On Systems of One-Vectors in Space of n Dimensions. By W. H. Young.

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In the present paper a system of one-vectors* in space of n dimensions is reduced to a normal form.

The space in question is supposed to be flat, *i.e.*, of zero curvature.

I use S_k to denote a flat space of k dimensions contained in the whole space S_n.

The normal form in S_{2m-1} consists of m one-vectors, the line of action of one being arbitrary. In S_{2m} the normal form again consists of m one-vectors, lying, however, in a covariant S_{2m-1}.

In the course of the discussion certain other properties of the system are obtained. The whole subject is closely connected with the theory of linear complexes in space of n dimensions. I here confine myself to giving a geometrical definition of line coordinates in such a space, and a simple geometrical deduction of the known quadratic equations between them.

It is well known that in S_n we can construct, in an infinite number of ways, a fundamental $(n+1)$-pyramid which does not lie in any space of lower dimensions.

Theorem I. *Any one-vector through an arbitrary point O may be replaced by n one-vectors along lines joining O to the n points of any fundamental n-pyramid lying in any S_{n-1} not passing through O.*

These n points together with O define a fundamental $(n+1)$-pyramid in the S_n.

Take the plane through the line of action of the one-vector and one of the edges meeting at O. This plane meets the S_{n-1} determined by the remaining $(n-1)$ edges through O in a straight line. We can then, by the parallelogram law, replace our one-vector by two components, one along the first edge, and one in the S_{n-1}. In this

* The reader may, if he please, substitute the word "force" for "one-vector," throughout the present paper, which is, in part, introductory to one on vectors of a more general nature. In a paper entitled "Sulla Statica dei Corpi Rigidi nello Spazio a Quattro Dimensioni," *Giornale di Battaglini*, xxxiv., 1896, some of the properties here given are discussed for four dimensions by De Francesco.

S_{n-1} the n vertices of the original $(n+1)$-pyramid determine a fundamental n-pyramid, and the component of our one-vector in this S_{n-1} goes through one of the vertices, viz., O. Thus, by induction, the theorem follows, it being certainly true when $n=2$.

The above tacitly assumes that the one-vector does not lie in any S_k of the fundamental $(n+1)$-pyramid. If this be the case, we only have to effect the reduction for the corresponding $(k+1)$-pyramid, the components corresponding to the remaining edges through O vanishing.

THEOREM II. *Any one-vector may be replaced by one-vectors along the edges of the fundamental* $(n+1)$*-pyramid.*

Let us assume that the line of action of the one-vector does not lie in any S_k of the fundamental $(n+1)$-pyramid. It must intersect at least one of the S_{n-1}'s of the pyramid in a point at a finite distance. Join this point of intersection to the vertex S_0 of the pyramid opposite to this S_{n-1}. The plane through the one-vector and this line will meet the S_{n-1} in question in a straight line through the same point of intersection. Hence, by the parallelogram law, the one-vector may be replaced by two one-vectors, one through the S_0 and the other in the S_{n-1}. By Theorem I. the former component may be replaced by components along the edges through the S_0. The possibility of the reduction then depends on replacing the one-vectors in the S_{n-1} by components along the edges of the fundamental $(n+1)$-pyramid which lie in that S_{n-1}. But these are the edges of an n-pyramid which is a fundamental one in that S_{n-1}. The required result follows by induction, for it obviously holds when $n=1$.

THEOREM III. *A given system of one-vectors may be replaced in one and only one way by one-vectors along the edges of a given fundamental* $(n+1)$*-pyramid.*

The possibility of this reduction follows from Theorem II.

To prove the uniqueness, we remark that, if two modes of reduction are possible, reversing one system so obtained and combining with the other, we obtain a system in equilibrium.* This system of one-vectors along the edges may be divided into two classes, viz., those through an arbitrarily chosen vertex and those in the opposite S_{n-1} of the fundamental $(n+1)$-pyramid. Unless the former all vanish, they can, by repeated application of the parallelogram law, be reduced

* A system is said to balance, or to be in equilibrium, when, by introducing suitable pairs of equal and opposite one-vectors and by repeated application of the parallelogram law, it can be reduced to two equal and opposite one-vectors.

to a single one-vector through the chosen vertex. This one-vector, lying outside the S_{n-1}, cannot be replaced by, and therefore also cannot balance, a system of one-vectors in it. The theorem now follows by induction.

COR. I. *A system of one-vectors has* $\frac{1}{2}(n+1)n$ *"coordinates," viz., the equivalent one-vectors along the edges of the fundamental* $(n+1)$*-pyramid. There are, moreover,* $\frac{1}{2}(n+1)n$ *necessary and sufficient conditions of equilibrium, viz., each coordinate must vanish.*

COR. II. *There are* $\frac{1}{2}(n-1)(n-2)$ *necessary and sufficient conditions that a given system of one-vectors should be equivalent to a single one-vector.*

This is evident if we reflect that there are ∞^{2n-2} straight lines in S_n, and that therefore a straight line has $(2n-2)$ independent coordinates; a single one-vector therefore $(2n-1)$. The required number is got by subtracting this number from the number of coordinates of a general system.

COR. III. *Any system of one-vectors in an* S_n *may be replaced by a one-vector through any arbitrary point* O, *and a system of one-vectors in an arbitrary* S_{n-1} *which does not pass through* O.

In a special case the one-vector may of course vanish, or the system in the S_{n-1} be in equilibrium.

We now proceed to generalize certain known theorems about systems of one-vectors in S_3. It will be convenient to consider first the cases of S_4 and S_5.

THEOREM IV. *Any system of one-vectors in* S_4 *is equivalent to two one-vectors, and has as covariant the* S_3 *containing them.*

By Cor. III. to Theorem III. such a system is equivalent to a one-vector through an arbitrary point O and a system of one-vectors in an S_3. This latter system, if not reducible to a single one-vector, is, as is well known, equivalent to two one-vectors, one of which may be chosen to act along any line we please in the S_3. Take it to act along a line through the point in which the one-vector through O meets the S_3. Then it may be compounded with the latter one-vector, and thus the system is reducible to two one-vectors, which, in general, do not intersect.

These two one-vectors may be replaced by any other suitable pair in the S_3 determined by them; viz., the line of action of one of the two one-vectors may be chosen arbitrarily in the S_3, the line of action of the other and the magnitude of both one-vectors being then determined. The S_3 itself is a covariant of the system.

In fact, if it were possible by reducing in another way to get a second pair lying in a different S_3, we could reverse this pair and obtain with the first pair a system in equilibrium. Since, however, two S_3's intersect in an S_2, we can arrange that one one-vector of our second pair should lie in this S_2, and the other accordingly not. We should then have a single one-vector not in the first S_3, balancing one-vectors lying in it, which is obviously absurd.

THEOREM V. *a. Any system of one-vectors in S_5 is equivalent to three one-vectors, one of which may be chosen to act through any point we choose.*

The reduction to three one-vectors follows from the previous theorem by means of Cor. III. to Theorem III. One of the one-vectors passes by construction through an arbitrary point O. If the S_3 determined by any two one-vectors be intersected by the line of action of the third, the three one-vectors would lie in an S_4. That this would be a special case is obvious, since we might choose as our system three straight lines not lying in an S_4. Omitting for the present the discussion of special cases, we may assume in what follows that the three one-vectors do not lie in an S_4.

THEOREM V. *b. Any arbitrary line being chosen, one of the three one-vectors can be made to act along it.*

We already have one arbitrary point O on the first one-vector. Take any other point P. The plane through P and the first one-vector meets the S_3 determined by the other two in a point Q. The first one-vector may now be replaced by two components, along OP and OQ. The other two one-vectors may be replaced by two one-vectors in their S_3, one of which passes through Q. Compounding the two one-vectors meeting at Q, we are left with three one-vectors, one of which acts along the arbitrary line OP.

THEOREM V. *c. Any arbitrary S_3 being chosen, two of the one-vectors can be made to act in it.*

By the above we may take any straight line in the S_3 as line of action of one of the one-vectors. The other two one-vectors lie in a second S_3 which will intersect the first in a line. One of the two one-vectors may then be taken to act along this line and the result follows.

THEOREM V. *d. The arbitrary straight line being chosen* (V. *b*) *the S_3 containing the other two one-vectors is determined, also the magnitude of the first one-vector and the system in the S_3.*

Suppose first, if possible, that a second reduction could lead to a different S_3. Reverse the three one-vectors so found, and we have with the first three a system in equilibrium. The two S_3's have a common straight line, and one one-vector of each pair may be taken to act along it. We have now two one-vectors in one of the original S_3's, one one-vector along the arbitrary line and another one-vector. These latter two one-vectors may be replaced by two others in the S_3 determined by them, one of which may, as before, be taken in the original S_3 in which the other pair of one-vectors lies. We are left with three one-vectors in this S_3 and one outside it. This system can evidently not be in equilibrium unless the last one-vector vanishes. This involves, however, the one-vector along the arbitrary line meeting one of the two S_3's, the possibility of which has been already excluded.

The first part of the theorem is therefore proved by a *reductio ad absurdum*. A moment's consideration shows that two different reductions to the same straight line and S_3 are impossible, for it would lead to a one-vector along the straight line balancing a system in the S_3. Thus the whole theorem is proved.

THEOREM V. *e. The arbitrary S_3 being chosen* (V. *c.*), *the line of action of the third one-vector is determined, also the magnitude of the third one-vector and the system in S_3.*

The proof, being similar to the last, may be omitted.

THEOREM V. *f. If the line of action* (S_1) *of one of the one-vectors pass through an arbitrary point* (S_0), *the two other one-vectors lie in a fixed S_4 passing through the S_0.*

Let O be the point. Choose any other straight line through O. The plane through this line and the line of action of the original one-vector meets the conjugate S_3 of this latter line in a point P. The one-vector may now be replaced by one along the new line and one along OP. Make one of the pair of one-vectors in the S_3 pass through P, and we have reduced to a one-vector along the new line and two one-vectors, which, by construction, lie in the S_4 determined by O and the first S_3. Hence by (V. *d.*) the result follows.

THEOREM V. *g. If the S_3 of two of the one-vectors pass through an arbitrary plane* (S_2), *the remaining one-vector lies in another plane which intersects the first, and therefore lies with it in an S_4.*

Choose any other S_3 through the plane. The S_4 containing the two S_3's meets the third one-vector in a point O. The first two one-vectors may be replaced by one in the common plane, and a second which, by a known theorem in S_3, however we choose the first one-vector, passes through a fixed point P of that plane. We may replace this second one-vector by a one-vector in the plane and one along OP. Compounding this with the one-vector already passing through P, we are left, outside the chosen S_3, with a one-vector in the plane determined by P and the original one-vector through O.

Hence, by V. *e.*, the theorem follows.

Similarly, the converse may be proved, and the theorem that, if the S_3 containing two of the one-vectors turn round a fixed point, the line of action of the third one-vector lies in a fixed S_4 passing through the point.

All these theorems for S_4 and S_5 will be found summed up in the following four theorems for the general case of S_n.

THEOREM VI. *Any system of one-vectors in "even space" S_n, where $n = 2m$, is reducible to $m\left(=\frac{n}{2}\right)$ one-vectors, lying in a covariant S_{n-1}.*

THEOREM VII. *a. Any system of one-vectors in "odd space" S_n, where $n = 2m-1$, is reducible to $m\left(=\frac{n+1}{2}\right)$ one-vectors, the line of action of one of which may be chosen arbitrarily. The S_{n-2} containing the remaining $(m-1)\left(=\frac{n-1}{2}\right)$ one-vectors is then determined, and so are the magnitude of the first one-vector and the system in the S_{n-2}.*

THEOREM VII. *b. Any S_{2k-1} in the S_{2m-1} being chosen, k of the one-vectors can be made to act in it. The $S_{2m-1-2k}$ containing the remaining $(m-k)$ one-vectors is then determined, and so are the systems in the S_{2k-1}, $S_{2m-1-2k}$ respectively.*

THEOREM VII. *c. Any S_{2p} in the S_{2m-1} being chosen, the S_{2k-1} containing k of the one-vectors can be made to pass through it, if $k > p$ (or to lie in it if $k < p$). The conjugate $S_{2m-1-2k}$ then lies in a fixed $S_{2m-2p-2}$ (or passes through a fixed $S_{2m-2p-2}$), intersecting the S_{2p} and therefore lying with it in an S_{2m-2}.*

Assuming these theorems to have been demonstrated for all values of n up to $(2m-2)$ inclusive, we can prove them generally by induction.

It is evident that, VII. being proved, VI. follows at once from Theorem III., Cor. III.

To prove Theorem VII. By Theorem III., Cor. III., the system may be replaced by a one-vector through an arbitrary point O, and a system in an S_{n-1} not passing through O. Since, therefore,

$n-1 = 2m-2$ the system is equivalent to m one-vectors, one of which passes through an arbitrary point O.

Choose any other point P. The plane through P and the one-vector through O meet the S_{n-1} of the other one-vectors in a point Q. Replace the one-vector through O by one-vectors along OP, OQ, and choose one of the remaining one-vectors to pass through Q. Compounding the two one-vectors through Q, the system is reduced to a one-vector along an arbitrary line OP and $(m-1)$ other one-vectors.

It is now obvious that, instead of a straight line, we may choose an S_3, and make two one-vectors act in it. Because, one one-vector being chosen in the S_3, the S_3 and the S_{n-2} containing the other $(m-2)$ one-vectors intersect in a straight line, which may be taken as line of action of one one-vector.

Similarly, an S_5, S_7, or any odd space may be arbitrarily chosen.

Next, to prove that the choice of one arbitrary line determines the conjugate S_{n-2}.

Suppose two reductions lead to two different S_{n-2}'s; reverse one system so obtained, and we have with the other a system in equilibrium. Choosing $(m-2)$ one-vectors of each system in the S_{n-4} common to both S_{n-2}'s, we have left three one-vectors in a perfectly determinate S_5; one of these three may, of course, be chosen in the S_{n-4}, since an S_5 and S_{n-4} have a common line. We are left with

$$\left.\begin{array}{ll} & \text{two one-vectors in an } S_3 \\ \text{and} & \text{a system in an } S_{n-4} \end{array}\right\}.$$

In the general case, the S_3 and S_{n-4} have no common point, and therefore the whole system can only be in equilibrium if each partial system is in equilibrium by itself. This, however, leads at once to a special system in which the arbitrarily chosen line intersects its polar S_{n-2}, and hence, by a *reductio ad absurdum*, the theorem is proved.

It is now obvious that the magnitude of the one-vector along the arbitrary line and the system in the polar S_{n-2} are determined; otherwise we could obtain a system of one-vectors in an S_{n-2}, balanced by a single one-vector whose line of action lies entirely outside that S_{n-2}.

The proof of Theorems *b* and *c*, being very simple and identical in principle with the proofs of the corresponding theorems for five dimensions, may be omitted.

Counting up the Constants.—A system of one-vectors in S_n has $\frac{1}{2}n(n+1)$ constants. If we divide this up into an equivalent system in an S_k and one in an S_{n-k-1}, we have

$$\tfrac{1}{2}n(n+1)-\tfrac{1}{2}k(k+1)-\tfrac{1}{2}(n-k-1)(n-k) = (k+1)(n-k)$$

constants still at our disposal; this is exactly the number of coordinates of an S_k, and we should therefore expect to be able to choose the S_k arbitrarily, as has been shown by the foregoing discussion to be the case.

THEOREM VIII. *When n is odd, and equal to $2m-1$, the m one-vectors to which a given (general) system can be reduced determines a fundamental $(n+1)$-pyramid of constant volume.*

We know that the theorem is true for space of three dimensions. We may therefore assume it to hold for S_{n-2}. It then follows for S_n. For, any arbitrary line being chosen, we have a determinate length along it representing the one-vector along it, and, corresponding to the other $(m-1)$ one-vectors, a fundamental $(n-1)$-pyramid of constant volume in a determinate S_{n-2} (polar S_{n-2}), not intersecting the line. We have, therefore, a fundamental $(n+1)$-pyramid whose volume remains constant, however we choose the $(n-1)$ one-vectors, as long as one line of action remains unaltered. Now take any other arbitrary line. Its polar S_{n-2} has an S_{n-4} common with the first S_{n-2}. Take any line in this S_{n-4}. We may arrange that a one-vector should act along this line in each reduction. It follows that the volume of the pyramid is the same for the two reductions.

COROLLARY. *When n is even, and equal to $2m$, the m one-vectors to which a given general system can be reduced determine an n-pyramid which is fundamental in the S_{n-1} containing it, and whose volume is constant.*

This last theorem puts us in a position to discuss the special cases in odd spaces. If the arbitrary line meets its polar S_{n-2}, the volume of the pyramid is zero, and the system can be reduced to one in space of $(n-2)$ dimensions, and therefore to $(m-1)$ one-vectors. Conversely, unless the volume of the pyramid is zero, the system cannot be reduced to $(m-1)$ one-vectors, or to a system in space of $(n-2)$ dimensions. Such a system we may call "once specialized." If the new fundamental $(n-1)$-pyramid collapse, we have a further reduction—a system twice specialized. In general, we may have a system specialized any number of times, up to $(m-1)$ times specialized, which would be a system equivalent to a single one-vector, or m times specialized, which would be a system in equilibrium.

THEOREM IX. *The necessary and sufficient conditions that a system should be equivalent to a single one-vector may be expressed in a single statement. The one-vectors along the edges of every tetrahedron of the fundamental $(n+1)$-pyramid must be equivalent to a single one-vector.*

That the conditions are necessary follows from the manner in which we replace a one-vector by one-vectors along the fundamental $(n+1)$-pyramid. We have, therefore, only to show that the conditions are sufficient. Consider the case $n = 4$. Evidently, if the system be not equivalent to a single one-vector, when the one-vectors in every S_3 of the 5-pyramid are equivalent to a single one-vector, the two one-vectors to which the system is equivalent must be such that one passes through any arbitrary vertex of the pyramid; that is, every vertex of the fundamental pyramid lies in the characteristic S_3 defined by the two one-vectors, which is, of course, impossible.

When $n = 5$, by what has just been proved, if the one-vectors in every S_3 of the pyramid are equivalent to a single one-vector, this is true for every S_4 of the pyramid. Hence, if the whole system be not equivalent to a single one-vector, we may replace the system by two one-vectors—one through any vertex of the fundamental pyramid, the other in the opposite S_4. But the S_3 defined by these two one-vectors cannot pass through all the vertices of the fundamental pyramid. Proceeding in this way we may obviously extend the theorem to n dimensions.

Absolute Line Coordinates.—Let us choose definite directions along the edges of the fundamental $(n+1)$-pyramid as positive (viz., 12, 23, 31, 41, 42, 43, 51, 52, 53, 54, and so on). Now let us suppose a one-vector of unit magnitude to act in a definite sense along any straight line, and let us denote the ratio of the component one-vectors along any edge $\overline{rs}$ to the one-vector represented by that edge, attention being paid to sign, by p_{rs}. We have then, in the case when the system is equivalent to a single one-vector, a convenient system of absolute coordinates for the line of action including its sense. The construction of a line whose coordinates are thus defined follows by repeated application of the parallelogram law, and the same is true of the determination of the coordinates of a line whose position and sense are given. The identical relations satisfied by these $\frac{1}{2}(n+1)n$ coordinates, of which only $2(n-1)$ are independent, will be obtained by applying Theorem IX.

They are the well known equations*:—

$$p_{12}p_{34}+p_{23}p_{14}+p_{31}p_{24} = 0,$$
$$p_{12}p_{3r}+p_{23}p_{1r}+p_{31}p_{2r} = 0 \qquad (r = 4, 5, \dots n),$$

* Each of these equations merely expresses the fact that the moment of the system constituted by the one-vectors acting along the edges of the corresponding tetrahedron about itself vanishes.

and similar equations for every four indices. There are evidently $\frac{1}{4!}(n+1)n(n-1)(n-2)$ such equations. Though necessary and sufficient, they are not all independent; there are syzygies between them. We have also a quadratic non-homogeneous equation, expressing the fact that the resultant of all the component one-vectors is of unit magnitude, viz.,

$$\Sigma a_{rs}^2 p_{rs}^2 + 2\Sigma a_{rs} a_{ij} \cos(rs, ij) p_{rs} p_{ij} = 1$$

with an obvious notation.

We may, of course, if we please, avoid this equation by using only the ratios of the p's.

It will be noted that this method of defining line coordinates is independent of point or plane coordinates; and the equations between them are obtained without any analysis, being immediate consequences of the definition.

An Essay towards the Generating Functions of Ternariants. *By* Professor A. R. Forsyth. Received March 17th, 1898. Communicated April 7th, 1898.

The present paper contains an attempt to apply the principle of the method devised by Cayley for the construction of generating functions for binary forms to the corresponding problem for ternary forms. In the case of ternariants, regarded as determined by their leading coefficients, there is the difficulty that the universal concomitant u_x has unity for its leading coefficient, and that therefore no change is made in the leading coefficient of a concomitant on multiplying the concomitant by any power of u_x. It is necessary to take account of this consideration in discussing the problem from the side of the leading coefficients; accordingly, instead of dealing solely with the asyzygetic concomitants for the construction of a generating function, I have rejected those which are reducible by means of a power of u_x. Those which remain I have called a complete set of *hyposyzygetic* concomitants: in terms of u_x and of the members of such a set, every