THE MULTIPLICATION OF CONDITIONALLY CONVERGENT SERIES

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1. Although much has been written concerning the multiplication of series according to Cauchy's rule, the last word has not yet been said upon the subject, and a number of interesting questions connected with it remain unanswered. In this paper I prove a few simple theorems which I believe to be new. In § 4 I prove that a sufficient condition for the multiplication of two convergent series Σa_n, Σb_n is that na_n and nb_n should each tend to zero as n tends to infinity. In § 8 I generalise this result by showing (by the aid of slightly more elaborate analysis) that it is sufficient that the absolute values of na_n and nb_n should have an upper limit. In § 7 I establish a generalisation of a somewhat different kind, showing that the conditions

$$n\phi(n)\, a_n \to 0, \qquad \frac{nb_n}{\phi(n)} \to 0,$$

where $\phi(n)$ is one of a general class of functions of which $\log n$ is typical, are sufficient.

I have also (§ 13) stated and indicated the proofs of some corresponding theorems for integrals, and I have added (§§ 12, 10) a generalisation of Mertens' theorem and new proofs of some results of Pringsheim's concerning series of a special form. I have thought it worth while to add this last section, although it contains no new results, because the class of series to which it refers is the most natural and important of all, and because, so far as I know, the results have never yet been proved with anything like the simplicity which is desirable and attainable.

I wish to state explicitly that I have not proved, either positively or negatively, but particularly negatively, as much as I think ought to be capable of proof. In § 11 I indicate some questions which seem to me of considerable interest, but which I am at present unable to answer.

2. I shall adopt the notation of Mr. Bromwich's *Infinite Series* (pp. 82 *et seq.*); *i.e.*, I shall denote by A, B, C the series

$$a_1+a_2+a_3+\ldots,\quad b_1+b_2+b_3+\ldots,\quad c_1+c_2+c_3+\ldots,$$

where
$$c_n = a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1.$$

I shall also use the letters A, B, C in equations or inequalities to denote the *sums* of the series, when they are convergent; and I shall denote the sums of the first n terms of the series by A_n, B_n, C_n, so that, *e.g.*,

$$A_n = a_1+a_2+\ldots+a_n.$$

3. The classical results in connection with the multiplication of series are the following:—

(1) **Abel's Theorem.**—*If all three series are convergent, then* $C = AB$.

(2) **Cauchy's Theorem.**—*If A and B are absolutely convergent, then C is absolutely convergent.*

(3) **Mertens' Theorem.**—*If A is absolutely and B conditionally convergent, then C is convergent.*

In addition to these results, a number of theorems have been proved by Pringsheim, Voss, and Cajori.* These relate to the case in which A and B are conditionally convergent, but one at least becomes absolutely convergent when its terms are associated in certain groups, the number in each group being less than some fixed number. I shall return to some of the simplest and most important of these theorems later on.

4. THEOREM A.—*If A and B are convergent, and*

$$na_n \to 0, \qquad nb_n \to 0$$

as $n \to \infty$, then C is convergent.

The proof is very simple. For

$$C_n = a_1B_n+a_2B_{n-1}+\ldots+a_nB_1 = a_1B_n+a_2B_{n-1}+\ldots+a_NB_{n+1-N}$$
$$+a_{N+1}B_{n-N}+a_{N+2}B_{n-N-1}+\ldots+a_nB_1.$$

Applying Abel's partial summation lemma to the first line, we obtain

$$C_n-A_NB_{n+1-N} = A_1b_n+A_2b_{n-1}+\ldots+A_{N-1}b_{n+2-N}$$
$$+B_1a_n+B_2a_{n-1}+\ldots+B_{n-N}a_{N+1}.$$

* For references, see Bromwich, *Infinite Series*, p. 87.

If N is such a function of n that N and $n-N$ tend to infinity with n, then

(1) $$A_N B_{n+1-N} \rightarrow AB.$$

This is certainly the case if $Gn < N < Hn$, where G and H are constants, and $0 < G < H < 1$. But then

$$| A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N} | < K(N-1)\beta,$$
$$| B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1} | < K(n-N)\alpha,$$

where K is a constant, and α and β are the greatest of the moduli of

$$a_{N+1},\ a_{N+2},\ \ldots,\ a_n \qquad b_{n+2-N},\ b_{n+3-N},\ \ldots,\ b_n$$

respectively. In virtue of the restriction imposed upon N, we have

$$N-1 < \lambda n, \qquad n-N < \lambda n,$$

where λ is a constant. And we can choose n_0 so that

$$| n\alpha | < \epsilon/\lambda K, \qquad | n\beta | < \epsilon/\lambda K,$$

for $n \geqslant n_0$. It follows that for $n \geqslant n_0$, we have

(2) $$\begin{cases} | A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N} | < \epsilon, \\ | B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1} | < \epsilon, \end{cases}$$

and from (1) and (2) the conclusion follows.

5. This theorem is not of very wide application, the range of series which are only conditionally convergent, and yet satisfy the condition $na_n \rightarrow 0$, being of course comparatively narrow. The simplest of such series are those of the type

$$\frac{1}{\phi(1)} - \frac{1}{2\phi(2)} + \frac{1}{3\phi(3)} - \ldots,$$

where $\phi(n)$ is any function which tends steadily to infinity with n, but (like $\log n$ or $\log n \log\log n$) so slowly that the series is not absolutely convergent. Or, again, the series

$$\Sigma n^{-1-ai} \quad (a \gtrless 0)$$

is known* to oscillate finitely, so that, if $\phi(n)$ is any function which tends steadily to infinity with n, the series

$$\Sigma \frac{1}{n^{1+ai}\phi(n)}, \quad \Sigma \frac{\cos(a \log n)}{n\phi(n)}, \quad \Sigma \frac{\sin(a \log n)}{n\phi(n)}$$

are convergent. This result may be extended (as in Mr. Bromwich's paper printed earlier in this volume) to such series as

$$\Sigma \frac{\frac{\cos}{\sin}(a \log_{k+1} n)}{n \log n \log_2 n \ldots \log_k n\, \phi(n)},$$

where $\log_2 n = \log\log n, \quad \log_3 n = \log\log_2 n, \quad \ldots,$

and generally to series of the type

$$\Sigma \frac{f'(n)}{\phi(n)} \frac{\cos}{\sin} \{f(n)\},$$

* See Landau, *Crelle*, Bd. cxxv., pp. 105–7, for references in connection with this series.

where $f(n)$ is a function of n such that $f(n)$, $f'(n)$ are monotonic, $f(n) \to \infty$, $f'(n) \to 0$, and

$$\Sigma \{f'(n)\}^2$$

is convergent. Another interesting type is

$$\Sigma \frac{\Gamma(i+n)}{\Gamma(1+n)} \frac{1}{\phi(n)}.$$

The theorem, however, seems to me of some interest in spite of its comparatively narrow range of applicability, on account of the simplicity of the conditions and the fact that no use whatever is made of the notion of absolute convergence. All of Pringsheim's theorems depend on the possibility of securing absolute convergence in one at least of the series A, B by the insertion of brackets in some prescribed manner.

6. Series for which $na_n \to 0$ have another interesting property first discovered by Tauber.* The converse of Abel's theorem on the continuity of power series holds for them—that is to say, the convergence of Σa_n may be *deduced* from the equations

$$\lim na_n = 0, \qquad \lim_{n\to 1} \Sigma a_n x^n = A.$$

The fact that the simplest proof of Abel's theorem on the multiplication of series is derived from his theorem on the continuity of power series suggests that Theorem A might be deduced from Tauber's theorem. But this proves not to be the case, for the equations

$$\lim na_n = 0, \qquad \lim nb_n = 0,$$

do *not* involve

$$\lim nc_n = 0.$$

Suppose, *e.g.*, that

$$a_n = b_n = \frac{(-1)^n}{(n+1)\sqrt{\{\log(n+1)\}}},$$

so that

$$c_n = (-1)^n \sum_1^n \frac{1}{(r+1)(n+2-r)\sqrt{\{\log(r+1)\log(n+2-r)\}}}.$$

It is easy to see that, if n is odd, the value of r which makes $\log(r+1)\log(n+2-r)$ greatest is $r = \frac{1}{2}(n+1)$, so that

$$c_n > \frac{1}{\log\{\frac{1}{2}(n+3)\}} \sum_1^n \frac{1}{(r+1)(n+2-r)} = \frac{2}{(n+3)\log\{\frac{1}{2}(n+3)\}} \sum_2^{n+1} \frac{1}{r} > \frac{K}{n},$$

and nc_n certainly does not tend to zero. In fact this line of argument suffices to prove that C is convergent only when the more stringent conditions

$$n\sqrt{(\log n)}\, a_n \to 0, \qquad n\sqrt{(\log n)}\, b_n \to 0$$

are satisfied.

7. Theorem A may be generalised as follows. It is easy to verify that if $\psi(n)$ is any function of the form

(1) $$(\log n)^\alpha (\log\log n)^\beta (\log\log\log n)^\gamma \dots$$

which tends to infinity with n, then

$$\lim_{n\to\infty} \frac{\psi\left\{\dfrac{n}{\psi(n)}\right\}}{\psi(n)} = 1;$$

we may indeed replace the $\psi(n)$ which occurs inside the curly bracket by any other function of n of the same type as $\psi(n)$.

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* For references see Bromwich, *Infinite Series*, p. 251.

THEOREM B.—*If A and B are convergent, and*

$$n\psi(n)\,a_n \to 0, \qquad \frac{nb_n}{\psi(n)} \to 0,$$

where $\psi(n)$ is any function of n of the form (1), *then C is convergent.*

We have, as in § 1 above,

$$C_n - A_N B_{n+1-N} = A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N}$$
$$+ B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1},$$

and
$$|\,C_n - A_N B_{n+1-N}\,| < K\{(n-N)\,\alpha + (N-1)\,\beta\},$$

where α and β are the greatest of the moduli of $a_{N+1}, a_{N+2}, \ldots, a_n$ and $b_{n+2-N}, b_{n+3-N}, \ldots, b_n$ respectively.

We choose N to be of the same order of greatness as $n/\psi(n)$. Then, given ϵ, we can choose n so that

$$\alpha < \frac{\epsilon}{(N+1)\,\psi(N+1)}, \qquad \beta < \frac{\epsilon\psi(n+2-N)}{n+2-N},$$

and so
$$|\,C_n - A_N B_{n+1-N}\,| < K\epsilon \left\{ \frac{n}{N\psi(N)} + \frac{N\psi(n)}{n} \right\}$$

$$< K\epsilon \left[1 + \frac{\psi(n)}{\psi\left\{\frac{n}{\psi(n)}\right\}} \right] < K\epsilon.*$$

From this the theorem follows. The simplest and most interesting case is that in which

$$n(\log n)^{\alpha}\, a_n \to 0, \qquad \frac{nb_n}{(\log n)^{\alpha}} \to 0,$$

where $0 \leqslant \alpha \leqslant 1$ (if $\alpha > 1$ the first series is absolutely convergent and the result is a mere corollary from Mertens' theorem).

8. Another generalisation of Theorem A, in a somewhat different direction, is the following :—

THEOREM C.—*If A and B are convergent, and*

$$|\,na_n\,| < K, \qquad |\,nb_n\,| < K,$$

for all values of n, then C is convergent.

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* Of course K is not the same constant in all these inequalities.

It is known* that

$$\lim_{n\to\infty} \frac{C_1+C_2+\ldots+C_n}{n} = AB.$$

It is also known† that, if a series Σc_n is such that $(C_1+C_2+\ldots+C_n)/n$ has a limit as $n\to\infty$, then the necessary and sufficient condition for the convergence of the series is

$$\lim \frac{c_1+2c_2+3c_3+\ldots+nc_n}{n} = 0:$$

this indeed follows at once from the identity

$$\frac{c_1+2c_2+3c_3+\ldots+nc_n}{n} = \frac{n+1}{n}C_n - \frac{C_1+C_2+\ldots+C_n}{n}.$$

Let us denote the sums

$$a_1+2a_2+\ldots+na_n, \quad b_1+2b_2+\ldots+nb_n, \quad c_1+2c_2+\ldots+nc_n$$

by $\bar{A}_n$, $\bar{B}_n$, $\bar{C}_n$ respectively. It is easy to verify the identity

$$\begin{aligned}\bar{C}_n+C_n = {} & a_1\bar{B}_n+a_2\bar{B}_{n-1}+\ldots+a_n\bar{B}_1 \\ & +b_1\bar{A}_n+b_2\bar{A}_{n-1}+\ldots+b_n\bar{A}_1.\end{aligned}$$

Also $$C_1+C_2+\ldots+C_n = n(AB+\gamma_n),$$

where $\gamma_n\to 0$ as $n\to\infty$, and so

$$\frac{C_n}{n} = \frac{AB}{n}+\gamma_n-\frac{n-1}{n}\gamma_{n-1}\to 0.$$

It follows that the necessary and sufficient condition for the convergence of C is that

(1) $$(X+Y)/n\to 0,$$

where

(2) $$\begin{cases} X = a_1\bar{B}_n+a_2\bar{B}_{n-1}+\ldots+a_n\bar{B}_1, \\ Y = b_1\bar{A}_n+b_2\bar{A}_{n-1}+\ldots+b_n\bar{A}_1. \end{cases}$$

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* Bromwich, *Infinite Series*, p. 83.

† This result is due to Tauber and Pringsheim. See Bromwich, *Infinite Series*, p. 251, for references.

This condition can be written in a variety of different forms. Thus, applying Abel's lemma to X and Y, we obtain

$$(3)\qquad \begin{cases} X = b_1 A_n + 2b_2 A_{n-1} + \dots + nb_n A_1, \\ Y = a_1 B_n + 2a_2 B_{n-1} + \dots + na_n B_1. \end{cases}$$

Further, if we put $A_n = A + \epsilon_n, \quad B_n = B + \eta_n,$

so that $\epsilon_n \to 0$, $\eta_n \to 0$, we see that

$$X = A\bar{B}_n + b_1 \epsilon_n + 2b_2 \epsilon_{n-1} + \dots + nb_n \epsilon_1,$$

$$Y = B\bar{A}_n + a_1 \eta_n + 2a_2 \eta_{n-1} + \dots + na_n \eta_1.$$

Since $\bar{A}_n/n$, $\bar{B}_n/n$ each tend to zero, we see that the necessary and sufficient condition for the convergence of C is that

$$(4)\qquad (X' + Y')/n \to 0,$$

where

$$(5)\quad X' = b_1 \epsilon_n + 2b_2 \epsilon_{n-1} + \dots + nb_n \epsilon_1, \quad Y' = a_1 \eta_n + 2a_2 \eta_{n-1} + \dots + na_n \eta_1.$$

But, if $|na_n| < K$ and $|nb_n| < K$, it is clear that

$$\left| \frac{X'}{n} \right| < K \frac{|\epsilon_1| + |\epsilon_2| + \dots + |\epsilon_n|}{n} \to 0,$$

and similarly $\qquad |Y'/n| \to 0.$

Hence the theorem is established.

9. The simplest example of the use of this theorem is obtained by applying it to the series

$$\pm \frac{1}{a} \pm \frac{1}{a+b} \pm \frac{1}{a+2b} \pm \dots \pm \frac{1}{a+nb} \pm \dots .$$

We see that *any* two series of this type, whatever be the law of arrangements of the signs, may be multiplied together, provided only they are convergent. A simple example is obtained by squaring the series

$$(1)\qquad 1 - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{7} - \dots,$$

in which the number of terms in each group of signs increases by one at each step. That the series is convergent is easily proved by observing that if we subtract from it the series

$$(1)\qquad 1 - \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{7} - \dots$$

we obtain an absolutely convergent series, and that the series (2) is convergent and equal to

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}(\nu+1)}{\frac{1}{2}\nu(\nu+1)+1}.$$

The corresponding series in which the numbers of terms in the groups are 1^k, 2^k, 3^k, ..., where k is any positive integer, is also convergent. On the other hand, if the numbers are k, k^2, k^3, ..., the series oscillates, behaving very much like the oscillatory series

$$\sum \frac{\cos(a \log n)}{n}, \quad \sum \frac{\sin(a \log n)}{n}.$$

10. It will be convenient to give at this stage the simple proof of some of Pringsheim's results to which I alluded in § 1. The most important case, and the only one which I shall consider here, is that in which

$$a_n = (-1)^{n-1} \alpha_n, \quad b_n = (-1)^{n-1} \beta_n,$$

where α_n and β_n are positive and decreasing. The generalisations of Cajori are rather artificial, and it seems to me worth while to establish the really important results in as simple a way as possible; and Pringsheim's own proofs are far from being the simplest possible.*

Pringsheim's results may be stated thus: *if α_n, β_n tend steadily to zero, we have the following alternative sets of conditions for the multiplication of*

$$\Sigma(-1)^{n-1}\alpha_n, \quad \Sigma(-1)^{n-1}\beta_n,$$

by Cauchy's rule :—

(1) *it is* **necessary and sufficient** *that*

$$\gamma_n = |c_n| = \alpha_1\beta_n + \alpha_2\beta_{n-1} + \ldots + \alpha_n\beta_1 \to 0;$$

(2) *it is* **necessary and sufficient** *that*

$$(\alpha_1+\alpha_2+\ldots+\alpha_n)\beta_n \to 0, \quad (\beta_1+\beta_2+\ldots+\beta_n)\alpha_n \to 0;$$

(3) *it is* **sufficient but not necessary** *that*

$$\Sigma \alpha_n \beta_n$$

should be convergent ;

* A simpler proof of one of them is given by Mr. Bromwich, *Infinite Series*, pp. 86, 87. Even this proof does not seem to me as simple as it may be made.

(4) *it is* **necessary but not sufficient** *that*

$$\Sigma(a_n\beta_n)^{1+s}$$

should be convergent for any positive value of s.

These results may be proved as follows. We observe first that, if

$$A_n = A+(-1)^n\rho_n, \qquad B_n = B+(-1)^n\sigma_n,$$

we have $$0<\rho_n<a_{n+1}, \qquad 0<\sigma_n<\beta_{n+1}.$$

Also $$C_n = a_1B_n+a_2B_{n-1}+\ldots+a_nB_1,$$

$$(-1)^n(C_n-A_nB) = a_1\sigma_n-a_2\sigma_{n-1}+\ldots+(-1)^{n-1}a_n\sigma_1,$$

and so $$|C_n-A_nB|<a_1\beta_{n+1}+a_2\beta_n+\ldots+a_n\beta_2 = \gamma_{n+1}-a_{n+1}\beta_1.$$

From this it follows that the condition $\gamma_n\to 0$ is *sufficient* to ensure $C_n\to AB$, and that the condition is *necessary* is obvious. This establishes Pringsheim's theorem (1).

Again $$\gamma_n = a_1\beta_n+a_2\beta_{n-1}+\ldots+a_n\beta_1>(a_1+\ldots+a_n)\beta_n,$$

and similarly $$\gamma_n>(\beta_1+\ldots+\beta_n)a_n.$$

Hence the conditions (2) are *necessary*.

Also, if $\nu=\frac{1}{2}(n+1)$ or $\frac{1}{2}n$, according as n is odd or even, we have

$$\gamma_n = a_1\beta_n+a_2\beta_{n-1}+\ldots+a_n\beta_1<(a_1+a_2+\ldots+a_\nu)\beta_{n+1-\nu} +(\beta_1+\beta_2+\ldots+\beta_{n-\nu})a_{\nu+1},$$

and from this it follows that the conditions (2) are *sufficient.*

Finally, if $\Sigma a_n\beta_n$ is convergent, we can choose μ_0 so that

$$a_\mu\beta_\mu+a_{\mu+1}\beta_{\mu+1}+\ldots+a_\nu\beta_\nu<\epsilon \quad (\mu_0<\mu<\nu),$$

and, *a fortiori*, $$(a_\mu+a_{\mu+1}+\ldots+a_\nu)\beta_\nu<\epsilon \quad (\mu_0<\mu<\nu).$$

But when μ is fixed we can choose ν_0, so that

$$(a_1+a_2+\ldots+a_{\mu-1})\beta_\nu<\epsilon \qquad (\nu_0<\nu),$$

and so $$(a_1+a_2+\ldots+a_\nu)\beta_\nu<2\epsilon \quad (\nu_0<\nu).$$

Similarly, we can prove that the second of the conditions (2) is satisfied. Hence condition (3) is *sufficient;* that it is not necessary has been shown by Pringsheim by an example.*

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* The example is given by $a_n=\beta_n=\{(n+1)\log(n+1)\}^{-1}$.

Finally, as regards (4), I have nothing to add to Pringsheim's own proof. Since

$$(a_1+a_2+\ldots+a_n)\,\beta_n > na_n\beta_n,$$

the condition

$$na_n\beta_n \to 0$$

is *necessary*. Thus

$$n^{1+s}(a_n\beta_n)^{1+s} \to 0,$$

and so $\Sigma(a_n\beta_n)^{1+s}$ is convergent; *i.e.*, (4) is a *necessary* condition.

11. Theorems A, B, and C, taken in connection with Pringsheim's theorems, suggest questions of some interest to which I am unable at present to give a definite answer.

Let us, for simplicity, consider the special problem of the multiplication of the two series

$$\pm 1^{-s} \pm 2^{-s} \pm 3^{-s} \pm \ldots, \qquad \pm 1^{-t} \pm 2^{-t} \pm 3^{-t} \pm \ldots,$$

where all that is known about the signs of the terms is that they are such as to ensure the convergence of each series.

If $0 < s \leqslant \frac{1}{2}$, $0 < t \leqslant \frac{1}{2}$, or more generally, if s, t and $s+t$ are all positive and not greater than unity, we can certainly choose the signs so that A and B are convergent and C oscillatory. It is enough to take the alternating series $1^{-s}-2^{-s}+\ldots$, $1^{-t}-2^{-t}+\ldots$. The modulus γ_n of the n-th term of the product series is

$$\sum_{r=1}^{n} r^{-s}(n+1-r)^{-t},$$

which tends to infinity with n, if $s+t<1$, and to the finite limit*

$$\int_0^1 \frac{dx}{x^s(1-x)^{1-s}} = \frac{\pi}{\sin s\pi},$$

if $s+t=1$.

On the other hand, if $s=1$, $t=1$, Theorem C shows that the product series is convergent for *all* arrangements of the signs. But the argument by which it was proved does not appear to be capable of extension.

Now let us consider such a case as that in which $s=t=\frac{3}{4}$, or $s=\frac{1}{2}$, $t=1$. Then either (*a*) the product series is always convergent, or (*b*) it is possible to choose the signs so that the product series is oscillatory. My own opinion is that (*b*) is true; *i.e.*, that when $s+t>1$, but at least one

* In connection with the representation of infinite integrals as the limits of finite sums, see a paper by Mr. Bromwich and myself, *Quarterly Journal*, Vol. XXXIX., p. 222.

of s and t is less than 1, we can make A and B convergent and C oscillatory by a proper choice of signs. But I am unable to support this conclusion by an actual example. I wish merely to point out the considerable margin of uncertainty that still remains. In all such cases as these, of course, Pringsheim's results show that the product of the *alternating* series is convergent.

It is easy to see that examples of the kind desired are not likely to be very readily found. For the conditions

$$\sqrt{n}\, a_n \to 0, \qquad \sqrt{n}\, b_n \to 0$$

are sufficient to ensure $$c_n \to 0,\text{*}$$

since $|c''|$ can never be greater than in the alternating case. Moreover, the series Σc_n is *in any case* summable by Cesàro's mean value, *i.e.*,

$$\lim_{n\to\infty} \frac{C_1+C_2+\ldots+C_n}{n}$$

exists. Now series whose n-th term tends to zero, and which are summable, but not convergent, certainly exist—examples are given by the series

$$\Sigma \frac{\sin\sqrt{n}}{\sqrt{n}}, \quad \Sigma \frac{\cos\sqrt{n}}{\sqrt{n}}, \quad \Sigma \frac{(-1)^{[\sqrt{n}]}}{\sqrt{n}}.$$

But such examples are not particularly obvious, much less is it obvious how to construct examples in which the general term is of the form of the general term of the product of two convergent series.

12. I take this opportunity of also stating the following generalisation of Mertens' theorem, which I have not seen before, although it is not strictly relevant to the main purpose of the paper.

If A is absolutely convergent, and B is a finitely oscillating series whose n-th term tends to zero, then C is a finitely oscillating series; and if the limits of oscillation of B are β_1 and β_2, those of C are $A\beta_1$ and $A\beta_2$.

To prove this, we go back to the equation

$$\begin{aligned} C_n - A_N B_{n+1-N} = {} & A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N} \\ & + B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1}. \end{aligned}$$

Let us suppose that A is absolutely convergent, and that $|B_\nu| < K$ for all values of ν.

* It will be remembered (§ 6) that the conditions

$$n\sqrt{(\log n)}\, a_n \to 0, \qquad n\sqrt{(\log n)}\, b_n \to 0$$

ensure $nc_n \to 0$.

First choose N_0 so that

(1) $$|a_{N+1}|+|a_{N+2}|+\ldots<\epsilon/K,$$

for $N \geqslant N_0$. *A fortiori*, we have also

(2) $$|A-A_N|<\epsilon/K.$$

When any value of N greater than N_0 has been determined, we can choose n_0 so that

(3) $$|A_1 b_n+A_2 b_{n-1}+\ldots+A_{N-1} b_{n+2-N}|<\epsilon,$$

for $n \geqslant n_0$. From (1), (2), and (3) it follows that

$$|C_n-A_N B_{n+1-N}|<2\epsilon,$$

$$|C_n-A B_{n+1-N}|<3\epsilon,$$

for $n \geqslant n_0$, which establishes the result. In the particular case in which $\beta_1=\beta_2$, we obtain Mertens' theorem. It should be observed that the theorem is *not* true if the condition $b_n \rightarrow 0$ is removed. Suppose, for example, that $a_n>0$, and form the product of

$$a_1+a_2+a_3+\ldots, \quad 1-1+1-\ldots.$$

We easily see that $$C_{2n}=a_2+a_4+\ldots+a_{2n},$$

$$C_{2n+1}=a_1+a_3+\ldots+a_{2n+1},$$

so that C oscillates, but not between the limits prescribed by the theorem. In particular the product of

$$\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots, \quad 1-1+1-\ldots,$$

converges to the sum 1.

13. I shall conclude by stating the theorems for integrals which are analogous to some of those for series discussed in the preceding pages. But, as these theorems are of much less importance, I shall only outline the proofs.

Suppose that $a(x)$ and $b(x)$ are continuous functions, such that

$$\int_0^\infty a(x)\,dx, \quad \int_0^\infty b(x)\,dx$$

are convergent and have the values A, B. And let

$$c(x)=\int_0^x a(t)\,b(x-t)\,dt=\int_0^x a(x-t)\,b(t)\,dt.$$

$$A(x)=\int_0^x a(t)\,dt, \quad B(x)=\int_0^x b(t)\,dt, \quad C(x)=\int_0^x c(t)\,dt.$$

Then it is easy to prove the formulæ

$$C(x) = \int_0^x A(t)\, b(x-t)\, dt = \int_0^x A(x-t)\, b(t)\, dt$$

$$= \int_0^x a(t) B(x-t)\, dt = \int_0^x a(x-t) B(t)\, dt,$$

$$\int_0^x C(t)\, dt = \int_0^x A(t) B(x-t)\, dt = \int_0^x A(x-t) B(t)\, dt.$$

It is moreover easy to prove that, if $A(x)$ and $B(x)$ tend, as $x \to \infty$, to limits A and B, then

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x A(t)\, B(x-t)\, dt = AB.$$

It follows that:—

(1) *If* $$\int_0^\infty a(x)\, dx = A, \qquad \int_0^\infty b(x)\, dx = B,$$

then $$\lim_{x\to\infty} \frac{1}{x} \int_0^x dt \int_0^t du \int_0^u a(w)\, b(u-w)\, dw = AB.$$

This is the analogue of Cesàro's theorem that

$$(C_1 + C_2 + \ldots + C_n)/n \to AB,$$

whenever A and B are convergent.

From this the analogue of Abel's theorem follows at once; viz.,

(2) *If* $$\int_0^\infty dx \int_0^x a(t)\, b(x-t)\, dt$$

is convergent, its value is AB.

There is no difficulty whatever in establishing the analogues of Cauchy's and Mertens' theorems, viz., that

(3) *If A and B are absolutely convergent, so is C;*

(4) *If A is absolutely and B conditionally convergent, C is (absolutely or conditionally) convergent.*

Corresponding to Theorem A we have

(5) *If A and B are convergent, and $xa(x) \to 0$, $xb(x) \to 0$, as $x \to \infty$, then C is convergent.*

Corresponding to Theorem B, we have

(6) *If* $\phi(x) = (\log x)^{\alpha}(\log_2 x)^{\beta} \ldots (\log_k x)^{\kappa} \to \infty$ *with* x, *and*

$$x\phi(x)\,a(x) \to 0, \qquad \frac{xb(x)}{\phi(x)} \to 0,$$

then C *is convergent.*

Finally, we can show that the necessary and sufficient conditions for the convergence of

$$\int_0^{\infty} f(x)\,dx,$$

are

$$\text{(i.)} \quad \frac{1}{x}\int_0^x dt \int_0^t f(u)\,du \to 0,$$

$$\text{(ii.)} \quad \frac{1}{x}\int_0^x tf(t)\,dt \to 0;$$

and from this we can deduce the analogue of Theorem C, viz.

(7) *If* $|xa(x)| < K$, $|xb(x)| < K$, *then* C *is convergent.*