"Physical Society of London—Proceedings," Vol. iv., Pt. ii., Jan. to April, 1881.

"Sitzungsberichte der Physikalisch-medicinischen Societät zu Erlangen," 12 Heft, Nov. 1879 bis Aug. 1880.

"Smithsonian Report .... for the year 1879;" Washington, 1880.

"Bulletin de la Société Mathématique de France," Tome ix., No. 3; Paris, 1881.

"Proceedings of Royal Society," Vol. xxxii., Nos. 212, 213.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome iv., Dec., 1880; Paris, 1880.

"Bulletin des Sciences Mathématiques et Astronomiques—Table des Matières et Noms d'Auteurs," Tome iv., deuxième Série, and Tome v., Janv., 1881.

"Monatsbericht," März, 1881.

"Ueber Lamé'sche Functionen," von F. Klein (pp. 213—246); and "Ueber Körper, welche von confocalen Flächen zweiten Grades begränzt sind," von F. Klein (pp. 410—427, Math. Annalen, Vol. xviii.).

On a System of Coordinates. By Prof. Genese, M.A.

[*Read June 9th*, 1881.]

1. Let APB connote a *positive* rotation from the direction PA to the direction PB, so that

$$APB+BPA = 2\pi.$$

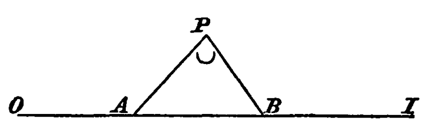

Fig. 1.

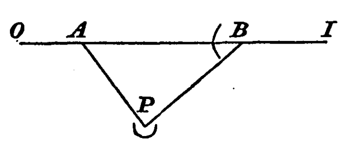

Fig. 2.

If A, B be points on an axis OI, I being at infinity,

$$IBP-IAP = APB \text{ (Fig. 1)},$$

or $\quad APB-2\pi$ (Fig. 2).

Now, taking O as origin, OI as axis of x, let x, y be the rectangular coordinates of P, and let a, b, c be the abscissæ of three points A, B, C on OI. Then

$$\cot IAP = \frac{x-a}{y}, \text{ \&c.},$$

therefore $\quad \cot APB = \dfrac{(x-a)(x-b)+y^2}{(b-a)\,y}, \ \cot BPC = \text{\&c.} \quad \text{.........(1)}.$

Let θ, ϕ, ψ denote the angles BPC, CPA, APB respectively, and λ, μ,

ν the cotangents of these angles; then

$$\theta+\phi+\psi = 2\pi, \text{ and } \mu\nu+\nu\lambda+\lambda\mu = 1 \quad \text{.............}(2).$$

The object of this paper is to exhibit results obtained by taking λ, μ, ν as the coordinates of P.

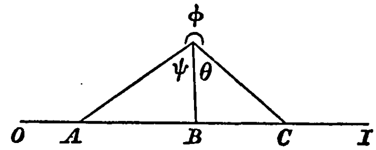

Fig. 3.

2. Solving for x^2+y^2, x, and y, in terms of λ, μ, ν,

$$\left.\begin{aligned} x^2+y^2 &= \frac{a^2(c-b)^2\lambda+\&c.}{(c-b)^2\lambda+\&c.} \\ x &= \frac{a(c-b)^2\lambda+\&c.}{(c-b)^2\lambda+\&c.} \\ y &= \frac{-(c-b)(a-c)(b-a)}{(c-b)^2\lambda+\&c.} \end{aligned}\right\} \quad \text{..............}(3).$$

And we notice that the degree of any locus in λ, μ, ν is not higher than its degree in x, y; but that curves passing through the circular points at infinity will be represented by equations of lower degree in λ, μ, ν than in x, y. On the other hand, we have one more coordinate, and the relation $\mu\nu+\nu\lambda+\lambda\mu = 1$; perhaps, however, analogy may suggest properties of the surface $xy+yz+zx = 1$.

3. The equation of the first degree,

$$A\lambda+B\mu+C\nu = D \quad \text{.........................}(4),$$

represents in general a circle.

But if $\dfrac{A}{c-b}+\dfrac{B}{a-c}+\dfrac{C}{b-a} = 0$, then (4) will represent a straight line.

The quantities $c-b$, $a-c$, $b-a$,—that is, with the usual sign-convention BC, CA, AB,—occurring frequently, it will be convenient to denote them by c_1, c_2, c_3. Thus, the condition that (4) should represent a straight line is

$$\frac{A}{c_1}+\frac{B}{c_2}+\frac{C}{c_3} = 0 \quad \text{.........................}(5).$$

4. If $A\lambda+B\mu+C\nu = D$ represent a straight line PT cutting the axis of x in T, and $ITP = \alpha$,

$$A : B : C : D :: c_1^2\,AT : c_2^2\,BT : c_3^2\,CT : c_1c_2c_3\cot\alpha \text{.........}(6);$$

and this establishes a relation between λ, μ, ν and *line-coordinates* referred to the intercepts p, q, r from ordinates through A, B, C, viz.,

$$\frac{A}{D} = \frac{c_1 p}{c_2 c_3}, \quad \frac{B}{D} = \&c. \quad \text{.......................}(7).$$

Cor. 1.—The straight line is an ordinate to the axis if

$$D = 0 \text{..................................}(8).$$

COR. 2.—Two straight lines,

$$A\lambda + B\mu + C\nu = D,$$

$$A'\lambda + B'\mu + C'\nu = D',$$

are parallel, if $$\frac{A-A'}{c_1^2} = \frac{B-B'}{c_2^2} = \frac{C-C'}{c_3^2} \quad\text{.......................(9).}$$

The equation to the straight line joining two points $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$ is

$$\begin{vmatrix} \lambda, & \mu, & \nu, & 1 \\ \lambda_1, & \mu_1, & \nu_1, & 1 \\ \lambda_2, & \mu_2, & \nu_2, & 1 \\ \frac{1}{c_1}, & \frac{1}{c_2}, & \frac{1}{c_3}, & 0 \end{vmatrix} = 0 \quad\text{.......................(10).}$$

The equation to a straight line through B is

$$c_1\lambda - c_3\nu = c_2 \cot \alpha \quad\text{..........................(11),}$$

a relation enabling us to change from λ, μ, ν to biangular coordinates.

The equation $$c_1^2\lambda + c_2^2\mu + c_3^2\nu = D \quad\text{..........................(12)}$$

represents the same as $$y = \frac{-c_1c_2c_3}{D},$$

i.e., a straight line parallel to axis of x.

If $D = 0$, $y = \infty$, thus

$$c_1^2\lambda + c_2^2\mu + c_3^2\nu = 0 \quad\text{..........................(13)}$$

represents the straight line at infinity.

Again, $$a\,c_1^2\lambda + b\,c_2^2\mu + c\,c_3^2\nu = D \quad\text{.......................(14)}$$

represents a straight line through O, and, in the case of $D=0$, it is the axis of y.

The Circle.

5. When $\frac{A}{c_1} + \frac{B}{c_2} + \frac{C}{c_3}$ is not zero, the equation

$$A\lambda + B\mu + C\nu = D$$

represents a circle.

By considering the intersections T, T' with OI, for which λ, μ, ν each $= \infty$, but are in the ratios $\frac{1}{\sin\theta} : \frac{1}{\sin\phi} : \frac{1}{\sin\psi}$, or as

$$\frac{1}{c_1AT} : \frac{1}{c_2BT} : \frac{1}{c_3CT},$$

we find that $A : B : C :: c_1^2\,AT\,.\,AT' : c_2^2\,BT\,.\,BT' : c_3^2\,CT\,.\,CT'$......(15).

Also it may be shown that

$$A : D :: c_1^2\,AT\,.\,AT' : c_1c_2c_3\,.\,2\rho \quad\text{.................(16),}$$

where β is the ordinate of the centre.

The coordinates a, β of the centre of the circle are

$$\left.\begin{aligned} a &= \frac{(b+c)\frac{A}{c_1}+\&c.}{2\left(\frac{A}{c_1}+\&c.\right)} \\ \beta &= \frac{D}{2\left(\frac{A}{c_1}+\&c.\right)} \\ \text{and} \qquad \text{radius} &= \frac{\sqrt{D^2+A^2+\&c.-2BC-\&c.}}{2\left(\frac{A}{c_1}+\&c.\right)} \end{aligned}\right\} \qquad \ldots\ldots\ldots(17).$$

From the geometrical interpretations of ABC, we see that the equation to any circle through B is $A\lambda+C\nu = D$ ....(18);
also that, if this circle touch the axis at B, the equation becomes

$$\lambda+\nu = k \qquad \ldots\ldots\ldots(19).$$

The equation to a circle through A and C is, of course,

$$\mu = d \qquad \ldots\ldots\ldots(20).$$

The condition that the last-mentioned circles should touch is [remembering that $\lambda\nu+\mu(\lambda+\nu) = 1$]

$$k^2 = 4(1-kd) \qquad \ldots\ldots\ldots(21).$$

This condition leads to an easy proof of Feuerbach's Theorem that the nine-point circle touches the inscribed and escribed circles;—at the point of contact $\lambda = \nu$ (Wolstenholme, Prob. 43).

The equation $\lambda = 0$, *i.e.* $\cot\theta = 0$, represents the circle on BC as diameter.

6. *The equation to a circle having its centre on the axis* is

$$A\lambda+B\mu+C\nu = 0 \qquad \ldots\ldots\ldots(22).$$

This will reduce to a point-circle if

$$A^2+\&c.-2BC-\&c. = 0,$$

or

$$\sqrt{A}+\sqrt{B}+\sqrt{C} = 0 \qquad \ldots\ldots\ldots(23).$$

In particular, $\mu+\nu = 0$ represents a point-circle at A ....(24).

The equation

$$\begin{vmatrix} \lambda, & \mu, & \nu \\ \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \end{vmatrix} = 0 \qquad \ldots\ldots\ldots(25)$$

represents the circle passing through the points $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, and having its centre on the axis ABC.

If P_1, P_2 be these points, P'_1, P'_2 their images with respect to the axis ABC, the circle passes through $P_1P_2P'_1P'_2$.

For brevity, such a circle will hereinafter be called the axial-circle (P_1P_2), or simply (P_1P_2).

The Equation of the Second Degree.

7. The equation $(A, B, C, D, F, G, H, L, M, N)(\lambda, \mu, \nu, 1)^2 = 0$ represents the general bicircular quartic.

In the present communication, the homogeneous equation

$$(A, B, C, F, G, H)(\lambda, \mu, \nu)^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

only will be considered.

Its equivalent in x, y coordinates is

$$l(x^2+y^2)^2+(mx+n)(x^2+y^2)+px^2+qx+r = 0\ldots\ldots\ldots(27).$$

Thus it may represent a *bicircular quartic with collinear foci*, a *circular cubic*, a *conic*, *two circles*, &c.

8. Considering the equations (22) and (25), we see that the analytical processes which demonstrate the non-metrical properties of the straight line and conic, may be interpreted, *mutatis mutandis*, to give properties of axial-circles and the curves represented by (26).

Ex. 1.—Two axial-circles may be drawn through any point P in a plane to touch any such curve at Q, R, say. The axial-circle (QR) may be called the polar of P. Then, if P move on a fixed axial-circle, the polar of P will pass through a fixed point (and its image).

Ex. 2.—From the analysis of Pascal's Theorem, if P, Q, R, S, T, U be points on a $(\lambda, \mu, \nu)^2$ curve, the axial-circles (PQ) and (ST), (QR) and (TU), (RS) and (UP) intersect in three points (and their images) lying on an axial-circle.

A closer study of the (λ, μ, ν) system may, perhaps, show a connection between the metrical properties of conics and $(\lambda, \mu, \nu)^2$ curves.

On the Generation of $(\lambda, \mu, \nu)^2$ Curves.

9. Let A_1, A_2 be two fixed points in the plane xy, C_1, C_2 two variable points on the axis of x, P the intersection of C_1A_1, C_2A_2; then, a *suitable* (2, 2) correspondence being found for C_1, C_2, P may be made to describe any conic.

By analogy, if circles be described with centres C_1, C_2, passing through A_1, A_2 respectively, and a *suitable* (2, 2) correspondence be found for C_1, C_2, the

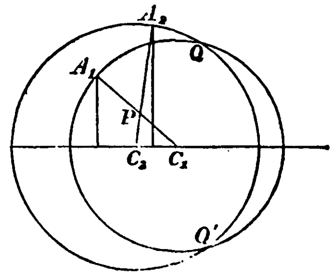

Fig. 4.

intersections Q, Q' of the circles will describe any given $(\lambda, \mu, \nu)^2$ curve.*

10. If a_1, a_2 be the distances of C_1, C_2 from the intersection of A_1, A_2 with the axis of x, it will be found that the most general correspondence which will make P describe a conic is

$$(lmn, pqr)\left(\frac{1}{a_1}, \frac{1}{a_2}, 1\right)^2 = 0 \quad \text{.................}(28).$$

Analogously, if a_1, a_2 be the distances of C_1, C_2 from the centre of the axial circle (A_1A_2), the same correspondence will make Q_1, Q_2 describe the general $(\lambda, \mu, \nu)^2$ curve.

11. If we put $l = 0$, $m = 0$, we get a (1, 1) correspondence; the locus of P is either a conic *passing through* A_1, A_2, or a straight line, and the locus of Q, Q' will be a $(\lambda, \mu, \nu)^2$ curve *passing through* A_1, A_2.†

12. Now, let A_1, A_2 be points of a $(\lambda, \mu, \nu)^2$ curve *lying on the axis*, then the following apparatus will trace the curve—

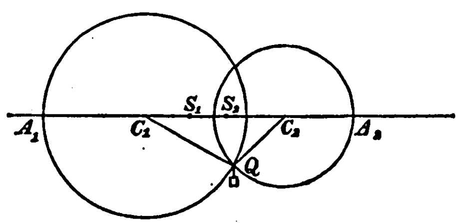

Fig. 6.

Two strings fastened to a pencil at Q with a weight attached, pass round C_1, C_2, and are fastened to fixed pegs S_1, S_2 in an axis. C_1, C_2 being made to move on this axis homographically, Q describes one of the $(\lambda, \mu, \nu)^2$ curves (S_1A_1, S_2A_2 are, of course, the lengths of the strings).

---

* It is to be observed that the general (2, 2) correspondence between C_1 and C_2 would make P describe a quartic having A_1 and A_2 for double points. For such a quartic *would* give a (2, 2) correspondence, and the degree n of the locus of P cannot be greater than 4, because then A_1, A_2 would be two multiple points of the $(n-2)^{\text{th}}$ order.

Also, a suitable (2, 2) correspondence will make P describe *any* given quartic having A_1, A_2 as double points. For eight other points completely determine the quartic, and eight positions of the pair C_1, C_2 determine the (2, 2) correspondence.

† Such a correspondence may be easily constructed practically.

Let F_1, F_2 be the foci of the given homography, I_1, I_2 the centres, *i.e.*, the positions of C_1 and C_2 when C_2 and C_1 are at infinity. Draw ordinates I_1B_1, I_2B_2 so that

$$(I_1B_1)^2 = (I_2B_2)^2 = I_1F_1 \cdot I_1F_2.$$

Then, if P be any point on a circle described on B_1B_2 as diameter, B_1P, B_2P determine corresponding positions of C_1, C_2; thus, *if two rods rigidly attached at right angles be made* to move round fixed pegs (B_1, B_2) suitably placed, they will determine *any given* homography on OI.

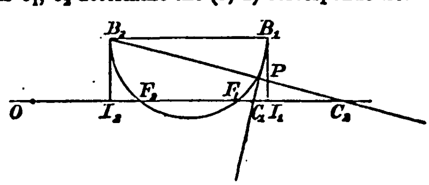

Fig. 5.

In particular, if C_1C_2 be a fixed length, Q will describe a central conic, whose eccentricity

$$= \sqrt{\frac{2C_1C_2}{A_1A_2}} \qquad \text{(29).*}$$

If $C_1C_2 = A_1A_2$, $e = \sqrt{2}$; and Q traces out a rectangular hyperbola.

In this case, the apparatus may be simplified, for $C_1Q = C_1A_1 = C_2A_2 = C_2Q$. Thus, if one arm of a rigid right angle C_1MQ slide on a horizontal axis, and a weight Q, attached to a string which passes over a pulley at C_1 (moving with C_1M) and is fastened to a fixed peg S_1 in the axis, be allowed to fall down MQ (vertical), Q will describe a rectangular hyperbola.

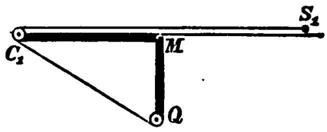

Fig. 7.

If the pulley C_1 be fixed to the axis, and S_1 move with the rigid angle, Q will trace out a parabola.

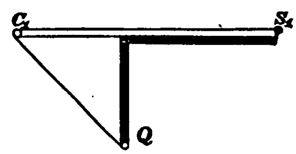

Fig. 8.

The $(\lambda, \mu, \nu)^2$ *Curve considered as an Envelope.*

If Q, R be two points on fixed straight lines a_1, a_2 respectively, and a suitable (2, 2) correspondence be established between Q and R, the join QR will envelope a conic.

Analogously, if Q, R move on two fixed axial circles a_1, a_2, the axial circle (QR) will, with a suitable (2, 2) correspondence, envelope a $(\lambda, \mu, \nu)^2$ curve.

14. If we substitute a (1, 1) correspondence, the envelope will touch a_1, a_2 in each case.

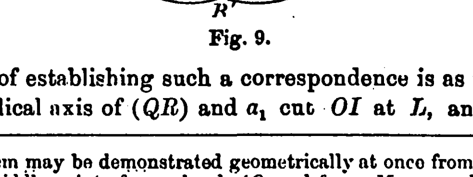

Fig. 9.

One mode of establishing such a correspondence is as follows :—

Let the radical axis of (QR) and a_1 cut OI at L, and let M be the

* This theorem may be demonstrated geometrically at once from the following,— "If M be the middle point of any chord AQ, and from M perpendiculars be drawn to the chord and the axis, meeting the axis respectively in G and N; then $CG = e^2 . CN$, C being the centre."

corresponding point for a_2; if L, M be homographically related, the envelope of the circle QR will be a $(\lambda, \mu, \nu)^2$ curve, touching a_1 and a_2.

In particular, *if LM be of constant length, the envelope will be a conic.*

It is remarkable that the eccentricity of this conic is

$$\sqrt{\frac{A_1A_2}{2LM}} \qquad (30),$$

where A_1, A_2 are the centres of a_1, a_2, *i.e.*, the inverse of the form (29) for the conic as a locus.

If the correspondence take the form

$$\frac{l}{OL}+\frac{m}{OM}=1,$$

the circle (QR) will pass through a fixed point.

15. Again, let the circles be replaced by two straight lines AQ, BR perpendicular to the axis OAB.

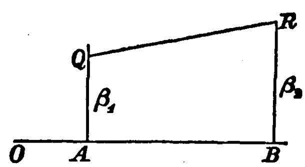

Fig. 10.

Then, a suitable (1, 1) correspondence being established between $AQ^2=\beta_1^2$ and $BR^2=\beta_2^2$, the axial circle (QR) will envelope a $(\lambda, \mu, \nu)^2$ curve touching AQ, BR. The general relation is between β_1^2, β_2^2 and not β_1, β_2, because the axial circles pass through the images of Q, R in the axis, and are equally well determined by $AQ=-\beta_1$, $BR=-\beta_2$.

(If we consider anallagmatic surfaces instead of curves, β_1^2, β_2^2 are proportional to the areas of the traces made by a movable sphere on two fixed planes.)

For the particular case $l\beta_1^2+m\beta_2^2=n$, the circle (QR) passes through two fixed points.

16. It may, however, be added that the correspondence $l\beta_1+m\beta_2=n$, for which the straight line QR passes through a fixed point, gives a circular cubic as the envelope of (QR). In particular, if $\beta_2\pm\beta_1=$ constant, the envelope is an ellipse having AB for minor axis; with upper sign, the join QR passes through the focus, with lower sign, QR is constant and equal to the major axis.

17. In order that the axial-circles should have a real envelope, consecutive members must intersect.

Let a, $a+da$ be the abscissæ of the centres of axial-circles of radii N, $N+dN$. By Euc. I. 20, the condition of intersection is $dN<da$ numerically. In fact,

$$\frac{dN}{da}=-\cos\psi \qquad (31),$$

where ψ is the angle which N makes with OI.

The Foci on the Axis.

18. The condition that the axial-circle

$$l\lambda+m\mu+n\nu=0$$

should touch $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2=0,$

is $l^2(BC-F^2)+\&c.+2mn(GH-AF)+\&c.=0$ ....(32).

Now the condition that the circle should reduce to a point is

$$l^2+\&c.-2mn-\&c.=0.$$

The two relations give, in general, four values of the ratios $l:m:n$, that is, *there are four foci on the axis.*

Cartesian Ovals.

19. If r_1, r_2, r_3 be the distances of any point P from A, B, C, we find

that $r_1:r_2:r_3::\frac{\sin\theta}{c_1}:\frac{\sin\phi}{c_2}:\frac{\sin\psi}{c_3}$ ....(33).

Thus the equation $lc_1r_1+mc_2r_2+nc_3r_3=0$

is equivalent to $l\sin\theta+m\sin\phi+n\sin\psi=0$

or
$$\frac{l}{\sqrt{1+\lambda^2}}+\frac{m}{\sqrt{1+\mu^2}}+\frac{n}{\sqrt{1+\nu^2}}=0.$$

But $\lambda^2+1=\lambda^2+\lambda\mu+\lambda\nu+\mu\nu=(\lambda+\mu)(\lambda+\nu).$

Thus our equation is

$$l\sqrt{\mu+\nu}+m\sqrt{\nu+\lambda}+n\sqrt{\lambda+\mu}=0 \quad \text{...............(34)};$$

as we should expect, for the point-circles $(\mu+\nu)=0$, $\nu+\lambda=0$, $\lambda+\mu=0$, at A, B, C, are foci.

It is easily shown that $l^2c_1+m^2c_2+n^2c_3=0$ ....(35)

is the *condition that the fourth focus should be at infinity.*

With this condition, $l\sqrt{\mu+\nu}+\&c.=0$

represents a Cartesian oval.

20. The condition that

$$p(\mu+\nu)+q(\nu+\lambda)+r(\lambda+\mu)=0$$

should touch the Cartesian is

$$\frac{l^2}{p}+\frac{m^2}{q}+\frac{n^2}{r}=0.$$

Now, returning to x, y coordinates, we find that

$$\mu+\nu:\nu+\lambda:\lambda+\mu::c_1^2\{(x-a)^2+y^2\}:\&c. \quad \text{............(36)}.$$

Therefore $p(\mu+\nu)+\&c. = 0$ represents the circle

$$L(x^2+y^2-2ax+t)=0,$$

where
$$pc_1^2+qc_2^2+rc_3^2=L,$$
$$pc_1^2a+\&c. \quad =La,$$
$$pc_1^2a^2+\&c. \quad =Lt,$$

whence
$$pc_1^2=\frac{L(bc-\overline{b+c}\,a+t)}{-c_2c_3},\quad qc_2^2=\&c.$$

If we replace t by a^2-N^2, N being the radius of the circle,

$$pc_1:qc_2:rc_3::(a-b)(a-c)-N^2:\&c.$$

Thus, if N be the length of a normal to a Cartesian drawn from the point G on the axis ($OG=a$),

$$\frac{l^2c_1}{(a-b)(a-c)-N^2}+\frac{m^2c_2}{(a-c)(a-a)-N^2}+\&c.=0;$$

or, since $l^2c_1+m^2c_2+n^2c_3=0$,

$$N^2=\frac{(a-a)(a-b)(a-c)}{a+\dfrac{l^2c_1bc+m^2c_2ca+n^2c_3ab}{l^2c_1a+m^2c_2b+n^2c_3c}}\quad\text{..................(37).}$$

If the origin O be so chosen that

$$l^2c_1bc+\&c.=0,$$

then
$$N^2=\frac{AG.BG.CG}{OG}\quad\text{.........................(38).}$$

For the determination of O, we get

$$\frac{OA}{l^2}=\frac{OB}{m^2}=\frac{OC}{n^2}\quad\text{...........................(39).}$$

i.e., it is the triple focus, and it is clearly the point in which *the asymptote to the evolute of the Cartesian cuts the axis.*

21. Since $\dfrac{dN}{da}=-\cos\psi$, ψ being the angle the normal makes with the axis of x, we get

$$-\cos\psi=\frac{2a-(a+b+c)+\dfrac{abc}{a^2}}{2\sqrt{\dfrac{(a-b)(a-c)(a-a)}{a}}}\quad\text{..............(40),}$$

which may be regarded as the tangential equation to the evolute of the Cartesian.

22. A simpler formula is obtainable from (38), if we take logarithms

of both sides before differentiation, viz.,

$$\frac{2}{N}(-\cos\psi) = \frac{1}{AG} + \frac{1}{BG} + \frac{1}{CG} - \frac{1}{OG};$$

or, if T be the point at which the tangent corresponding to N meets the axis,

$$\frac{2}{TG} = \frac{1}{AG} + \frac{1}{BG} + \frac{1}{CG} - \frac{1}{OG} \quad \text{..................(41)},$$

and this is easily changed into

$$\frac{OT}{OG} = \frac{AT}{AG} + \frac{BT}{BG} + \frac{CT}{CG} \quad \text{....................(42).*}$$

Inversion.

The formulæ for inversion in λ, μ, ν coordinates are very simple, viz., if (λ', μ', ν') be the inverse of (λ, μ, ν) with respect to any centre O on the axis, the radius of inversion being taken $= \sqrt{OA \cdot OC}$, then

$$\mu' = \mu,$$
$$\lambda' = \left(\frac{r}{p} - 1\right)\mu + \frac{r}{p}\nu,$$
$$\nu' = \left(\frac{p}{r} - 1\right)\mu + \frac{p}{r}\lambda,$$

where $\quad p = c_1^2 \cdot OA, \quad r = c_3^2 \cdot OC.$

In particular, if $OA : OC :: c_3^2 : c_1^2$ (or, which is the same thing, $OB^2 = OA \cdot OC$), then

$$\mu' = \mu,$$
$$\nu' = \lambda,$$
$$\lambda' = \nu.$$

Note on Apparatus for describing Conics.

If C_1, C_2 be made to move with uniform velocity u, then Q will have a constant parallel velocity $= \frac{A_1A_2'}{C_1C_2}u$, and therefore (Newton, Prop. viii.) a vertical acceleration varying inversely as the cube of the distance from A_1A_2.

It will be observed that only a part of the curve can be described, the apparatus failing to work when the angle QC_1C_2, or QC_2C_1, be-

* If C be taken at infinity, the system of coordinates reduces to a form of biangular coordinates: $\theta = \beta$, $\phi = \pi + \alpha$, and $\psi = \beta - \alpha$. Also, $\sin\theta : \sin\phi : \sin\psi :: r_1 : -r_2 : c_3$. Thus equation (34) will represent the curve $lr_1 - mr_2 + nc_3 = 0$. When $l^2 = m^2$, this represents a central conic; the point O is at infinity, and equation (42) becomes $0 = \frac{AT}{AG} + \frac{BT}{BG}$, and expresses that the tangent and normal to a conic divide the line joining the foci harmonically.

comes a right angle. This remark does not apply to the modification for the rectangular hyperbola or to the apparatus for the parabola.

Additional Note.

I have found, since writing the above, that the method of Art. 10, *as regards the* (1, 1) *correspondence*, is not new. It is used in Dr. Casey's elegant memoir on "Bicircular Quartics," Art. 10, and was originally given by Chasles, "Comptes Rendus," 1853. The extension, by using curves of Index 1 in place of conics passing through four points, is due to M. Terquem; and not, as stated by Clebsch (p. 376, "Vorlesungen über Geometrie") to M. de Jonquières. See "Théorèmes segmentaires," *Nouvelles Annales*, 1853, p. 358.

To prevent misconception I have to add, respecting the system of coordinates, that I had in the first instance reinvented Mr. Walton's "Trigonic Coordinates," *Quarterly Journal*, Vol. ix., p. 340. I suppressed my paper on the subject, and went on with the work only on perceiving the advantages mentioned in § 2.

On Spherical Quartics, with a Quadruple Cyclic Arc, and a Triple Focus. By Henry M. Jeffery.

[*Read June 9th*, 1881.]

1. In general, quartics with quadruple cyclic arcs have also quadruple foci in their quadrantal poles; but if a cyclic arc of the satellite-conic coincide with the quadruple cyclic arc of the quartic, its quadrantal pole is a triple focus. (§ 11.)

Such a group of spherical quartics may be thus defined:

$$\kappa = (1+dx)(1+x^2+y^2)+\lambda(1+x^2+y^2)^2 \qquad \text{(A)},$$

if x, y denote by Gudermann's system of spherical coordinates the tangents of arcs, intercepted on the arcs of coordinates by great circles drawn from their quadrantal poles.*

These quartics are of the eighth class, if they are non-singular, and may have five single foci, collinear with the triple focus. If the quartics are nodal, two single foci unite, as *in plano*, at a node and disappear; such quartics are of the sixth class (§ 12.)

If $d^2 < 1$, there may be two cusped quartics in the group, pear-shaped (Fig. 1), and cardioidal (Fig. 2); these are of the fifth class.

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* If α, β be arcs drawn perpendicularly from any point of a quartic to the cyclic arcs, their geometrical relation is thus expressed:

$$\kappa(\sin\alpha)^4 = \sin\alpha\sin\beta+\lambda \qquad \text{(A)}.$$