C. $f(\theta) = \frac{3^3 \cdot 5^3}{2^4}$; 6 roots, viz.—

$$\begin{aligned} 16\theta^2 - 16\theta + 1 &= 0, \quad \text{in (2)}; \\ \theta^2 + 14\theta + 1 &= 0, \quad \text{in (3)}; \\ \theta^2 - 16\theta + 16 &= 0, \quad \text{in (6)}. \end{aligned}$$

D. $f(\theta) = -128$; 12 roots, viz.—

$$(\theta^2 - \theta + 1)^3 + 128\theta^2 (\theta - 1)^2 = 0, \text{ in (4) and (5)}.$$

E. $f(\theta) = \frac{5^3}{2^2}$; 12 roots, viz.—

$$\begin{aligned} (4\theta^2 - 4\theta - 1)^2 &= 0, \quad \text{in (2)}; \\ (\theta^2 - 6\theta + 1)^2 &= 0, \quad \text{in (3)}; \\ (\theta^2 + 4\theta - 4)^2 &= 0, \quad \text{in (6)}. \end{aligned}$$

Hence the roots of $F(y, y) = 0$ are $0, \frac{3^3 \cdot 5^3}{2^4}$, each once; ∞, -2^7, $\frac{5^3}{2^2}$, each twice; *i.e.*—

$$F(y, y) = -2^{16} y \left(y - \frac{3^3 \cdot 5^3}{2^4}\right) \left(y - \frac{5^3}{2^2}\right)^2 (y + 2^7)^2.$$

The multiplicity of the roots of $F(y, y) = 0$ may be otherwise, and more simply, determined as follows. The form of $F(x, y)$ (see the equation A, *supra*) shows that $F(y, y)$ has one root equal to zero and two roots equal to infinity; the multiplicities of the finite roots are determined by the equation

$$2^{16} \left[\frac{3^3 \cdot 5^3}{2^4} + 2 \times \frac{5^3}{2^2} - 2 \times 2^7\right] = 2A_{3,2} = 2^{12} \cdot 3^2 \cdot 31.$$

Note on the Formula for the Multiplication of Four Theta Functions. By HENRY J. STEPHEN SMITH, Savilian Professor of Geometry in the University of Oxford.

[*Read February 13th,* 1879.]

The normal formula for the multiplication of four Theta Functions is ("Proceedings of the London Mathematical Society," Vol. I., Part viii., p. 4)

$$(1) \ldots \left\{ \begin{aligned} & 2\theta_{\mu_1, \mu_1'}(x_1)\, \theta_{\mu_2, \mu_2'}(x_2)\, \theta_{\mu_3, \mu_3'}(x_3)\, \theta_{\mu_4, \mu_4'}(x_4) \\ & = \prod_{j=1}^{j=4} \theta_{\sigma - \mu_j, \sigma' - \mu'_j}(s - x_j) + \prod_{j=1}^{j=4} \theta_{\sigma - \mu_j, \sigma' - \mu'_j + 1}(s - x_j) \\ & + (-1)^{\sigma'} \prod_{j=1}^{j=4} \theta_{\sigma - \mu_j + 1, \sigma' - \mu'_j}(s - x_j) + (-1)^{\sigma' + 1} \prod_{j=1}^{j=4} \theta_{\sigma - \mu_j + 1, \sigma' - \mu'_j + 1}(s - x_j). \end{aligned} \right.$$

In this formula we have

$$(2) \ldots\ldots \begin{cases} \theta_{\mu,\mu'}(x) = \sum\limits_{m=-\infty}^{m=+\infty} (-1)^{m\mu'} q^{\frac{1}{4}(2m+\mu)^2} e^{i(2m+\mu)x} \\ = \sum\limits_{m=-\infty}^{m=+\infty} (-1)^{m\mu'} \epsilon^{\frac{1}{4}i\pi\omega(2m+\mu)^2} e^{i(2m+\mu)x}; \end{cases}$$

$$(3) \ldots\ldots \begin{cases} 2s = x_1 + x_2 + x_3 + x_4, \\ 2\sigma = \mu_1 + \mu_2 + \mu_3 + \mu_4, \\ 2\sigma' = \mu_1' + \mu_2' + \mu_3' + \mu_4'; \end{cases}$$

$q = \epsilon^{i\pi\omega}$ being a constant of which the analytical modulus is inferior to unity, and the indices μ, μ' being integral numbers, which render the sums 2σ, $2\sigma'$ even.

I. *The Eleven Cases of the Formula.*

Since
$$(4) \ldots\ldots \begin{cases} \theta_{\mu+2,\mu'}(x) = (-1)^{\mu'} \theta_{\mu,\mu'}(x), \\ \theta_{\mu,\mu'+2}(x) = \theta_{\mu,\mu'}(x), \end{cases}$$

we need only attribute the values 1, 0 to the indices μ, μ'. We may also permute the four arguments x_1, x_2, x_3, x_4 in any way we please; thus the matrix $\begin{vmatrix} \mu_1, \mu_2, \mu_3, \mu_4 \\ \mu_1', \mu_2', \mu_3', \mu_4' \end{vmatrix}$ may have eleven different values, which are enumerated in the following table:—

Table I.

A. $\sigma \equiv 0$, $\sigma' \equiv 0$, mod. 2: Cases i.—iv.

$$\begin{vmatrix} 1\,1\,1\,1 \\ 1\,1\,1\,1 \end{vmatrix}, \quad \begin{vmatrix} 1\,1\,1\,1 \\ 0\,0\,0\,0 \end{vmatrix}, \quad \begin{vmatrix} 0\,0\,0\,0 \\ 1\,1\,1\,1 \end{vmatrix}, \quad \begin{vmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{vmatrix}.$$

B. $\sigma \equiv 0$, $\sigma' \equiv 1$, mod. 2: Cases v., vi.

$$\begin{vmatrix} 1\,1\,1\,1 \\ 1\,1\,0\,0 \end{vmatrix}, \quad \begin{vmatrix} 0\,0\,0\,0 \\ 1\,1\,0\,0 \end{vmatrix}.$$

C. $\sigma \equiv 1$, $\sigma' \equiv 0$, mod. 2: Cases vii., viii.

$$\begin{vmatrix} 1\,1\,0\,0 \\ 1\,1\,1\,1 \end{vmatrix}, \quad \begin{vmatrix} 1\,1\,0\,0 \\ 0\,0\,0\,0 \end{vmatrix}.$$

D. $\sigma \equiv 1$, $\sigma' \equiv 1$, mod. 2: Cases ix.—xi.

$$\begin{vmatrix} 1\,1\,0\,0 \\ 1\,1\,0\,0 \end{vmatrix}, \quad \begin{vmatrix} 1\,1\,0\,0 \\ 1\,0\,1\,0 \end{vmatrix}, \quad \begin{vmatrix} 1\,1\,0\,0 \\ 0\,0\,1\,1 \end{vmatrix}.$$

The formulæ appertaining to these eleven cases are all different from one another; *i.e.*, none of them can be derived from any other by a permutation of the four arguments x; we can, however, pass from any one of them to any other by means of the formula which connects any two different Theta functions; viz.,

$$(5) \ldots\ldots \theta_{\mu+\nu,\mu'+\nu'}(x) = \epsilon^{i\pi\left[\nu\frac{x}{\pi} + \nu^2\frac{\omega}{4} - \frac{\mu\nu}{2}\right]} \times \theta_{\mu,\mu'}\left[x + \frac{\pi}{2}(\nu\omega + \nu')\right].$$

Using a notation with a single suffix, and writing

$$\theta_{0,1}(x) = \vartheta_0(x), \quad \theta_{1,1}(x) = \vartheta_1(x),$$

$$\theta_{1,0}(x) = \vartheta_2(x), \quad \theta_{0,0}(x) = \vartheta_3(x),$$

we may exhibit the eleven formulæ as follows.

TABLE II. *Formulæ for the Multiplication of Four Theta Functions.*

A.

(i.) $$2\vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1 = \vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1 + \vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2 + \vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0 - \vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3.$$

(ii.) $$2\vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2 = \vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2 + \vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1 + \vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3 - \vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0.$$

(iii.) $$2\vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0 = \vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0 + \vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3 + \vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1 - \vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2.$$

(iv.) $$2\vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_3\times\vartheta_3 + \vartheta_0\times\vartheta_0\times\vartheta_0\times\vartheta_0 + \vartheta_2\times\vartheta_2\times\vartheta_2\times\vartheta_2 - \vartheta_1\times\vartheta_1\times\vartheta_1\times\vartheta_1.$$

B.

(v.) $$2\vartheta_1\times\vartheta_1\times\vartheta_2\times\vartheta_2 = \vartheta_2\times\vartheta_2\times\vartheta_1\times\vartheta_1 + \vartheta_1\times\vartheta_1\times\vartheta_2\times\vartheta_2 - \vartheta_3\times\vartheta_3\times\vartheta_0\times\vartheta_0 + \vartheta_0\times\vartheta_0\times\vartheta_3\times\vartheta_3.$$

(vi.) $$2\vartheta_0\times\vartheta_0\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_0\times\vartheta_0 + \vartheta_0\times\vartheta_0\times\vartheta_3\times\vartheta_3 - \vartheta_2\times\vartheta_2\times\vartheta_1\times\vartheta_1 + \vartheta_1\times\vartheta_1\times\vartheta_2\times\vartheta_2.$$

C.

(vii.) $$2\vartheta_1\times\vartheta_1\times\vartheta_0\times\vartheta_0 = \vartheta_0\times\vartheta_0\times\vartheta_1\times\vartheta_1 + \vartheta_3\times\vartheta_3\times\vartheta_2\times\vartheta_2 + \vartheta_1\times\vartheta_1\times\vartheta_0\times\vartheta_0 - \vartheta_2\times\vartheta_2\times\vartheta_3\times\vartheta_3.$$

(viii.) $$2\vartheta_2\times\vartheta_2\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_2\times\vartheta_2 + \vartheta_0\times\vartheta_0\times\vartheta_1\times\vartheta_1 + \vartheta_2\times\vartheta_2\times\vartheta_3\times\vartheta_3 - \vartheta_1\times\vartheta_1\times\vartheta_0\times\vartheta_0.$$

D.

(ix.) $$2\vartheta_1\times\vartheta_1\times\vartheta_3\times\vartheta_3 = \vartheta_3\times\vartheta_3\times\vartheta_1\times\vartheta_1+\vartheta_0\times\vartheta_0\times\vartheta_2\times\vartheta_2 - \vartheta_2\times\vartheta_2\times\vartheta_0\times\vartheta_0+\vartheta_1\times\vartheta_1\times\vartheta_3\times\vartheta_3.$$

(x.) $$2\vartheta_1\times\vartheta_2\times\vartheta_0\times\vartheta_3 = \vartheta_3\times\vartheta_0\times\vartheta_2\times\vartheta_1+\vartheta_0\times\vartheta_3\times\vartheta_1\times\vartheta_2 + \vartheta_2\times\vartheta_1\times\vartheta_3\times\vartheta_0-\vartheta_1\times\vartheta_2\times\vartheta_0\times\vartheta_3.$$

(xi.) $$2\vartheta_2\times\vartheta_2\times\vartheta_0\times\vartheta_0 = \vartheta_0\times\vartheta_0\times\vartheta_2\times\vartheta_2+\vartheta_3\times\vartheta_3\times\vartheta_1\times\vartheta_1 - \vartheta_1\times\vartheta_1\times\vartheta_3\times\vartheta_3+\vartheta_2\times\vartheta_2\times\vartheta_0\times\vartheta_0.$$

For brevity, the arguments x_1, x_2, x_3, x_4, and the arguments $s-x_1$, $s-x_2$, $s-x_3$, $s-x_4$ are omitted in the left and right hand members respectively.

It will be observed that of the 256 sets of values which may be attributed to the constituents of the matrix

$$\begin{vmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 \end{vmatrix},$$

192 are excluded by the condition that 2σ and $2\sigma'$ are even; of the remaining 64, 4 are represented by the formulæ i.—iv., which are symmetrical with respect to all the arguments; 24 by the formula x., which is entirely unsymmetrical; and 6 by each of the formulæ v.—ix. and xi., which are symmetrical with respect to the arguments taken in pairs.

II. *Application of the Formula to the Abelian Functions.*

The Abelian functions are defined by the equations

$$(6) \ldots\ldots \begin{cases} Al_1(x) = \epsilon^{-\zeta\left(\frac{\pi x}{2K}\right)^2}\times\dfrac{\vartheta_1\left(\frac{\pi x}{2K}\right)}{\frac{\pi}{2K}\vartheta_1'(0)}, \\ Al_2(x) = \epsilon^{-\zeta\left(\frac{\pi x}{2K}\right)^2}\times\dfrac{\vartheta_2\left(\frac{\pi x}{2K}\right)}{\vartheta_2(0)}, \\ Al_0(x) = \epsilon^{-\zeta\left(\frac{\pi x}{2K}\right)^2}\times\dfrac{\vartheta_0\left(\frac{\pi x}{2K}\right)}{\vartheta_0(0)}, \\ Al_3(x) = \epsilon^{-\zeta\left(\frac{\pi x}{2K}\right)^2}\times\dfrac{\vartheta_3\left(\frac{\pi x}{2K}\right)}{\vartheta_3(0)}; \end{cases}$$

the constants K and ζ being determined by the equations

$$(7) \;\ldots\ldots\; \sqrt{\frac{2K}{\pi}} = 1+2q+2q^4+2q^9+2q^{16}+\ldots,$$

$$(8) \;\ldots\ldots\; \tfrac{1}{4}\zeta = \frac{q-4q^4+9q^9-16q^{16}+\ldots}{1-2q+2q^4-2q^9+2q^{16}-\ldots}.$$

The formulæ of Table II. give rise to a corresponding system of formulæ for the multiplication of four Abelian functions. To obtain this second system, we have only to express the Theta functions as Abelian functions, and to attend to the equations

$$(9) \;\ldots\ldots\; \begin{cases} k^2 = \dfrac{\vartheta_2^4(0)}{\vartheta_3^4(0)}, \quad k'^2 = \dfrac{\vartheta_0^4(0)}{\vartheta_3^4(0)}, \\ \dfrac{\pi}{2K}\vartheta'_{1,1}(0) = i\,\dfrac{\vartheta_0(0)\,\vartheta_2(0)}{\vartheta_3(0)}. \end{cases}$$

TABLE III. *Formulæ for the Multiplication of Four Abelian Functions.*

A.

(i.) $$2Al_1\times Al_1\times Al_1\times Al_1 = Al_1\times Al_1\times Al_1\times Al_1 + \frac{1}{k'^2}Al_2\times Al_2\times Al_2\times Al_2 + \frac{1}{k^2}Al_0\times Al_0\times Al_0\times Al_0 - \frac{1}{k^2k'^2}Al_3\times Al_3\times Al_3\times Al_3.$$

(ii.) $$2Al_2\times Al_2\times Al_2\times Al_2 = Al_2\times Al_2\times Al_2\times Al_2 + k'^2\,Al_1\times Al_1\times Al_1\times Al_1 + \frac{1}{k^2}Al_3\times Al_3\times Al_3\times Al_3 - \frac{k'^2}{k^2}Al_0\times Al_0\times Al_0\times Al_0.$$

(iii.) $$2Al_0\times Al_0\times Al_0\times Al_0 = Al_0\times Al_0\times Al_0\times Al_0 + \frac{1}{k'^2}Al_3\times Al_3\times Al_3\times Al_3 + k^2Al_1\times Al_1\times Al_1\times Al_1 - \frac{k^2}{k'^2}Al_2\times Al_2\times Al_2\times Al_2.$$

(iv.) $$2Al_3\times Al_3\times Al_3\times Al_3 = Al_3\times Al_3\times Al_3\times Al_3 + k'^2\,Al_0\times Al_0\times Al_0\times Al_0 + k^2Al_2\times Al_2\times Al_2\times Al_2 - k^2k'^2Al_1\times Al_1\times Al_1\times Al_1.$$

B.

(v.) $$2Al_1 \times Al_1 \times Al_2 \times Al_2 =$$
$$Al_2 \times Al_2 \times Al_1 \times Al_1 + Al_1 \times Al_1 \times Al_2 \times Al_2$$
$$+ \frac{1}{k^2} Al_3 \times Al_3 \times Al_0 \times Al_0 - \frac{1}{k^2} Al_0 \times Al_0 \times Al_3 \times Al_3.$$

(vi.) $$2Al_0 \times Al_0 \times Al_3 \times Al_3 =$$
$$Al_3 \times Al_3 \times Al_0 \times Al_0 + Al_0 \times Al_0 \times Al_3 \times Al_3$$
$$+ k^2 Al_2 \times Al_2 \times Al_1 \times Al_1 - k^2 Al_1 \times Al_1 \times Al_2 \times Al_2.$$

C.

(vii.) $$2Al_1 \times Al_1 \times Al_0 \times Al_0 =$$
$$Al_0 \times Al_0 \times Al_1 \times Al_1 - \frac{1}{k'^2} Al_3 \times Al_3 \times Al_2 \times Al_2$$
$$+ Al_1 \times Al_1 \times Al_0 \times Al_0 + \frac{1}{k'^2} Al_2 \times Al_2 \times Al_3 \times Al_3.$$

(viii.) $$2Al_2 \times Al_2 \times Al_3 \times Al_3 =$$
$$Al_3 \times Al_3 \times Al_2 \times Al_2 - k'^2 Al_0 \times Al_0 \times Al_1 \times Al_1$$
$$+ Al_2 \times Al_2 \times Al_3 \times Al_3 + k'^2 Al_1 \times Al_1 \times Al_0 \times Al_0.$$

D.

(ix.) $$2Al_1 \times Al_1 \times Al_3 \times Al_3 =$$
$$+ Al_3 \times Al_3 \times Al_1 \times Al_1 - Al_0 \times Al_0 \times Al_2 \times Al_2$$
$$+ Al_2 \times Al_2 \times Al_0 \times Al_0 + Al_1 \times Al_1 \times Al_3 \times Al_3.$$

(x.) $$2Al_1 \times Al_2 \times Al_0 \times Al_3 =$$
$$Al_3 \times Al_0 \times Al_2 \times Al_1 + Al_0 \times Al_3 \times Al_1 \times Al_2$$
$$+ Al_2 \times Al_1 \times Al_3 \times Al_0 - Al_1 \times Al_2 \times Al_0 \times Al_3$$

(xi.) $$2Al_2 \times Al_2 \times Al_0 \times Al_0 =$$
$$Al_0 \times Al_0 \times Al_2 \times Al_2 - Al_3 \times Al_3 \times Al_1 \times Al_1$$
$$+ Al_1 \times Al_1 \times Al_3 \times Al_3 + Al_2 \times Al_2 \times Al_0 \times Al_0.$$

III. *Case when the Sum of the Four Arguments is zero.*

Putting $$s = x_1 + x_2 + x_3 + x_4 = 0,$$
and attending to the equation

$$(10) \ldots\ldots \; \theta_{\mu, \mu'}(-x) = (-1)^{\mu\mu'} \theta_{\mu, \mu'}(x),$$

we find that in each of the formulæ i.—xi., one of the terms on the right-hand side cancels one of the two equal terms on the left, and that the four formulæ A., and the formulæ of the three pairs B., C., D., ix. and xi., become respectively coincident; the formula D. x. remains *sui generis.* Introducing the elliptic functions

$$\operatorname{sn} x = \frac{Al_1(x)}{Al(x)}, \quad \operatorname{cn} x = \frac{Al_2(x)}{Al(x)}, \quad \operatorname{dn} x = \frac{Al_3(x)}{Al(x)}$$

into the five formulæ thus obtained, we arrive at the following system:—

$$\text{(i.)}\quad k^2k'^2\Pi\,.\,\operatorname{sn} x - k^2\Pi\,.\,\operatorname{cn} x + \Pi\,.\,\operatorname{dn} x - k'^2 = 0,$$

$$\text{(ii.)}\quad k^2 \operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{cn} x_3 \operatorname{cn} x_4 - k^2 \operatorname{cn} x_1 \operatorname{cn} x_2 \operatorname{sn} x_3 \operatorname{sn} x_4 - \operatorname{dn} x_1 \operatorname{dn} x_2 + \operatorname{dn} x_3 \operatorname{dn} x_4 = 0,$$

$$\text{(iii.)}\quad k'^2 \operatorname{sn} x_1 \operatorname{sn} x_2 - k'^2 \operatorname{sn} x_3 \operatorname{sn} x_4 + \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{cn} x_3 \operatorname{cn} x_4 - \operatorname{cn} x_1 \operatorname{cn} x_2 \operatorname{dn} x_3 \operatorname{dn} x_4 = 0,$$

$$\text{(iv.)}\quad \operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{dn} x_3 \operatorname{dn} x_4 - \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{sn} x_3 \operatorname{sn} x_4 + \operatorname{cn} x_3 \operatorname{cn} x_4 - \operatorname{cn} x_1 \operatorname{cn} x_2 = 0,$$

$$\text{(v.)}\quad \operatorname{sn} x_1 \operatorname{cn} x_2 \operatorname{dn} x_4 + \operatorname{dn} x_1 \operatorname{cn} x_3 \operatorname{sn} x_4 + \operatorname{dn} x_2 \operatorname{sn} x_3 \operatorname{cn} x_4 + \operatorname{cn} x_1 \operatorname{sn} x_2 \operatorname{dn} x_3 = 0.$$

The first of these, which alone is symmetrical with respect to the four arguments, is the formula given by Professor Cayley ("Proceedings," p. 43). The formulæ (ii.), (iii.), (iv.) are symmetrical with respect to the two pairs x_1x_2, x_3x_4, and with respect to the two arguments of each pair; thus each of these formulæ represents a set of three. Lastly, the formula (v.) remains unchanged when any two of the arguments are interchanged, provided that the other two are interchanged at the same time; *i.e.*, there are six formulæ of this type.

As a verification of these formulæ in a particular case, let $x_4 = 0$; we find

$$(11)\ \ldots\ldots \begin{cases} Dd_1d_2 - k^2Cc_1c_2 = k'^2, \\ D + k^2Cs_1s_2 = d_1d_2, \\ Dc_1c_2 - Cd_1d_2 = k'^2s_1s_2, \\ C + Ds_1s_2 = c_1c_2, \\ Dc_1s_2 - Sd_2 = -s_1c_2, \end{cases}$$

(where we have written, for brevity, $s_1 = \operatorname{sn} x_1$, $c_1 = \operatorname{cn} x_1$, $d_1 = \operatorname{dn} x_1$; $s_2c_2d_2$ and SCD having similar meanings with respect to the arguments x_2 and $x_1 + x_2 = -x_3$); and these equations are easily shown to be true by means of the formulæ for the addition of elliptic functions.

IV. *Formula for the Multiplication of Four Multiple Theta Functions.*

We define a double Theta function by the equation

$$(12) \ldots\ldots \begin{cases} \theta(\mu, \mu'; \nu, \nu'; x, y) = \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} (-1)^{m\mu'+n\nu'} \\ \times A^{\frac{1}{4}(2m+\mu)^2} \times B^{\frac{1}{2}(2m+\mu)(2n+\nu)} \times C^{\frac{1}{4}(2n+\nu)^2} \\ \qquad \times e^{(2m+\mu)ix+(2n+\nu)iy} \\ = \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} (-1)^{m\mu'+n\nu'} e^{\frac{i\pi}{4}\phi(2m+\mu, 2n+\nu)} \\ \qquad \times e^{i(2m+\mu)x+i(2n+\nu)y} \end{cases}$$

where ϕ is the quadratic form (a, b, c); $i\pi a$, $i\pi b$, $i\pi c$ being the hyperbolic logarithms of A, B, C; and where the condition of convergence is that the real part of $i\phi$ must be a negative form of negative determinant.

Considering four different Theta functions, and writing

$$(13) \ldots\ldots \begin{cases} 2s = x_1+x_2+x_3+x_4, & 2t = y_1+y_2+y_3+y_4, \\ 2\sigma = \mu_1+\mu_2+\mu_3+\mu_4, & 2\tau = \nu_1+\nu_2+\nu_3+\nu_4, \\ 2\sigma' = \mu'_1+\mu'_2+\mu'_3+\mu'_4, & 2\tau' = \nu'_1+\nu'_2+\nu'_3+\nu'_4, \end{cases}$$

we have the formula

$$(14) \ldots \begin{cases} 4\Pi \,.\, \theta(\mu_j, \mu'_j; \nu_j, \nu'_j; x_j, y_j) \\ = \Sigma(-1)^{\alpha\sigma'+\alpha\alpha'+\beta\tau'+\beta\beta'} \\ \times \Pi \,.\, \theta(\sigma-\mu_j+\alpha, \sigma'-\mu'_j+\alpha'; \tau-\nu_j+\beta, \tau'-\nu'_j+\beta'; s-x_j, t-y_j) \end{cases}$$

the signs of multiplication Π referring to the four values 1, 2, 3, 4 of the index j, and the sign of summation Σ in the right-hand member referring to the symbols α, α', β, β', each of which is to have the values 0, 1; so that the right-hand member contains sixteen terms. The formula may be demonstrated in the same manner as the corresponding formula for the single Theta functions (see the note already cited, "Proceedings," Vol. I., Part viii., Art. 2).

If, instead of a double Theta function, we consider a multiple Theta function containing λ arguments x, y, ..., and depending on λ pairs of indices $\mu\mu'$, $\nu\nu'$, $\rho\rho'$, ..., we have the equation of definition

$$(15) \ldots\ldots \begin{cases} \theta(|\mu, \mu'|; |x|) = \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \sum_{r=-\infty}^{r=+\infty} \ldots \\ (-1)^{m\mu'+n\nu'+r\rho'+\ldots} \times e^{\frac{i\pi}{4}\phi(2m+\mu, 2n+\nu, 2r+\rho, \ldots)} \\ \qquad \times e^{i(2m+\mu)x+i(2n+\nu)y+i(2r+\rho)z+\ldots} \end{cases}$$

where ϕ is a quadratic form, containing λ indeterminates, and such that the real part of $i\phi$ is definite and negative; and we obtain a formula similar to (14); viz., taking four Theta functions, such that the sums of the homologous indices are all even, we have

$$(16) \ldots\ldots \begin{cases} 2^{\lambda}\Pi.\theta(|\mu_j, \mu_j'|; |x_j|) = \\ \Sigma(-1)^{|a\sigma'+aa'|}\times\Pi.\theta(|\sigma-\mu_j+a, \sigma'-\mu_j'+a'|; |s-x_j|), \end{cases}$$

where the symbols $|\mu_j, \mu_j'|$, $|\sigma-\mu_j+a, \sigma'-\mu_j'+a'|$ are placed by abbreviation for the λ pairs

$$\mu_j, \mu_j'; \nu_j, \nu_j'; \ldots$$

and $\sigma-\mu_j+a, \sigma'-\mu_j'+a'; \tau-\nu_j+\beta, \tau'-\nu_j'+\beta'; \ldots,$

respectively; and the symbol $|a\sigma'+aa'|$ represents the sum of the λ terms $(a\sigma'+aa')+(\beta\tau'+\beta\beta')+\ldots$

Each of the 2λ indices aa', $\beta\beta'$, ... is to receive successively the values 0, 1; and the sign of summation Σ extends to every combination of these values, so that the right-hand member consists of $2^{2\lambda}$ products, each affected with its proper sign, of four Theta functions.

The multiple Theta function (15) satisfies the equations

$$(17) \ldots\ldots \begin{cases} \theta(\mu+2, \mu'; \ldots; x\ldots) = (-1)^{\mu'}\theta(\mu, \mu'; \ldots; x\ldots), \\ \theta(\mu, \mu'+2; \ldots; x\ldots) = \theta(\mu, \mu'; \ldots; x\ldots); \end{cases}$$

$$(18) \ldots\ldots \theta(\mu, \mu'; \ldots; -x\ldots) = (-1)^{\mu\mu'}\theta(\mu, \mu'; \ldots; x\ldots);$$

$$(19) \ldots\ldots \begin{cases} \theta(\mu, \mu'; \nu, \nu'; \rho, \rho'; \ldots x, y, z\ldots) \\ \qquad = e^{\frac{i\pi}{4}\phi(\mu,\nu,\rho\ldots)}\times e^{i(\mu x+\nu y+\rho z+\ldots)} \\ \qquad\qquad \times\theta(0, 0; 0, 0; 0, 0; \ldots x_1, y_1, z_1, \ldots), \\ x_1 = x+\mu'\frac{\pi}{2}+\frac{\pi}{4}\frac{d\phi}{d\mu}, \quad y_1 = y+\nu'\frac{\pi}{2}+\frac{\pi}{4}\frac{d\phi}{d\nu}, \\ z_1 = z+\rho'\frac{\pi}{2}+\frac{\pi}{4}\frac{d\phi}{d\rho}, \quad \ldots\ldots, \end{cases}$$

which correspond to the equations (4), (10), (5), relating to a Theta function of one argument.

The equations (17) show that we need only attribute to the indices the values 0, 1; thus, the formula (16) may be regarded as comprehending 11^{λ} different equations, if we regard two equations as identical which may be deduced from one another by permutation of the four indices j; or, as comprehending $\frac{11.12.13\ldots10+\lambda}{1.2.3\ldots\lambda}$ different equations, if we also regard as identical two equations which may be deduced from one another by a simultaneous permutation of the λ arguments, and corresponding pairs of indices, in each of the Theta functions; viz., counting the different equations on this principle, we have as many of them as there are λ-combinations, with repetition, of the eleven matrices of Table I. But we can always pass from any one of the equations (16) to any other by means of the formula (19), which may be employed to express any one of the 4^{λ} Theta functions as a product of any other by an exponential factor; although (for brevity) we have supposed that the indices of one of the Theta functions compared in that formula are all equal to zero.

Quaternion Proof of Minding's Theorem. *By* J. J. WALKER, M.A.

[*Read February* 13*th*, 1879.]

At the conclusion of his very able paper on the Conditions of Astatic Equilibrium (Proc. of Lond. Math. Soc., Vol. IX., pp. 116—118), Professor Minchin has given a demonstration of this theorem,* in which recourse has been had to Cartesian coordinates, whereby the unity of treatment of the subject by Quaternions, elsewhere adopted throughout the paper, and to the use of which it owes much of its interest, has been marred. A short note, showing how the pure Quaternion proof may be completed, may, perhaps, be thought worthy of insertion.

Starting with Professor Minchin's equation (32) of the line in which the single resultant and Poinsot's axis combine, viz.,

$$\rho = \lambda i + \frac{1}{a} V(Jk - Kj),$$

with (31) $$SJk = SKj \qquad (31),$$

which give $$\rho = \lambda i + \frac{1}{a}(Jk - Kj) \qquad (32'),$$

wherein λ is an arbitrary scalar, the problem is to find the vector of the point in which that line meets the plane of IJ or IK.

Let $i' = UI$, $j' = UJ$, $k' = UK$, while $t_2 = TJ$, $t_3 = TK$.

To determine the λ of ρ in the plane IJ,

$$S\rho k' = \lambda Sik' + \frac{t_2}{a} Sj'kk',$$

i.e., $$0 = \lambda Sik' - \frac{t_2}{a} Ski'.$$

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* Originally given in a paper which appeared in *Crelle*, Bd. XV. (1836), to which notice appears to have been more recently drawn by Moigno, *Mec. Anal.*, Ch. ix. (1868).