon = \frac{\cosh c \sinh b \cdot \beta + \cosh b \sinh c \cdot \gamma}{\sinh a \sinh \theta.}$$

As ϵ is of unit length, it may be expressed as $\cos \phi \cdot j + \sin \phi \cdot k$, and if i denotes the axis of revolution

$$\xi = \cosh \theta \,(\cos \phi \cdot j + \sin \phi \cdot k) + \sqrt{-1} \sinh \theta \cdot i.$$

The axis ξ is evidently a vector to a point in the conjugate hyperboloid of one sheet.

In the above investigation it is assumed that the magnitude of the perpendicular component of the Sinh is necessarily greater than the component parallel to the axis of revolution. This means that

$$\cosh^2 c \sinh^2 b + \cosh^2 b \sinh^2 c + 2 \cosh b \cosh c \sinh b \sinh c \cos \beta\gamma$$
$$> \sinh^2 b \sinh^2 c \sin^2 \beta\gamma.$$

Let $\sin \beta\gamma = 1$, $\cos \beta\gamma = 0$; then each of the two terms on the left is greater than the term on the right of the inequality. Let sin $\beta\gamma = 0$ and $\cos \beta\gamma = -1$, then the above expression reduces to the well known inequality $a^2 + b^2 > 2\, ab$. Hence the terms on the left are always greater than the term on the right.

In the case when the two versors are equal, we can verify that it is the line of intersection of the central plane with the equilateral hyperboloid which is indicated by the product of the versors.

As the two versors are equal they might be denoted by $e^{b\beta}$ and $e^{b\gamma}$. Let $\cosh b = x$, $\sinh b = y$. Then according to the theorem

$$e^{b\beta}\, e^{b\gamma} = x^2 + y^2 \cos \beta\gamma + xy\ (\beta + \gamma) + \sqrt{-1}\, y^2 \sin \beta\gamma \cdot a$$

As (fig. 6), OB the semi-transverse axis of the hyperbola PXQ is 1, NQ represents the sinh of half of the product angle. Now by the geometry of the construction

$$\frac{\text{NQ}}{\text{OB}} = \tfrac{1}{2}\sqrt{2y^2 + 2y^2 \cos \beta\gamma}$$

$$= \frac{y}{\sqrt{2}}\sqrt{1 + \cos \beta\gamma}.$$

Again $\dfrac{\text{ON}}{\text{OX}} = \dfrac{x}{\cosh \theta}$

$$= \frac{x\sqrt{(x^2 + y^2 \cos \beta\gamma)^2 - 1}}{\sqrt{x^2\, y^2\, 2\, (1 + \cos \beta\gamma)}}$$

$$= \sqrt{1 + \frac{y^2}{2}(1 + \cos\ \beta\gamma)}.$$

Now $\cosh 2\text{XOQ} = (\cosh \text{XOQ})^2 + (\sinh \text{XOQ})^2$

$$= \left(\frac{\text{NQ}}{\text{OB}}\right)^2 + \left(\frac{\text{ON}}{\text{OX}}\right)^2$$

$$= \frac{y^2}{2}(1 + \cos \beta\gamma) + 1 + \frac{y^2}{2}(1 + \cos \beta\gamma)$$

$$= 1 + y^2 + y^2 \cos \beta\gamma$$

$$= x^2 + y^2 \cos \beta\gamma$$

which agrees with the above theorem.

We have seen that the general spherical versor is denoted by $e^{a\sqrt{-1}\xi}$, where

$$\xi = -\sin \theta \cdot a + \cos \theta \cdot \epsilon,$$

a denoting the axis of revolution and ϵ an axis in the perpendicular plane. Similarly a general versor for the equilateral hyperboloid of two sheets is denoted by $e^{a\xi}$, where

$$\xi = \sqrt{-1}\, \sinh \theta \cdot a + \cosh \theta \cdot \epsilon,$$

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a and ϵ denoting the same kind of axes as before. This leads us to the consideration of hyperboloidal axes. Let ξ_1 denote a radius to the double sheet (fig. 7);

$$\xi_1 = \cosh\theta\cdot a + \sqrt{-1}\sinh\theta\cdot\epsilon.$$

The length of ξ_1 is

$$\sqrt{\cosh^2\theta + \sinh^2\theta}$$

but its modulus is $\sqrt{\cosh^2\theta - \sinh^2\theta}$, which is 1. Let ξ_2 denote a radius to the single sheet;

$$\xi_2 = \sqrt{-1}\sinh\theta\cdot a + \cosh\theta\cdot\epsilon.$$

The corresponding axes for the unit sphere are

$$\xi_1 = \cos\theta\cdot a + \sin\theta\cdot\epsilon$$

$$\text{and } \xi_2 = -\sin\theta\cdot a + \cos\theta\cdot\epsilon.$$

Just as a spherical vector is expressed by $r\sqrt{-1}\xi$, so a hyperboloidal vector is expressed by $r\xi$, where r denotes the modulus and ξ the axis. The principal difference is that in the case of the sphere ξ is of constant length, whereas in the case of the hyperboloid the length of the axis depends on its position relative to the axis of revolution.

Consider now a general triangle on the hyperboloid of two sheets (fig. 8). Let the axes to the three points be denoted by

$$\xi = \cosh\theta\cdot a + \sqrt{-1}\sinh\theta\cdot\beta$$

$$\eta = \cosh\theta'\cdot a + \sqrt{-1}\sinh\theta'\cdot\gamma$$

$$\zeta = \cosh\theta''\cdot a + \sqrt{-1}\sinh\theta''\cdot\delta.$$

Then $\xi\eta = \cosh\theta\cosh\theta' - \sinh\theta\sinh\theta'\cos\beta\gamma$ (1)

$- \cosh\theta\sinh\theta'\cdot\overline{a\gamma} - \sinh\theta\cosh\theta'\cdot\overline{\beta a}$ (2)

$- \sqrt{-1}\sinh\theta\sinh\theta'\sin\beta\gamma\cdot a$ (3)

Hence $\cosh\xi\eta = (1)$

and $\text{Sinh}\,\xi\eta = (2) + (3)$.

We have proved that the length of (3) is always less than the length of (2); hence $\overline{\xi\eta}$ has the form

$$\sinh\phi\cdot a + \sqrt{-1}\cosh\phi\cdot\epsilon.$$

And the same is true for $\eta\zeta$ and $\zeta\xi$. The central section is always hyperbolic.

Now $\xi\zeta = (\xi\eta)(\eta\zeta)$.

Therefore $\cosh\xi\zeta = \cosh\xi\eta\cosh\eta\zeta + \cosh(\text{Sinh}\,\xi\eta\ \text{Sinh}\,\eta\zeta)$ and $\text{Sinh}\,\xi\zeta = \cos\eta\zeta\ \text{Sinh}\,\xi\eta + \cosh\xi\eta\ \text{Sinh}\,\zeta\eta$

$$+ \text{Sinh}\{\text{Sinh}\,\xi\eta\ \text{Sinh}\,\eta\zeta\}.$$

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Consider now a general triangle on the hyberboloid of one sheet (fig. 9).

Let the three axes be

$$\xi = \cosh\theta\cdot\beta + \sqrt{-1}\sinh\theta\cdot\alpha$$
$$\eta = \cosh\theta'\cdot\gamma + \sqrt{-1}\sinh\theta'\cdot\alpha$$
$$\zeta = \cosh\theta''\cdot\delta + \sqrt{-1}\sinh\theta''\cdot\alpha.$$

Then $\xi\eta = \cosh\theta\cosh\theta'\cos\beta\gamma - \sinh\theta\sinh\theta'$ (1)

$- \cosh\theta\sinh\theta'\cdot\overline{\beta\alpha} - \cosh\theta'\sinh\theta\cdot\overline{\alpha\gamma}$ (2)

$+ \sqrt{-1}\cosh\theta\cosh\theta'\sin\beta\gamma\cdot\alpha$ (3)

In this case the length of the normal part of the Sinh may be greater than, equal to, or less than the length of the components along the axis of revolution. For we have to compare—

$\cosh^2\theta\sinh^2\theta' + \cosh^2\theta'\sinh^2\theta - 2\cosh\theta\cosh\theta'\sinh\theta\sinh\theta'\cos\beta\gamma$ with $\cosh^2\theta\cosh^2\theta'\sin^2\beta\gamma$. Let $\sin\beta\gamma = 0$, $\cos\beta\gamma = -1$; then the former term is the greater. Let $\cos\beta\gamma = 0$, $\sin\beta\gamma = 1$; then the former term is the less. And the terms may be equal. In the former case the axis of $\xi\eta$ has the form

$$\cosh\phi\cdot\epsilon + \sqrt{-1}\sinh\phi\cdot\alpha$$

and the section is hyperbolic. In the latter case the axis of $\xi\eta$ has the form

$$\sqrt{-1}\,\{\cosh\theta\cosh\theta'\sin\beta\gamma\cdot\alpha + \sqrt{-1}\,(\cosh\theta\sinh\theta'\cdot\overline{\beta\alpha} + \cosh\theta'\sinh\theta\cdot\overline{\alpha\gamma})\}.$$

The axis inside the brackets denotes an axis of the equilateral hyperboloid of two sheets, and the section is elliptic.

As before

$$\xi\zeta = (\xi\eta)\,(\eta\zeta)$$

therefore $\cosh\xi\zeta = \cosh\xi\eta\cosh\eta\zeta + \cosh\{\text{Sinh}\,\xi\eta\,\text{Sinh}\,\eta\zeta\}$

and

$\text{Sinh}\,\xi\eta = \cosh\eta\zeta\,\text{Sinh}\,\xi\eta + \cosh\xi\eta\,\text{Sinh}\,\eta\zeta + \text{Sinh}\,\{\text{Sinh}\,\xi\eta\,\text{Sinh}\,\eta\zeta\}.$

Hyperbolic Quaternions.

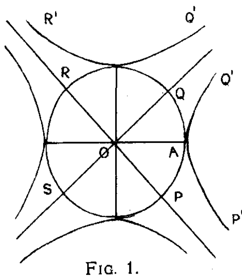

Fig. 1.

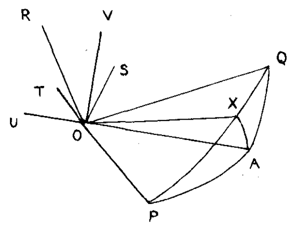

Fig. 2.

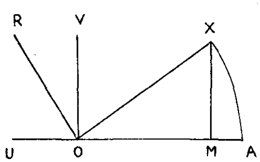

Fig. 3.

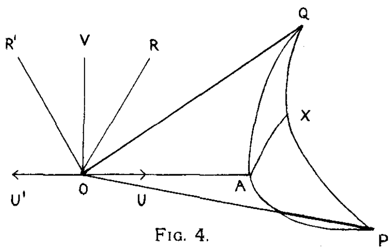

Fig. 4.

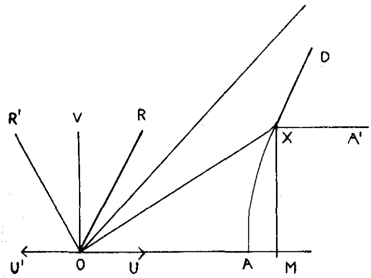

Fig. 5.

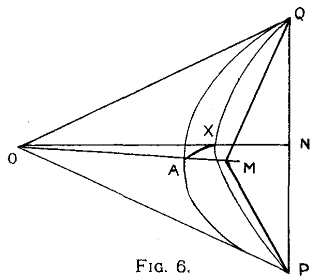

Fig. 6.

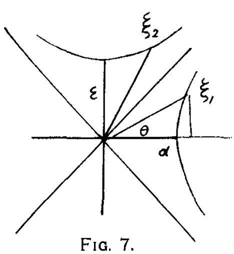

Fig. 7.

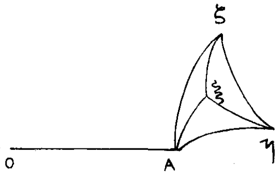

Fig. 8.

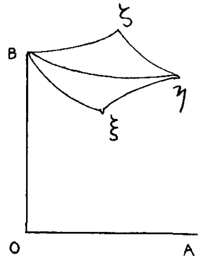

Fig. 9.

A. Ritchie & Son Edin.