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LI. *The Reaction upon the Driving-Point of a System executing Forced Harmonic Oscillations of various Periods, with Applications to Electricity.* By Lord RAYLEIGH, D.C.L., F.R.S.*

THE object of the present communication is to prove some general mechanical theorems, which may be regarded as in some sort extensions of that of Thomson relating to the energy of initial motions. The question involved in the latter may be thus stated†:—

“Given any material system at rest. Let any parts of it be set in motion suddenly with any specified velocities possible, according to the connections of the system; and let its other parts be influenced only by its connections with these. It is required to find the motion.” And the solution is “that the motion actually taken by the system is that which has less kinetic energy than any other motion fulfilling the prescribed velocity conditions.” On the other hand, if the impulses are given, a theorem of Bertrand tells us that the kinetic energy is the *greatest* possible.

For our present purpose we suppose the system to be set in motion by an impulse of one particular type, which we may call the first. The impulse itself may be denoted by $\int \Psi_1 dt$, and the corresponding velocity generated by $\dot{\psi}_1$. Under any given circumstances as to constraint, the velocity and the impulse are in proportion to one another; and the

* Communicated by the Author.

† Thomson and Tait's 'Natural Philosophy,' §§ 316, 317.

resulting kinetic energy T is proportional to the square of either, being equal to $\frac{1}{2}\dot{\psi}_1 \int \Psi_1 dt$. Now Thomson's theorem asserts that the introduction of a constraint can only increase the value of T when $\dot{\psi}_1$ is given. Hence, whether $\dot{\psi}_1$ be given or not, the constraint can only increase the ratio of $\frac{1}{2}T$ to $\dot{\psi}_1^2$, or of $\int \Psi_1 dt$ to $\dot{\psi}_1$. This form of the statement virtually includes both Bertrand's and Thomson's theorems, which are thus seen to be merely different aspects of the same truth. If the velocity be given, the impulse is a minimum in the absence of constraint. If the impulse be given, the velocity is under the same circumstances a maximum. Calling the ratio of $\int \Psi_1 dt$ to $\dot{\psi}_1$ the moment of inertia of the system when subjected to forces of the type in question, we may say that this moment can only be increased by the introduction of a constraint forcing the motion to follow a different law from that natural to it.

In close analogy to this theorem there are two others, relating to equilibrium and to steady motion resisted by viscous forces, of at least equal importance*. They may be thus stated.

Conceive a system to be displaced from stable equilibrium by a force of specified type. If the corresponding displacement be given in magnitude, the force is a minimum—or if the magnitude of the force be given, the displacement is a maximum,—when there is no constraint. Or we may say that the *stiffness* of the system, with respect to the kind of force in question, is increased by constraint. Examples, in illustration of the general proposition, are given in the papers already cited.

The third theorem depends upon the properties of the dissipation-function, and its most interesting application is to the conduction of heat and electricity. To take the latter case, if an electromotive force be applied to any system of conductors, the "resistance" to steady currents can only be increased by the imposition of a constraint, such for example as the rupture of a contact.

Hitherto we have supposed the forces to be either instantaneous or steady; and the three theorems depend upon the functions T , F , and V , expressing respectively kinetic energy, dissipation, and potential energy, only one of them being supposed to come into consideration at a time. We have now

* Phil. Mag. Dec. 1874, "A Statical Theorem"; March 1875, "General Theorems relating to Equilibrium and to Initial and Steady Motion." See also 'Theory of Sound,' ch. iv.

to inquire under what conditions the theorems remain intact when the impressed force is a harmonic function of the time.

As regards the first theorem, the justification for the neglect of F and V may be that they are non-existent, as in many problems of ordinary hydrodynamics. In such cases the motion is at any instant of the same character as if it had been generated impulsively from rest, and the moment of inertia is a minimum. But even when F and V are generally sensible, their influence tends to diminish as the frequency of alternation increases, and we approach at last a state of things in which they may be neglected. From this point onwards we may say that the moment of inertia is a minimum in the unconstrained condition. Thus in a system of electrical conductors subject to a rapidly periodic electromotive force, the distribution of currents is ultimately independent of the resistances, and the *self-induction* is a minimum in the absence of constraint.

In like manner, even when T and F are sensible, the motion tends to be more and more determined by V , as the frequency of the vibrations is imagined to *diminish*. An "equilibrium theory" ultimately becomes applicable, and the "stiffness" is a minimum when there are no constraints.

The theorem in which F is mainly concerned stands in a somewhat different position. If T and V are both sensible, we cannot find an extreme case, in respect of the frequency of the vibration, which shall annul their influence. If, however, V vanish, we can make F paramount by taking the period sufficiently long; and if T vanish, we can attain the same object by limiting ourselves to the case when the period is very short. If T and V both vanish, the theorem of minimum resistance in the absence of constraint holds good for all periods of vibration. In the application to a system of electrical conductors which possess resistance and induction, but no capacity for *charge* needing to be regarded, we find that while (as already stated) the induction becomes paramount when the vibrations are very rapid, on the other hand when they are very slow the distribution is determined ultimately by the resistances only. In the first case the self-induction, and in the second the effective resistance, is a minimum in the absence of constraints.

We are now prepared to enter upon the consideration of the problem which is the main subject of the present paper, viz. the behaviour of systems in which F , and one or other of the two remaining functions T and V , are sensible, but without the restriction to very rapid or to very slow motions by

which the influence of the second function may be got rid of. The investigation is almost the same whether it be T or V that enters; for the sake of definiteness I will take the first alternative.

Consider then a system, devoid of potential energy, in which the coordinate ψ_1 is made to vary by the operation of the harmonic force Ψ_1 , proportional to e^{ipt} . The other coordinates ψ_2, ψ_3, \dots may be chosen arbitrarily, and it will be very convenient to choose them (as may always be done) so that no product of them enters into the expressions for T and V. They would be in fact the principal or normal coordinates of the system on the supposition that ψ_1 is constrained (by a suitable force of its own type) to remain zero. The expressions for T and F thus take the following forms:—

$$T = \frac{1}{2}a_{11}\dot{\psi}_1^2 + \frac{1}{2}a_{22}\dot{\psi}_2^2 + \frac{1}{2}a_{33}\dot{\psi}_3^2 + \dots \\ + a_{12}\dot{\psi}_1\dot{\psi}_2 + a_{13}\dot{\psi}_1\dot{\psi}_3 + a_{14}\dot{\psi}_1\dot{\psi}_4 + \dots \quad (1)$$

$$F = \frac{1}{2}b_{11}\dot{\psi}_1^2 + \frac{1}{2}b_{22}\dot{\psi}_2^2 + \frac{1}{2}b_{33}\dot{\psi}_3^2 + \dots \\ + b_{12}\dot{\psi}_1\dot{\psi}_2 + b_{13}\dot{\psi}_1\dot{\psi}_3 + b_{14}\dot{\psi}_1\dot{\psi}_4 + \dots \quad (2)$$

from which we get the equations of motion

$$a_{11}\ddot{\psi}_1 + a_{12}\ddot{\psi}_2 + a_{13}\ddot{\psi}_3 + \dots + b_{11}\dot{\psi}_1 + b_{12}\dot{\psi}_2 + \dots = \Psi_1,$$

$$a_{12}\ddot{\psi}_1 + a_{22}\ddot{\psi}_2 + b_{12}\dot{\psi}_1 + b_{22}\dot{\psi}_2 = 0,$$

$$a_{13}\ddot{\psi}_1 + a_{33}\ddot{\psi}_3 + b_{13}\dot{\psi}_1 + b_{33}\dot{\psi}_3 = 0,$$

$$\dots \dots \dots$$

since there are no forces other than Ψ_1 . We now introduce the supposition that the whole motion is harmonic in response to Ψ_1 . Thus the above equations may be replaced by

$$(ip a_{11} + b_{11})\dot{\psi}_1 + (ip a_{12} + b_{12})\dot{\psi}_2 + (ip a_{13} + b_{13})\dot{\psi}_3 + \dots = \Psi_1,$$

$$(ip a_{12} + b_{12})\dot{\psi}_1 + (ip a_{22} + b_{22})\dot{\psi}_2 = 0,$$

$$(ip a_{13} + b_{13})\dot{\psi}_1 + (ip a_{33} + b_{33})\dot{\psi}_3 = 0.$$

$$\dots \dots \dots$$

By means of the second and following equations, $\dot{\psi}_2, \dot{\psi}_3, \dots$ are expressed in terms of $\dot{\psi}_1$. Introducing these values into the first, we get

$$\frac{\Psi_1}{\dot{\psi}_1} = ip a_{11} + b_{11} - \frac{(ip a_{12} + b_{12})^2}{ip a_{22} + b_{22}} - \frac{(ip a_{13} + b_{13})^2}{ip a_{33} + b_{33}} - \dots \quad (3)$$

The ratio $\Psi_1 : \dot{\psi}_1$ is a complex quantity, of which the real part corresponds to the work done by the force in a complete period, and dissipated in the system. By an extension of

electrical language we may call it the *resistance* of the system and denote it by the letter R' . The other part of the ratio is imaginary. If we denote it by $ipL'\dot{\psi}_1$, or $L'\dot{\psi}_1$, L' will be the moment of inertia, or self-induction of electrical theory. We write therefore

$$\Psi_1 = (R' + ipL')\dot{\psi}_1; \quad . \quad . \quad . \quad (4)$$

and the values of R' and L' are to be deduced by separation of the real and imaginary parts of the right-hand member of (3).

$$\begin{aligned} \text{Now the real part of } \frac{(ip a_{12} + b_{12})^2}{ip a_{22} + b_{22}} \\ &= \frac{b_{12}^2 b_{22} + p^2 a_{12} (2 b_{12} a_{22} - a_{12} b_{22})}{b_{22}^2 + p^2 a_{22}^2} \\ &= \frac{b_{12}^2}{b_{22}} - \frac{p^2 (a_{12} b_{22} - a_{22} b_{12})^2}{b_{22} (b_{22}^2 + p^2 a_{22}^2)}; \quad . \quad . \quad . \quad (5) \end{aligned}$$

so that

$$R' = b_{11} - \Sigma \frac{b_{12}^2}{b_{22}} + p^2 \Sigma \frac{(a_{12} b_{22} - a_{22} b_{12})^2}{b_{22} (b_{22}^2 + p^2 a_{22}^2)}. \quad . \quad . \quad (6)$$

This is the value of the resistance as determined by the constitution of the system, and by the frequency of the imposed vibration. Each component of the latter series (which alone involves p) is of the form $\alpha p^2 / (\beta + \gamma p^2)$, where α , β , γ are all positive, and (as may be seen most easily by considering its reciprocal) increases continuously as p^2 increases from zero to infinity. We conclude that as the frequency of vibration increases, the value of R' increases continuously with it. At the lower limit the motion is determined sensibly by the quantities b (the resistances) only, and the corresponding resultant resistance R' is an absolute minimum, whose value is

$$b_{11} - \Sigma \frac{b_{12}^2}{b_{22}}. \quad . \quad . \quad . \quad (7)$$

At the upper limit the motion is determined by the inertia of the component parts without regard to resistances, and the value of R' is

$$b_{11} - \Sigma \frac{b_{12}^2}{b_{22}} + \Sigma \frac{(a_{12} b_{22} - a_{22} b_{12})^2}{b_{22} a_{22}^2}. \quad . \quad . \quad . \quad (8)$$

That the resistance in this case would exceed that expressed by (7) might have been anticipated from the analogue of Thomson's theorem; but we now learn in addition that at every stage of the transition, during which in general the motions of the various parts disagree in phase, every incre-

ment of frequency of vibration is accompanied by a corresponding increment of resistance.

$$\begin{aligned} \text{Again, the imaginary part of } & \frac{(ip a_{12} + b_{12})^2}{ip a_{22} + b_{22}} \\ &= ip \frac{b_{12}(2a_{12}b_{22} - a_{22}b_{12}) + p^2 a_{22}a_{12}^2}{b_{22}^2 + p^2 a_{22}^2} \\ &= ip \frac{a_{12}^2}{a_{22}} - ip \frac{(a_{12}b_{22} - a_{22}b_{12})^2}{a_{22}(b_{22}^2 + p^2 a_{22}^2)}, \quad \cdot \cdot \cdot \cdot \cdot \quad (9) \end{aligned}$$

so that

$$L' = a_{11} - \sum \frac{a_{12}^2}{a_{22}} + \sum \frac{(a_{12}b_{22} - a_{22}b_{12})^2}{a_{22}(b_{22}^2 + p^2 a_{22}^2)}. \quad \cdot \cdot \cdot \cdot \quad (10)$$

In the latter series each term is positive, and continually diminishes as p^2 increases. Hence every increase of frequency is attended by a diminution of the moment of inertia, which tends ultimately to the minimum corresponding to disappearance of the dissipative terms.

Certain very particular cases in which R' and L' remain constant do not require more than a passing allusion. If T and F are of the same form, every such quantity as $(a_{12}b_{22} - a_{22}b_{12})^2$ vanishes.

As examples of the general theorem may be mentioned the problems considered by Prof. Stokes in his well-known paper upon "The Effect of the Internal Friction of Fluids on the Motion of Pendulums"*. Consider, for instance, the result for a sphere of radius a , vibrating (according to e^{ipt}) in a fluid for which the kinematic coefficient of viscosity is μ' . M' denoting the mass of the fluid displaced by the sphere, Prof. Stokes's results may be written

$$\begin{aligned} R' &= \frac{9M'}{4a} \left\{ \frac{2\mu'}{a} + \sqrt{(2\mu'p)} \right\}, \\ L' &= M' \left\{ \frac{1}{2} + \frac{9\sqrt{(2\mu')}}{4a\sqrt{p}} \right\}. \end{aligned}$$

When p is zero, which represents uniform motion of the sphere,

$$R' = \frac{9\mu'M'}{4a^2}, \quad L' = \infty \dagger.$$

As p increases, the expressions show that, in agreement with the theorem, R' continually increases and L' continually diminishes. In fact R' tends to become infinite, and L' to

* Camb. Trans. vol. ix., 1850.

† That the energy of the motion is infinite in this case does not appear to have been noticed.

assume the value ($\frac{1}{2}M'$) given by ordinary hydrodynamics, in which viscosity is not regarded.

The use of the principal coordinates would not often be advantageous when the object is a special calculation of L' and R' , rather than the establishment of a general theorem. In one very important case—that of two degrees of freedom only—the question does not arise, since but one other coordinate ψ_2 enters in addition to ψ_1 . Under this head we may take the problem of the reaction upon the primary circuit of the electric currents induced in a neighbouring secondary circuit. In this case the coordinates (or rather their rates of increase) are naturally taken to be the currents themselves, so that $\dot{\psi}_1$ is the primary, and $\dot{\psi}_2$ the secondary current.

In usual electrical notation we represent the coefficients of self and mutual induction by L , N , M , so that

$$T = \frac{1}{2} L \dot{\psi}^2 + M \dot{\psi}_1 \dot{\psi}_2 + \frac{1}{2} N \dot{\psi}_2^2,$$

and the resistances by R and S . Thus

$$a_{11} = L, \quad a_{12} = M, \quad a_{22} = N;$$

$$b_{11} = R, \quad b_{12} = 0, \quad b_{22} = S;$$

and (6) and (10) become at once

$$R' = R + \frac{p^2 M^2 S}{S^2 + p^2 N^2}, \quad \dots \dots \dots (11)$$

$$L' = L - \frac{p^2 M^2 N}{S^2 + p^2 N^2}. \quad \dots \dots \dots (12)$$

These formulæ were given long ago by Maxwell*, who remarks that the reaction of the currents in the secondary has the effect of increasing the effective resistance and diminishing the effective self-induction of the primary circuit.

If the rate of alternation be very slow, the secondary circuit is without influence. If, on the other hand, the rate be very rapid,

$$R' = R + \frac{M^2 S}{N^2}, \quad L' = \frac{LN - M^2}{N}.$$

The formulæ (11) and (12) may be applied to deal with a more general problem of considerable interest, which arises when the secondary circuit acts upon a third, this upon a fourth, and so on, the only condition being that there must be no mutual induction except between immediate neighbours in

* Phil. Trans. 1865; M is misprinted for M^2 .

the series. Thus $a_{13}, a_{14}, a_{24}, \dots$ (or, as we should here call them, $M_{13}, M_{14}, M_{24}, \dots$) are supposed to vanish, as would usually happen in experiment. For the sake of distinctness we will limit ourselves to four circuits.

In the fourth circuit the current is due *ex hypothesi* only to induction from the third. Its reaction upon the third, for the rate of alternation under contemplation, is given at once by (11) and (12); and if we use the complete values applicable to the third circuit under these conditions, we may thenceforth ignore the fourth circuit. In like manner we can now deduce the reaction upon the secondary, giving the effective resistance and self-induction of that circuit under the influence of the third and fourth circuits; and then, by another step of the same kind, we may arrive at the values applicable to the primary circuit, under the influence of all the others. The process is evidently general; and we know by the theorem that, however numerous the train of circuits, the influence of the others upon the first must be to increase its effective resistance and diminish its effective inertia, in greater and greater degree as the rapidity of alternations increases.

In the limit, when the rapidity of alternation increases indefinitely, the distribution of currents is determined by the induction-coefficients irrespective of resistance, and it is of such a character that the currents are alternately opposite in sign as we pass along the series*.

As another example under the head of two degrees of freedom, we will take the case of two electrical conductors in parallel. It is not necessary to include the influence of the leads outside the points of bifurcation. Provided there be no mutual induction between these parts and the remainder, their induction and resistance enter into the result by simple addition.

Under the operation of resistance only, the total current $\dot{\psi}_1$ would divide itself between the conductors R and S in the parts

$$\frac{S\dot{\psi}_1}{R+S}, \text{ and } \frac{R\dot{\psi}_1}{R+S}.$$

We may conveniently take the second coordinate $\dot{\psi}_2$ so that the currents in the two conductors are

$$\frac{S}{R+S} \dot{\psi}_1 + \dot{\psi}_2, \text{ and } \frac{R}{R+S} \dot{\psi}_1 - \dot{\psi}_2,$$

$\dot{\psi}_1$ still representing the total current.

* See a paper, "On some Electromagnetic Phenomena considered in connection with the Dynamical Theory," Phil. Mag. July 1869.

Thus,

$$F = \frac{1}{2}R \left(\frac{S}{R+S} \dot{\psi}_1 + \dot{\psi}_2 \right)^2 + \frac{1}{2}S \left(\frac{R}{R+S} \dot{\psi}_1 - \dot{\psi}_2 \right)^2 \\ = \frac{1}{2}\dot{\psi}_1^2 \frac{SR}{R+S} + \frac{1}{2}\dot{\psi}_2^2 (R+S);$$

and if L, M, N be the induction-coefficients of the two conductors,

$$T = \frac{1}{2}\dot{\psi}_1^2 \frac{LS^2 + 2MSR + NR^2}{(R+S)^2} \\ + \dot{\psi}_1 \dot{\psi}_2 \frac{(L-M)S + (M-N)R}{R+S} + \frac{1}{2}\dot{\psi}_2^2 (L-2M+N).$$

Accordingly,

$$a_{11} = \frac{LS^2 + 2MSR + NR^2}{(R+S)^2}, \quad a_{12} = \frac{(L-M)S + (M-N)R}{R+S}, \\ a_{22} = L - 2M + N;$$

$$b_{11} = \frac{SR}{R+S}, \quad b_{12} = 0, \quad b_{22} = R+S;$$

and thus by (6), (10),

$$R' = \frac{SR}{R+S} + \frac{p^2}{R+S} \frac{\{(L-M)S + (M-N)R\}^2}{(R+S)^2 + p^2(L-2M+N)^2}, \quad (13)$$

$$L' = \frac{LS^2 + 2MSR + NR^2}{(R+S)^2} - \frac{\{(L-M)S + (M-N)R\}^2}{(R+S)^2(L-2M+N)} \\ + \frac{\{(L-M)S + (M-N)R\}^2}{(L-2M+N)\{(R+S)^2 + p^2(L-2M+N)^2\}}. \quad (14).$$

It should be remarked that $(L-2M+N)$ is necessarily positive, representing twice the kinetic energy of the system when the current in the first conductor is $+1$ and in the second -1 .

Of the three terms in (14) the second and third cancel one another when p vanishes, and when p is very great the third term tends to disappear. The first and second terms together may be put into the form

$$\frac{LN - M^2}{L - 2M + N}, \quad \dots \dots \dots (15)$$

independent (as it should be) of the resistances. In this $(LN - M^2)$ is necessarily positive, but may be relatively small when the wires are wound together. The energy of the system is then very small, when the currents are so rapid that their distribution is determined by induction.

There is an interesting distinction to be noted here dependent upon the manner in which the connections are made. Consider, for example, the case of a bundle of five contiguous wires wound into a coil, of which three wires connected in series (so as to give maximum self-induction) constitute one of the branches in parallel, and the other two, connected similarly in series, constitute the other branch. There is still an alternative in respect to the manner of connection of the two branches. If steady currents would circulate opposite ways (M negative), the total current is divided into two parts in the ratio of 3 : 2, in such a manner that the more powerful current in the double wire nearly neutralizes at external points the magnetic effects of the less powerful current in the triple wire, and the total energy of the system is very small. But now suppose that the connections are such that steady currents would pass the same way round in both branches (M positive). It is evident that the condition of minimum energy cannot be satisfied if the currents are in the same direction, but requires that the smaller current in the triple wire should be in the opposite direction to the larger current in the double wire. In fact the ratio of currents must be 3 : -2; so that (as on the same scale the total current is 1) the component currents in the branches are both numerically greater than the total current which is divided between them. And this peculiar feature becomes more and more strongly marked the nearer L and N approach to equality*.

When there are several conductors in parallel, the results would in general be very complicated. When, however, there is no mutual induction between the various members, a simplification occurs. If the currents be denoted by $\dot{\psi}_1, \dot{\psi}_2, \dot{\psi}_3 \dots$, the difference of potentials at the common terminals is

$$E = (ipL_1 + R_1)\dot{\psi}_1 = (ipL_2 + R_2)\dot{\psi}_2 = \dots,$$

so that

$$\frac{E}{\dot{\psi}_1 + \dot{\psi}_2 + \dots} = \frac{1}{\Sigma(ipL + R)^{-1}}.$$

But if R' and L' be the effective resistance and self-induction respectively of the combination,

$$\frac{E}{\dot{\psi}_1 + \dot{\psi}_2 + \dots} = R' + ipL',$$

* The reader who is interested in this subject is referred to my papers in the *Phil. Mag.* July 1869, June 1870, "On Some Electromagnetic Phenomena," &c.

so that

$$\frac{1}{R' + ipL'} = \Sigma \frac{1}{R + ipL} \cdot \cdot \cdot \cdot \cdot \quad (16)$$

Now

$$\Sigma \frac{1}{R + ipL} = \Sigma \frac{R - ipL}{R^2 + p^2 L^2};$$

or, if we write

$$\Sigma \frac{R}{R^2 + p^2 L^2} = A, \quad \Sigma \frac{L}{R^2 + p^2 L^2} = B, \quad \cdot \cdot \cdot \quad (17)$$

$$\Sigma \frac{1}{R + ipL} = A - ipB = \frac{1}{(A + ipB)/(A^2 + p^2 B^2)}.$$

Hence

$$R' = \frac{A}{A^2 + p^2 B^2}, \quad L' = \frac{B}{A^2 + p^2 B^2} \cdot \cdot \cdot \quad (18)$$

Equations (17) and (18) contain the solution of the problem. When $p=0$,

$$R' = \frac{1}{\Sigma(R^{-1})}, \quad L' = \frac{\Sigma(LR^{-2})}{(\Sigma R^{-1})^2} \cdot \cdot \cdot \quad (19)$$

When $p=\infty$,

$$R' = \frac{\Sigma(RL^{-2})}{(\Sigma L^{-1})^2}, \quad L' = \frac{1}{\Sigma(L^{-1})} \cdot \cdot \cdot \quad (20)$$

These examples will suffice.

The relation between Ψ_1 and $\dot{\psi}_1$ expressed in (4) may be exhibited in another way in terms of the phase difference (ϵ) and the ratio of maxima. Thus if

$$\Psi_1 = P e^{i\epsilon} \dot{\psi}_1,$$

we have

$$P = \sqrt{(R'^2 + p^2 L'^2)}, \quad \tan \epsilon = \frac{pL'}{R'} \cdot \cdot \cdot \quad (21)$$

As p increases from 0 to ∞ , ϵ usually ranges from 0 to $\frac{1}{2}\pi$. At first sight it might appear probable that every increment of p would involve an increment of ϵ , but this seems not to be generally true. For consider a case in which

$$a_{11}=0, \quad a_{12}=0, \quad a_{13}=0, \dots$$

so that by (10)

$$pL' = \Sigma \frac{a_{22}b_{12}^2 p}{b_{22}^2 + p^2 a_{22}^2}.$$

Here pL' begins (as usual) at zero and ends at zero. During part of the range, therefore, it falls; and thus since R' rises throughout, it follows that ϵ does not rise throughout.

It may be worth while to remark that in some cases, where we cannot deal with phases, we are concerned principally with the value of $\sqrt{(R'^2 + p^2 L'^2)}$, a quantity which practical electricians are then tempted to call the resistance of the system. This temptation should be overcome, and the name reserved for R' , on which depends the amount of energy dissipated. It must be admitted, however, that a name for $\sqrt{(R'^2 + p^2 L'^2)}$ is badly required. Perhaps it might be called the "throttling."

The corresponding theorem in cases when T vanishes is deduced in a similar manner with use of the potential energy,

$$V = \frac{1}{2} c_{11} \psi_1^2 + \frac{1}{2} c_{22} \psi_2^2 + \frac{1}{2} c_{33} \psi_3^2 + \dots \\ + c_{12} \psi_1 \psi_2 + c_{13} \psi_1 \psi_3 + c_{14} \psi_1 \psi_4 + \dots$$

Thus, if we write

$$\Psi_1 = \mu' \psi_1 + R' \frac{d\psi_1}{dt} \dots \dots \dots (22)$$

we find

$$\mu' = c_{11} - \Sigma \frac{c_{12}^2}{c_{22}} + p^2 \Sigma \frac{(b_{12} c_{22} - b_{22} c_{12})^2}{c_{22} (c_{22}^2 + p^2 b_{22}^2)}, \dots \dots (23)$$

$$R' = b_{11} - \Sigma \frac{b_{12}^2}{b_{22}} + \Sigma \frac{(b_{12} c_{22} - b_{22} c_{12})^2}{b_{22} (c_{22}^2 + p^2 b_{22}^2)}. \dots \dots (24)$$

As p^2 increases, the "stiffness" (represented by μ') increases, and the "resistance" diminishes.

After what has been said it will not be necessary to occupy space with illustrations of the present theorem. Indeed its applications seem to afford less interest. It is curious that here, again, the easiest examples would be taken from electricity, although the principle itself is one of general mechanics. These (relating to the periodic charge and discharge of condensers through high resistances) may be left to the reader who wishes to pursue the subject further. The application to the theory of the conduction of heat may also be noticed.

When the three functions T , F , and V are all sensible, it is not generally possible to make the transformation to sums of squares upon which our process was founded. There are, however, special cases in which the same transformation which is required to simplify T and V is successful also as regards F . Among these are of course to be reckoned cases in which F does not appear, and those where there is but one other coordinate besides ψ_1 . Assuming that b_{23} , b_{34} , \dots vanish, we have

$$\frac{\Psi_1}{\psi_1} = c_{11} - p^2 a_{11} + ip b_{11} - \frac{(c_{12} - p^2 a_{12} + ip b_{12})^2}{c_{22} - p^2 a_{22} + ip b_{22}} - \dots \dots (25)$$

If we put

$$\Psi_1 = \mu' \psi_1 + R' \frac{d\psi_1}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

we obtain the values of μ' and R' by writing in (23) and (24) throughout $c_{11} - p^2 a_{11}$, $c_{22} - p^2 a_{22}$, $c_{12} - p^2 a_{12}$, . . . for c_{11} , c_{22} , c_{12} , . . . respectively.

A simpler case, which may be worth special mention, arises when all the coefficients b_{12} , b_{13} , . . . vanish. We have then

$$\mu' = c_{11} - p^2 a_{11} - \Sigma \frac{(c_{12} - p^2 a_{12})^2 (c_{22} - p^2 a_{22})}{(c_{22} - p^2 a_{22})^2 + p^2 b_{22}^2}, \quad . \quad . \quad (27)$$

$$R' = b_{11} + \Sigma \frac{(c_{12} - p^2 a_{12})^2 b_{22}}{(c_{22} - p^2 a_{22})^2 + p^2 b_{22}^2}. \quad . \quad . \quad . \quad . \quad (28)$$

This case, when there are two degrees of freedom, is considered in my book on the 'Theory of Sound,' § 117.

If all the frictional coefficients b_{22} , b_{33} , . . . disappear, we have

$$R' = b_{11}, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and

$$\mu' = c_{11} - p^2 a_{11} - \Sigma \frac{(c_{12} - p^2 a_{12})^2}{c_{22} - p^2 a_{22}}. \quad . \quad . \quad . \quad . \quad (30)$$

Whenever, during its increase, p approaches and passes through one of the values proper to the free vibrations of the system supposed to be vibrating under the condition that ψ_1 is constrained by a suitable force Ψ_1 to remain zero, μ' rises to $-\infty$ and passes through to $+\infty$.

LII. *On the Self-induction and Resistance of Straight Conductors.* By Lord RAYLEIGH, Sec. R.S., D.C.L.*

IN connection with the experimental results of Professor Hughes†, I have recently been led to examine more minutely the chapter in Maxwell's 'Electricity and Magnetism' (vol. ii. ch. xiii.) in which the author calculates the self-induction of cylindrical conductors of finite section. The problems being virtually in two dimensions, the results give the ratio $L : l$, where L is the coefficient of self-induction, and l the length considered. And since both these quantities are linear, the ratio is purely numerical. In some details the formulæ, as given by Maxwell, require correction, and in some directions the method used by him may usefully be

* Communicated by the Author.

† Inaugural Address to the Society of Telegraph Engineers, January 1886.