If however, $\gamma^4 = -1$, then

$$\frac{u^2}{\gamma} = \tfrac{1}{4}(-2 \pm \sqrt{-12}),$$

viz., this is
$$\frac{u^2}{\gamma} = \tfrac{1}{2}(-1 \pm i\sqrt{3}) = \omega$$

if ω be an imaginary cube root of unity $(\omega^2+\omega+1=0)$; hence

$$u^8 = (\gamma\omega)^4 = -\omega.$$

Moreover,
$$1+\frac{2u^8}{v} = 1+\frac{2u^2}{\gamma}, \;= 1+2\omega,$$

or say,
$$= \omega-\omega^2, \quad [= \sqrt{-3} \text{ if } \omega = \tfrac{1}{2}(-1+i\sqrt{3})];$$

and we thus have, as in the above-mentioned Note,

$$y = \frac{(\omega-\omega^2)\,x+\omega^2 x^3}{1-\omega^2(\omega-\omega^2)\,x^2}, \text{ giving } \frac{dy}{\sqrt{1-y^2\,.\,1+\omega y^2}} = \frac{(\omega-\omega^2)\,dx}{\sqrt{1-x^2\,.\,1+\omega x^2}};$$

or, what is the same thing, for the modulus $k^2 = -\omega$, we have

$$\text{sn}\,(\omega-\omega^2)\,\theta = \frac{(\omega-\omega^2)\,\text{sn}\,\theta+\omega^2\,\text{sn}^3\,\theta}{1-\omega^2(\omega-\omega^2)\,\text{sn}^2\,\theta};$$

the values of $\text{cn}\,(\omega-\omega^2)\,\theta$ and $\text{dn}\,(\omega-\omega^2)\,\theta$ are thence found to be

$$\text{cn}\,(\omega-\omega^2)\,\theta = \frac{\text{cn}\,\theta\,(1-\omega^2\,\text{sn}^2\,\theta)}{1-\omega^2(\omega-\omega^2)\,\text{sn}^2\theta};$$

and
$$\text{dn}\,(\omega-\omega^2)\,\theta = \frac{\text{dn}\,\theta\,(1+\omega^2\,\text{sn}^2\,\theta)}{1-\omega^2(\omega-\omega^2)\,\text{sn}^2\,\theta};$$

which are the formulæ of transformation for the elliptic functions.

Complex Multiplication Moduli of Elliptic Functions.

By A. G. Greenhill.

[*Read March 8th*, 1888.]

The problem of the Complex Multiplication of Elliptic Functions is the determination of the elliptic functions of the complex argument $(a+b\sqrt{\Delta}\,i)\,u$, in terms of the elliptic functions of the argument u, where the ratio of the periods $K'/K = \sqrt{\Delta}$, and Δ is a prime number;

but, if Δ is a composite number mn, then we can have

$$K'/K = \sqrt{(m/n)};$$

but in this case b must contain the factor n.

The coefficients in the expression of an elliptic function of the argument $(a+b\sqrt{\Delta}\,i)\,u$ in terms of the elliptic functions of u will involve the values of the *modular functions* corresponding to

$$K'/K = \sqrt{\Delta},$$

and thus the *modular equation* in some shape requires solution; and it is the chief object of this paper to make a collection of all the numerical solutions hitherto obtained, for integral values of Δ.

According to a remark of Abel (*Œuvres*, t. 1, p. 272, 1st edition), quoted by Kronecker (*Berlin Sitz*, 1857), the modular equation in such cases is always soluble by radicals.

A few numerical cases are given by Legendre and Abel, but the first important collection of results is due to Kronecker (*Berlin Sitz*, 1862), who gives the numerical values of Legendre's modulus κ, or in some cases of $\kappa\kappa'$, for a series of values of Δ, and promises a more complete collection, which has not yet appeared.

According to the form of Δ with respect to the modulus 4 or 8, it will be convenient to consider four classes, and to choose the *absolutely simplest numerical invariant* appropriate to each class, which classes are distinguished as follows—

Class *A*. $\Delta \equiv 3$, mod. 8.
Class *B*. $\Delta \equiv 7$, mod. 8.
Class *C*. $\Delta \equiv 1$, mod. 4.
Class *D*. $\Delta \equiv 2$, mod. 4.

The class for $\Delta \equiv 0$, mod. 4, does not require special treatment, as it can be made to depend on one of the previous classes by means of the quadric transformation.

The article, "*Neue Untersuchungen im Gebiete der elliptischen Functionen*," of F. Klein (*Math. Ann.*, Bd. xxvi., 1886), gives references to the most recent researches on modular equations; and in the course of this paper great use will be made of the following articles:—

Sohnke, "*Æquationes modulares pro transformatione Functionum Ellipticarum*," Crelle, 16.

Schröter, "*Dissertatio inauguralis de Æquationibus modularibus*," *Regiomonti*, 1854; also *Liouville*, 1858; and *Acta Mathematica*, 1882.

Hermite, "*Théorie des Équations modulaires*;" Paris, 1859.

Klein and Kiepert, "*Ueber die Transformation der elliptischen Functionen*," *Math. Ann.*, xiv., p. 111; xxvi., p. 369; xxxii., p. 1.

G. H. Stuart, "*Complex Multiplication of Elliptic Functions,*" *Quar. Jour. of Math.*, Vol. xx., p. 18.

E. W. Fielder, "*Ueber eine besondere Classe irrationalem Modulargleichungen;*" Zürich, 1885.

R. Russell, "*On* $\kappa\lambda$, $\kappa'\lambda'$ *Modular Equations,*" *Proc. of the Lond. Math. Soc.*, Nov. 10, 1887.

The general expressions for the formulas of Complex Multiplication are also given by the author in an article in the *Quarterly Journal of Mathematics*, Vol. xxii.

CLASS *A*.

$\Delta \equiv 3$, mod. 8.

The absolutely simplest numerical invariant to choose for this class is Klein's *absolute invariant* J, the same as Dedekind's *Valenz* (*Crelle*, 83), and connected with Hermite's a by the equation

$$J = -\tfrac{4}{27}\, a$$

(*Théorie des Équations modulaires*); but it is convenient to use Kiepert's form in terms of Legendre's moduli κ and κ' (*Math. Ann.*, Vol. xxvi.),

$$J = -\frac{(1-16\kappa^2\kappa'^2)^3}{108\kappa^2\kappa'^4},$$

obtained by a quadric transformation from Klein's form

$$J = \frac{4}{27}\,\frac{(1-\kappa^2\kappa'^2)^3}{\kappa^4\kappa'^4},$$

so that Kiepert's J is a "*Modul-function zweiter Stufe.*"

Then, if we work with Weierstrass's canonical first elliptic integral

$$\int_x^\infty \frac{dx}{\sqrt{(4x^3-g_2x-g_3)}},$$

and *normalize* it by multiplying by the twelfth root of the discriminant

$$D = g_2^3 - 27g_3^2,$$

so that it becomes
$$\int \frac{dy}{\sqrt{(4y^3-\gamma_2 y-\gamma_3)}},$$

we can make the new discriminant

$$\gamma_2^3 - 27\gamma_3^2 = -1,$$

and thus the absolute invariant

$$J = \frac{g_2^3}{D} = -\gamma_2^3,$$

$$J-1 = \frac{27g_3^2}{D} = -27\gamma_3^2.$$

In this Class A, the simplest formula of complex multiplication connects

$$x = \wp u, \text{ and } y = \wp \frac{u}{M},$$

where $$\frac{1}{M} = \tfrac{1}{2}(-1 + \sqrt{\Delta}\, i),$$

leading to the differential relation

$$\frac{M\,dy}{\sqrt{(4y^3 - g_2 y - g_3)}} = \frac{dx}{\sqrt{(4x^3 - g_2 x - g_3)}},$$

by an equation of the form (*Quar. Jour. of Math.*, xxii., p. 127)

$$y = M^2 \frac{x^n - A_1 x^{n-1} + A_2 x^{n-2} \dots}{(x^m - G_1 x^{m-1} + G_2 x^{m-2} \dots)^2},$$

where $$n = 2m+1, \quad \Delta = 4n - 1 = 8n + 3,$$

and $$A_1 = 2G_1;$$

the A's and G's being certain modular functions, to be subsequently determined. (Kiepert, *Math. Ann.* XXVI., p. 398.)

The determination of G_1 is the most trouble, so it is important to notice that it is a numerical factor of $\sqrt{\Delta} + i$.

For, if we denote by M' and G_1' the conjugate imaginaries of M and G_1, and if we put

$$z = \wp n u = \wp \frac{u}{MM'},$$

then $$z = M'^2 \frac{y^n - 2G_1' y^{n-1}}{(y^m - G_1' y^{m-1} \dots)^2}$$

$$= MM'^2 \frac{(x^n - 2G_1 x^{n-1} \dots)^n - 2G_1' M^{-2} (x^n \dots)^{n-1} (x^m \dots)^2 \dots}{(x^m - G_1 x^{m-1} \dots)^2 \left\{ (x^n - 2G_1 x^{n-1} \dots)^m - G_1' M^{-2} (x^n \dots)^{m-1} (x^n \dots)^2 \right\}^2}$$

$$= \frac{1}{n^2} \frac{x^{n^2} - 2(nG_1 + G_1' M^{-2}) x^{n^2-1} \dots}{x^{n^2-1} - 2(nG_1 + G_1' M^{-1}) x^{n^2-1} \dots};$$

and then, from the known expansion of $\wp u$ in powers of u,

$$\wp u = \frac{1}{u^2} + * + \frac{g_2}{20} u^2 + \dots$$

The absent terms denoted by the $*$ require that

$$nG_1 + G_1' M^{-2} = 0;$$

so that, if we put $G_1 = a\sqrt{\Delta} + bi, \quad G_1' = a\sqrt{\Delta} - bi,$

then $$na\sqrt{\Delta}+nbi+(a\sqrt{\Delta}-bi)\tfrac{1}{2}(-2n+1+\sqrt{\Delta}\,i)=0,$$

or $$(a-b)\left\{\sqrt{\Delta}-(4n-1)i\right\}=0,$$

or $$a=b;$$

but, to determine a, the expansions of y and z in terms of x must be carried out further.

When once G_1, and therefore $A_1 = 2G_1$, has been determined, the remaining G's and A's are found by the recurring formulas given by Kiepert, *Math. Ann.*, xxvi., p. 399.

The preceding equation connecting y and x is equivalent to any one of the three following equations (*Quar. Jour. of Math.*, XXII., p. 125)—

$$y-e_1 = M^2(x-e_2)\prod_{r=1}^{r=m}\left\{x-\wp(\omega_2+2r\omega_1/n)\right\}^2 \div D,$$

$$y-e_2 = M^2(x-e_3)\prod\left\{x-\wp\left[(n-2r)\,\omega_1/n\right]\right\}^2 \div D,$$

$$y-e_3 = M^2(x-e_1)\prod\left\{x-\wp\left[\omega_2+(n-2r)\,\omega_1/n\right]\right\}^2 \div D,$$

$$D = \prod\left\{x-\wp(2r\omega_1/n)\right\}^2;$$

where $$\frac{\omega_2'}{\omega_2}=\frac{K'}{K}i=\sqrt{\Delta}\,i,$$

and $$\omega_1=\tfrac{1}{2}(\omega_2+\omega_2'),\quad \omega_3=\tfrac{1}{2}(\omega_2-\omega_2');$$

so that $$G_1=\sum_{r=1}^{r=m}\wp(2r\omega_1/n).$$

Thus, for example, when $\Delta = 51$,

$$G_1=\wp\frac{2\omega_1}{13}+\wp\frac{4\omega_1}{13}+\wp\frac{8\omega_1}{13}+\wp\frac{16\omega_1}{13}+\wp\frac{32\omega_1}{13}+\wp\frac{64\omega_1}{13}$$

(Kiepert, *Math. Ann.*, XXVI., p. 381).

But suppose the complex multiplier $\dfrac{1}{M}$, instead of being

$$\tfrac{1}{2}(-1+\sqrt{\Delta}\,i),$$

had been $$\tfrac{1}{2}(-\rho+\sqrt{\Delta}\,i),$$

where ρ is an odd integer; then we should have to put

$$n=\tfrac{1}{4}(\Delta+\rho^2)$$

in the above formulas; and this explains why in Hermite's *Équations Modulaires*, p. 44, Class 3° (our Class A), Δ has the values

$$4n-\rho^2=4n-1,\quad 4n-9,\quad 4n-25,\ \dots.$$

CLASS *A*.

$\Delta \equiv 3$, mod. 8.

$\Delta = 3$. From Jacobi's modular equation of the third order,

$$\sqrt{\kappa\lambda} + \sqrt{\kappa'\lambda'} = 1,$$

we obtain, putting $\kappa = \lambda'$, $\kappa' = \lambda$,

$$2\sqrt{\kappa\kappa'} = 1,$$

or $$2\kappa\kappa' = \tfrac{1}{2} = \sin 30^\circ,$$

so that the modular angle is 15°, and

$$\kappa = \sin 15^\circ, \qquad \kappa' = \cos 15^\circ.$$

Then the absolute invariant $J = 0$, and

$$\gamma_2 = 0, \quad 27\gamma_3^2 = 1, \quad \text{or} \quad 3\sqrt{3}\,\gamma_3 = 1.$$

Also $$\frac{1}{M} = \omega,$$

an imaginary cube root of unity; and

$$\wp\,\omega u = \omega \wp u,$$

the simplest case of *Complex Multiplication*, required in the reduction of the elliptic integrals considered by Legendre (*Fonctions Elliptiques*, t. I., cap. XXVI.).

$\Delta = 11$. Taking Schröter's or Russell's form of the modular equation of the 11^{th} order,

$$\sqrt{\kappa\lambda} + \sqrt{\kappa'\lambda'} + 2\sqrt[6]{4\kappa\lambda\,\kappa'\lambda'} = 1,$$

and putting $\kappa = \lambda'$, $\kappa' = \lambda$; then

$$2\sqrt{\kappa\kappa'} + 2\sqrt[3]{2\kappa\kappa'} = 1.$$

Forming the equation in $\kappa^2\kappa'^2$, and putting

$$J = -\frac{(1-16\kappa^2\kappa'^2)^3}{108\kappa^2\kappa'^2},$$

we find $$J = -\frac{2^9}{3^3}, \quad J - 1 = -\frac{7^2 \times 11}{3^3},$$

$$\gamma_2 = \frac{8}{3}, \quad \gamma_3 = \frac{7\sqrt{11}}{27}.$$

Here Hermite's $a = 2^7$, the value of which could be inferred from his equations at the foot of p. 47 of the *Équations Modulaires.*

Also $G_1 = -\frac{1}{6}(\sqrt{11}+i)$, $A_1 = -\frac{1}{3}(\sqrt{11}+i)$,

and the values of A_2 and A_3 are given in the *Quar. Jour. of Math.*, XXII., p. 134.

$\Delta = 19$. We shall find

$$J = -2^9, \quad J-1 = -3^3 \times 19;$$

$$\gamma_2 = 8, \quad \gamma_3 = \sqrt{19}, \quad \gamma_2+1 = 3^2, \quad \gamma_2^2-\gamma_2+1 = 3\times 19.$$

These values are obtained from Hermite's *Théorie des Équations Modulaires*, p. 47, where it is shown that, α being the equivalent of $-\frac{27}{4}J$,

$$\Delta = 3, \quad \alpha = 0;$$

and $$\Delta = 11, \quad \alpha = 2^7, \text{ as before};$$

$$\Delta = 19, \quad \alpha = 2^7\times 3^3;$$

$$\Delta = 27, \quad \alpha = 2^7\times 3\times 5^3;$$

$$\Delta = 43, \quad \alpha = 2^{10}\times 3^3\times 5^3;$$

and by inference from the approximate equation (*Équations Modulaires*, p. 48),

$$2^8\alpha = e^{\pi\sqrt{\Delta}} - 744 + 196880e^{-\pi\sqrt{\Delta}} + \dots,$$

we obtain $$\Delta = 67, \quad \alpha = 2^7\times 3^3\times 5^3\times 11^3,$$

$$\Delta = 163, \quad \alpha = 2^{10}\times 3^3\times 5^3\times 23^3\times 29^3.$$

According to Hermite (*Equations Modulaires*, p. 47) the value of α is an integer when there is only one improperly primitive class of the determinant $-\Delta$; and $\Delta = 163$ is probably the highest number of this nature.

Hermite points out that in these cases $e^{\pi\sqrt{\Delta}}$ is very nearly an integer; for instance, in $e^{\pi\sqrt{163}}$ the decimal part begins with a series of twelve 9's.

Using the symbol $\approx$ for approximate equality,

$$1728J \approx -\frac{1}{q} \approx -e^{\pi\sqrt{\Delta}},$$

so that $$12\gamma_2 \approx e^{\frac{1}{3}\pi\sqrt{\Delta}}, \quad \text{also} \quad 216\gamma_3 \approx e^{\frac{1}{2}\pi\sqrt{\Delta}};$$

therefore $e^{\frac{1}{3}\pi\sqrt{\Delta}}$ is also very nearly an integer, a multiple of 12; while $e^{\frac{1}{2}\pi\sqrt{\Delta}} \div \sqrt{\Delta}$ is also very nearly an integer, a multiple of 216. (H. J. S. Smith, *Report on the Theory of Numbers to the British Association*, 1865, p. 374.)

Thus
$$e^{\frac{1}{3}\pi\sqrt{19}} \approx 96 \quad = 12\,(3^2-1),$$
$$e^{\frac{1}{3}\pi\sqrt{43}} \approx 960 \quad = 12\,(9^2-1),$$
$$e^{\frac{1}{3}\pi\sqrt{67}} \approx 5280 \quad = 12\,(21^2-1),$$
$$e^{\frac{1}{3}\pi\sqrt{163}} \approx 640320 = 12\,(231^2-1);$$
while
$$e^{\frac{1}{2}\pi\sqrt{19}} \div \sqrt{19} \approx 216,$$
$$e^{\frac{1}{2}\pi\sqrt{43}} \div \sqrt{43} \approx 216\times 21 \quad = 216\times 7\times 3,$$
$$e^{\frac{1}{2}\pi\sqrt{67}} \div \sqrt{67} \approx 216\times 217 \quad = 216\times 7\times 31,$$
$$e^{\frac{1}{2}\pi\sqrt{163}} \div \sqrt{163} \approx 216\times 185801 = 216\times 7\times 11\times 19\times 127.$$

The values of J corresponding to $\Delta=3$, 11, and 19 afford interesting numerical applications of Klein's *ikosahedron equation*, the corresponding r resolvent equation (*Ikosaëder*, p. 102) having a root $r=3$, 11, 19, respectively. The determination of z, the corresponding *ikosahedron irrationality*, is then an interesting numerical exercise.

$\Delta = 27$. Here $J = -\dfrac{2^9\times 5^3}{3^2}$, $J-1 = -\dfrac{11^2\times 23^2}{3^2}$;
$$\gamma_2 = \frac{40}{3}\,3^{\frac{1}{3}}, \quad \gamma_3 = \frac{253}{27}\,3^{\frac{1}{2}}.$$

These values can be obtained by the cubic transformation of Klein (*Math. Ann.*, XIV., p. 143),
$$J : J-1 : 1 = (\tau-1)(9\tau-1)^3 : (27\tau^2-18\tau-1)^2 : -64\tau,$$
and J' the same function of τ', with $\tau\tau'=1$.

Putting
$$J'=0,$$
then
$$9\tau'=1, \quad \tau=9;$$
and
$$J = -\frac{2^9\times 5^3}{3^2};$$
also
$$G_1 = -\tfrac{1}{2}\,3^{\frac{1}{3}}\,(\sqrt{27}+i),$$
(*Quar. Jour. of Math.*, XXII., p. 136).

$\Delta = 35$. Here $J = -\gamma_2^3$, $J-1 = -27\gamma_3^2$,

where
$$\gamma_2 = \tfrac{8}{3}\sqrt{5}\left\{\tfrac{1}{2}(\sqrt{5}+1)\right\}^4,$$
$$\gamma_3 = \frac{256+115\sqrt{5}}{27}\sqrt{7},$$
(*Quar. Jour. of Math.*, Vol. XXII., p. 137, 1887);

also $$G_1 = -\tfrac{1}{8}\{\tfrac{1}{2}(\sqrt{5}+1)\}^6(\sqrt{35}+i).$$

The manner in which these numerical values were obtained from Kiepert's L-equation for $n = 9$ is there explained.

The values of γ_2 and γ_3 above correspond to the case of
$$K'/K = \sqrt{35}\,; \quad \text{but when } K'/K = \sqrt{(7 \div 5)},$$
we must change the sign of $\sqrt{5}$.

We might have obtained the same value of J by employing Fiedler's modular equation of the 35^{th} order (*Irrationale Modulargleichungen*, p. 97), by putting $\lambda = \kappa'$, $\lambda' = \kappa$, and $x = 2\sqrt{\kappa\kappa'}$; then, in Fiedler's notation,
$$Z_1' = x-1, \quad Z_2' = \tfrac{1}{4}x^2 - x, \quad Z_3' = -\tfrac{1}{4}x^2,$$
and $$Z_0^{(4)} = -\tfrac{1}{2}x + \sqrt{(2x-x^3)}\,;$$
so that, substituting in his equation, we obtain
$$x^3-5x^2+3x+1+4\sqrt{(2x-x^3)} = 0,$$
$$x^6-10x^5+31x^4-12x^3-x^2-26x+1 = 0,$$
$$(x^3-5x^2+13x-1)^2-20\,(x^2-3x)^2 = 0,$$
$$x^3-(5+2\sqrt{5})\,x^2+(13+6\sqrt{5})\,x-1 = 0\,;$$
and forming the equations for x^2 and x^4,
$$x^6-(19+8\sqrt{5})\,x^4+(339+152\sqrt{5})\,x^2-1 = 0,$$
$$x^{12}-3x^8+(230403+103040\sqrt{5})\,x^4-1 = 0,$$
$$(x^4-1)^3+(230400+103040\sqrt{5})\,x^4 = 0.$$
Then $$J = -\frac{4}{27}\frac{(1-x^4)^3}{x^4} = -\frac{4}{27}(230400+103040\sqrt{5})$$
$$= -\frac{2^9\times 5\sqrt{5}}{3^3}\left(\frac{\sqrt{5}+1}{2}\right)^{12}.$$

The same values could also have been obtained by combining Schröter's or Russell's modular equation of the 5^{th} order with Gutzlaff's of the 7^{th} order, but examples of this method will occur hereafter.

$\Delta = 43$. Here $J = -2^{12}\times 5^3$, $J-1 = -3^3\times 21^2\times 43$;

and $$\gamma_2 = -\sqrt[3]{J} = 80, \quad \gamma_3 = 21\sqrt{43}\,;$$
$$\gamma_2+1 = 3^4; \quad \gamma_2^2-\gamma_2+1 = 3\times 7^2\times 43\,;$$

obtained in Hermite's manner by approximate numerical calculation, or obtainable in his manner from the modular equation for $n = 11$.

We notice that when J is an integer, then γ_2+1 is the square of a number which is a multiple of 3, while γ_3 has a factor 7; these considerations are useful in determining the value of J by approximate numerical calculation for high values of Δ.

For $$\Delta = 43, \quad G_1 = -3(\sqrt{43}+i),$$

(*Quar. Jour. of Math.*, XXII., p. 171).

$\Delta = 51$. The value obtained by Dr. L. Kiepert for J is

$$J = -64(5+\sqrt{17})^3(\sqrt{17}+4)^2$$

$$= -256(3\sqrt{17}+11)(\sqrt{17}+4)^3,$$

and then $$J-1 = -7^2(128+31\sqrt{17})^2,$$

$$\gamma_3 = \frac{7\sqrt{3}}{3^2}(128+31\sqrt{17}).$$

The modular functions for this transformation are intimately associated with Kiepert's functions for $n = 13$ (*Math. Ann.*, XXVI., p. 381); Kiepert's L-equation (p. 425) having a factor of the form

$$L^4+aL^2+13 = 0,$$

where, according to Kiepert,

$$a = -\tfrac{1}{2}(3\sqrt{17}+1);$$

so that $$L^2 = \tfrac{1}{2}(1+\sqrt{51}\,i)\,\omega,$$

where ω is an imaginary cube root of unity; and the corresponding modular functions will depend on arguments of the 13th part of multiples of the periods.

$\Delta = 59$. Here we shall find it most convenient to employ Hermite's method in *Équations Modulaires*, p. 44, for class 3°, with $n = 17$.

The number of improperly primitive classes, which we shall denote henceforth by the letter p, for the determinant $-\Delta$, is in this case of $\Delta = 59$ equal to 3 (Gauss, *Werke*, t. II., p. 287), so that a cubic equation for α must be expected.

Putting $u/v = t$, we shall find, with Hermite's notation,

$$u^8 = \frac{1}{1-v^8}, \quad u^8 = x,$$

$$\alpha = -\frac{(1-x+x^2)^3}{(x-x^2)^2} = \frac{(1-t^8)^3}{t^8}.$$

Then Sohnke's modular equation of the 17^{th} order (*Crelle*, t. 16),

$$(v-u)^{18}-16uv\,(1-u^8)(1-v^8)\,\{17uv\,(v-u)^6-(v^4-u^4)^2$$
$$+16\,(1+u^4v^4)^2\} = 0,$$

becomes an equation of the 18^{th} order in t,

$$(t-1)^{18}+16\times 17t^2\,(t-1)^6+15\times 16t^9+16\times 34t^5-16t = 0.$$

The corresponding values of Δ are $4n-\rho^2 = 67$, 59, 43, and 19; and from the known integral values of α for $\Delta = 19$, 43, and 67 given previously, we infer the corresponding factors, for

$$\Delta = 19,\quad t^3-\ t^2+\ 3t-1 = 0,$$
$$\Delta = 43,\quad t^3-3t^2+\ 7t-1 = 0,$$
$$\Delta = 67,\quad t^3-7t^2+13t-1 = 0;$$

leaving the factor of the 9^{th} degree

$$t^9-7t^8+22t^7-34t^6+40t^5-28t^4+22t^3-10t^2+11t-1 = 0,$$

for $\Delta = 59$.

Forming from this equation the corresponding equation in t^8, we shall find on putting

$$\frac{(1-t^8)^3}{t^8} = \alpha,$$

a cubic equation for α.

We might also have employed Fiedler's modular equation for $n = 15$, and then the corresponding values of Δ are 59, 51, 35, and 11; and the factors for 11, 35, and 51 can be inferred from the preceding values of α.

Also the modular equation for $n = 13$ might have been employed in Hermite's manner for the case of $\Delta = 51$, solved above by Kiepert, the extraneous factors corresponding to $\Delta = 43$, 27 and 3 being known, and easily divided out.

$\Delta = 67$. Here

$$J = -2^9\times 5^3\times 11^3,\quad J-1 = -27\times 7^2\times 31^2\times 67;$$

obtained from Hermite's *Équations Modulaires*, p. 48;

then $$\gamma_2 = 2^3\times 5\times 11,\quad \gamma_3 = 217\sqrt{67},$$

$$\gamma_2+1 = 3^2\times 7^2,\quad \gamma_2^2-\gamma_2+1 = 3\times 7^2\times 31^2\times 67.$$

The modular functions in this transformation correspond to Kiepert's case of $n = 17$ (*Math. Ann.*, XXVI., p. 428), the corresponding L-equation having the factor

$$L^4+L^3+17\,;$$

and the associated modular functions have as arguments the 17th parts of multiples of the periods.

$\Delta = 75 = 3\times 5^2$. Here

$$J = -64\sqrt{5}\,(31\sqrt{5}+69)^3.$$

This is obtained by Klein's quintic transformation (*Math. Ann.*, XIV., p. 143; *Proc. Lond. Math. Soc.*, Vol. IX., p. 126),

$$J : J-1 : 1 = (\tau^2-10\tau+5)^3 : (\tau^2-22\tau+125)(\tau^2-4\tau-1)^2 : -1728\tau\,;$$

and J' the same function of τ', with $\tau\tau' = 125$.

Putting $J' = 0$, then

$$\tau'^2-10\tau'+5 = 0,$$

$$\tau' = 5-2\sqrt{5},$$

$$\tau = 25\sqrt{5}\,(\sqrt{5}+2),$$

leading to the value of J above.

Then $$\gamma_2 = 4\times 5^3\,(69+31\sqrt{5}),$$

$$\gamma_3 = \tfrac{1}{9}\sqrt{3}\,(4352\sqrt{5}+9729).$$

This transformation is associated with Kiepert's transformation for $n = 19$ (*Math. Ann.*, XXVI., p. 428), and the corresponding L-equation has a factor of the form

$$L^4+aL^2+19 = 0.$$

$\Delta = 83$. Employing Hermite's method with $n=23$, and Schröter's or Russell's modular equation

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt[3]{4}\,\sqrt[12]{(\kappa\lambda\kappa'\lambda')} = 1,$$

then, putting $$\sqrt[3]{4}\,\sqrt[12]{(\kappa\lambda\,\kappa'\lambda')} = 2s,$$

from Hermite's equations (*Équations Modulaires*, p. 44),

$$u^8 = \kappa^2 = x,\quad 1-v^8 = \lambda'^2 = \frac{1}{x},$$

$$\kappa'^2 = 1-x,\quad \lambda^2 = \frac{x-1}{x},$$

so that $$\kappa^2\lambda'^2 = 1,$$

we find $$\kappa^2\lambda^2+\kappa'^2\lambda'^2 = x+\frac{1}{x}-2 = -\kappa^2\lambda^2\kappa'^2\lambda'^2$$
$$= -256s^{24};$$
$$\kappa\lambda+\kappa'\lambda' = \sqrt{(32s^{12}-256s^{24})};$$
$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = \sqrt{\left[4s^3+\sqrt{\{8s^6+\sqrt{(32s^{12}-256s^{24})}\}}\right]};$$
so that $$\sqrt{\{8s^6+\sqrt{(32s^{12}-256s^{24})}\}} = 1-4s+4s^2-4s^3,$$
or $$\sqrt{(32s^{12}-256s^{24})} = 1-8s+24s^2-40s^3+48s^4-32s^5+8s^6,$$
or $$1-16s+112s^2-464s^3+1312s^4-2752s^5+4432s^6$$
$$-5504s^7+5248s^8-3712s^9+1792s^{10}-512s^{11}$$
$$+32s^{12}+256s^{24} = 0,$$
an equation of the 24^{th} degree in s, for $\Delta = 11$, 43, 67, 83, and 91.

Putting $$\beta = \frac{1-2s-2s^2-2s^3}{2s^2},$$
we shall find that this becomes an equation of the 8^{th} degree in β, and that

$$\beta = 0, \quad \text{for} \quad \Delta = 67;$$
$$\beta = -1, \quad \text{for} \quad \Delta = 43;$$
$$\beta = -2, \quad \text{for} \quad \Delta = 11;$$
$$\beta = \tfrac{1}{2}(\sqrt{13}-1), \quad \text{for} \quad \Delta = 91.$$

The equation in β will therefore be of the form
$$\beta(\beta+1)(\beta+2)(\beta^2+\beta-3)(\beta^3+A\beta^2+B\beta+C) = 0;$$
and we easily find $A = 4, \quad B = 2, \quad C = -5;$
so that the cubic equation
$$\beta^3+4\beta^2+2\beta-5 = 0$$
having the discriminant $83 \div 27$, gives the value of β for $\Delta = 83$; and, forming the equation in t^8 or s^{24}, t being connected with s by the equations
$$x+\frac{1}{x}-2 = -256s^{24} = -t^8,$$
$$x+\frac{1}{x}-1 = 1-256s^{24} = 1-t^8,$$
we obtain a cubic equation for
$$a = \frac{(1-t^8)^3}{t^8} = \frac{(1-256s^{24})^3}{256s^{24}}.$$
A cubic for a was to be expected, as $p = 3$ for the determinant -83.

$\Delta = 91 = 7 \times 13$. Here

$$J = -\gamma_2^3, \quad J - 1 = -27\gamma_3^2,$$

where $\gamma_2 = 908 + 252\sqrt{13}$, $\gamma_3 = 11\sqrt{7}\,(2\sqrt{13}+7)(5\sqrt{13}+18)$.

These values were obtained originally by calculating the approximate values of

$$\gamma_2 + 1 = (6\sqrt{13}+21)^2 = 9\,(2\sqrt{13}+7)^2,$$

$$\gamma_2^2 - \gamma_2 + 1 = 3 \times 7 \times 11^2 \left(\frac{\sqrt{13}+3}{2}\right)^6,$$

and the values of γ_2' corresponding to a change of sign of $\sqrt{13}$, and

$$K'/K = \sqrt{(13 \div 7)}.$$

Calculating the approximate values of

$$12\gamma_2 \approx e^{\frac{1}{3}\pi\sqrt{(91)}}, \quad 12\gamma_2' \approx e^{\frac{1}{3}\pi\sqrt{(13 \div 7)}},$$

we find
$$\gamma_2 + \gamma_2' \approx 1816,$$

$$\gamma_2\gamma_2' \approx -1088,$$

so that we may guess that γ_2, γ_2' are the roots of the quadratic

$$\gamma^2 - 1816\gamma - 1088 = 0.$$

These values of γ_2 or γ_2' in Kiepert's L-equation for $n = 23$ will make the equation have a factor of the form

$$L^2 + aL + 23 = 0.$$

$\Delta = 99 = 3^2 \times 11$. Here J is obtained by performing the cubic transformation on

$$J' = -\frac{2^9}{3^3},$$

corresponding to
$$\Delta = 11.$$

With Klein's form, putting $\tau' = \dfrac{x}{27}$,

$$\frac{(x-27)(x-243)^3}{2^6 \times 3^3 x^3} = \frac{2^9}{3^3},$$

or
$$(x-27)(x-243)^3 = 2^{15}x^3.$$

Put
$$x - 27 = y^3,$$

and extract the cube root; then

$$y^4 - 32y^3 - 216y - 864 = 0,$$

or
$$(y^2-16y-30)^2 = 196\,(y+3)^2,$$
$$y^2-16y-30 = \pm 14\,(y+3),$$
$$(y^2-2y+12)(y^2-30y-72) = 0,$$
$$y = 1+\sqrt{11}\,i, \quad \text{or} \quad 3\,(\sqrt{33}+5);$$
$$r' = \frac{x}{27} = 1+\frac{y^3}{27} = 1+(\sqrt{33}+5)^3 = 27\,(23+4\sqrt{33});$$
$$r = \frac{23-4\sqrt{33}}{27} = \frac{(2\sqrt{3}-\sqrt{11})^2}{27},$$

and these values of r will presumably give the required value of J.

$\Delta = 107$, a prime, notyet solved, but depending on $n = 27$ (Kiepert, *Math. Ann.*, XXXII., p. 67).

$\Delta = 115 = 5\times 23$. Here
$$J = -\gamma_2^3, \quad J-1 = -27\gamma_3^2,$$
where
$$\gamma_2 = 3140+1404\sqrt{5},$$
$$\gamma_3 = (6\sqrt{5}+13)(378+169\sqrt{5})\sqrt{23},$$
$$\gamma_2+1 = (18\sqrt{5}+39)^2 = 9\,(6\sqrt{5}+13)^2,$$
$$\gamma_2^2-\gamma_2+1 = 3\times 23\,(378+169\sqrt{5})^2.$$

These numerical values are obtained by the combination of the modular equations of the 5th and 23rd order, as explained below.

Combine Mr. Russell's modular equations of the 5th and 23rd order,
$$\kappa\lambda'+\kappa'\lambda+2\sqrt[3]{(4\kappa\lambda\kappa'\lambda')} = 1,$$
$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt[3]{4}\,\sqrt[12]{(\kappa\lambda\kappa'\lambda')} = 1;$$
putting
$$4\kappa\lambda\kappa'\lambda' = x^{12}.$$

Then, from the equation of the 5th order,
$$\kappa\lambda'+\kappa'\lambda = 1-2x^4,$$
$$(\kappa'\lambda+\kappa'\lambda')^2 = 1+4\kappa\lambda\kappa'\lambda'-(\kappa\lambda'+\kappa'\lambda)^2$$
$$= 1+x^{12}-(1-2x^4)^2 = 4x^4-4x^8+x^{12};$$
$$\kappa\lambda+\kappa'\lambda' = 2x^2-x^6,$$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=\sqrt{2}\,x,$$

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}=\sqrt[4]{2}\,\sqrt{(x+x^3)}=1-\sqrt{2}\,x,$$

from the equation of the 23rd order.

Therefore $$\sqrt{2}\,x^3-2x^2+3\sqrt{2}\,x-1=0.$$

Now, $$\kappa'\lambda+\kappa\lambda'=1-2x^4,$$
$$\kappa\lambda+\kappa'\lambda'=2x^2-x^6;$$

so that, by multiplication,

$$\kappa\kappa'+\lambda\lambda'=(1-2x^4)(2x^2-x^6)=x^2(2-5x^4+2x^8);$$

$$(\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'})^2=2x^2(1-x^4)^2,$$

$$(-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'})^2=2x^2\left\{(1-x^4)^2-x^4\right\}.$$

Therefore $$\sqrt{2\kappa\kappa'}=x-x^5-\sqrt{(x^2-3x^6+x^{10})},$$

$$\sqrt{2\lambda\lambda'}=x-x^5+\sqrt{(x^2-3x^6+x^{10})};$$

$$\sqrt[4]{2\kappa\kappa'}=\sqrt{\frac{x+x^3-x^5}{2}}-\sqrt{\frac{x-x^3-x^5}{2}},$$

$$\sqrt[4]{2\lambda\lambda'}=\sqrt{\frac{x+x^3-x^5}{2}}+\sqrt{\frac{x-x^3-x^5}{2}}.$$

Now, since $$\sqrt{2}\,x^3-2x^2+3\sqrt{2}\,x-1=0,$$

therefore we shall obtain the cubic equations for $t=\sqrt[4]{4\kappa\kappa'}=2s^3$,

$$t^3-\sqrt{5}\,t^2+(18+7\sqrt{5})\,t-1=0,$$

$$2s^3-(3+\sqrt{5})\,s^2+(3+\sqrt{5})\,s-1=0;$$

also $$\sqrt{2}\,x=\left(\frac{\sqrt{5}+1}{2}\right)^2(1+\sqrt[4]{4\kappa\kappa'})=\left(\frac{\sqrt{5}-1}{2}\right)^2(1+\sqrt[4]{4\lambda\lambda'}).$$

Forming the equations for t^2, t^4, and t^8,

$$t^6+(31+14\sqrt{5})\,t^4+(569+254\sqrt{5})\,t^2-1=0,$$

$$t^{12}-(803+360\sqrt{5})\,t^8+(646403+23908\sqrt{5})\,t^4-1=0,$$

$$t^{24}-3t^{16}+(835673068803+373724357760\sqrt{5})\,t^8-1=0;$$

so that $$J=-\frac{4}{27}\,\frac{(1-t^8)^3}{t^8}$$

$$=-\frac{4}{27}(835673068800+373724357760\sqrt{5})$$

$$=-2^6\times5\sqrt{5}\,(157\sqrt{5}+351)^3,$$

$$\gamma_2 = 3140+1404\sqrt{5},$$
$$\gamma_2+1 = 9\,(6\sqrt{5}+13)^2,$$
$$\gamma_2^2-\gamma_2+1 = 3\times 23\,(378+169\sqrt{5})^2,$$
$$\gamma_3 = (6\sqrt{5}+13)(378+169\sqrt{5})\sqrt{23}.$$

The corresponding modular functions are those of the 29^{th} part of multiples of the periods, and Kiepert's L-equation for $n = 29$ with these values of γ_2 and γ_3 has a factor of the form L^4+aL^2+29.

$\Delta = 123 = 3\times 41$ can be solved by a combination of the modular equations of the 3^{rd} and 41^{st} order, or by using Hermite's method with $n = 23$.

$\Delta = 131$, a prime, not yet solved.

$\Delta = 139$, a prime, not yet solved.

$\Delta = 147 = 3\times 7^2$ can be solved by employing Klein's transformation of the 7^{th} order (*Proc. Lond. Math. Soc.*, Vol. IX., p. 125; *Math. Ann.*, XIV., p. 143) with $J' = 0$.

$\Delta = 155 = 5\times 31$; combine the modular equations of the 5^{th} and 31^{st} order. Then, as for $\Delta = 115$, if

$$x^{12} = 4\kappa\lambda\,\kappa'\lambda',$$
$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = \sqrt[4]{2}\,\sqrt{(x+x^5)}.$$

Then, in Russell's modular equation of the 31^{st} order (*Proc. Lond. Math. Soc.*, Nov. 10, 1887),

$$(P^2-4Q)^2-4PR = 0,$$

we must put
$$P = \sqrt[4]{2}\,\sqrt{(x+x^5)}+1,$$
$$Q = \frac{x^6}{\sqrt{2}}+\sqrt[4]{2}\,\sqrt{(x+x^5)},$$
$$R = \frac{x^3}{\sqrt{2}};$$

which will lead to the required result.

$\Delta = 163$. Here $J = -\gamma_2^3,\quad J-1 = -27\gamma_3^2,$

where
$$\gamma_2 = 53360,\quad \gamma_3 = 185801\sqrt{163},$$
$$\gamma_2+1 = 3^2\times 7^2\times 11^2,\quad \gamma_2^2-\gamma_2+1 = 3\times 19^2\times 127^2\times 163;$$

and then
$$2s^3-4s^2+6s-1 = 0.$$

These values were inferred by approximate calculation from

Hermite's formula (*Équations Modulaires*, p. 48; H. J. S. Smith, *Report* (1865) *on the Theory of Numbers*, p. 374), the calculation being very much abbreviated from the consideration that γ_2+1 is the square of a number which is a multiple of 3.

The value of $\Delta=163$ appears to be the highest for which, according to Hermite's canon, the absolute invariant J is an integer, and so for the present we terminate at this point the series of values of Δ in Class A.

Class *B*.

$\Delta \equiv 7$, mod. 8.

This is the class, Hermite's class 4°, for which no simple numerical invariant has yet been discovered; and, according to Hermite and Joubert, the only modular function to seek to determine numerically is $\sqrt[4]{(\kappa\kappa')}$, or sometimes $\sqrt[12]{(\kappa\kappa')}$.

Jacobi's modular equations between u and v are not suitable for this purpose, but the $\kappa\lambda-\kappa'\lambda'$ equations of Mr. Robert Russell become immediately of the requisite form on putting $\lambda=\kappa'$, $\lambda'=\kappa$.

The corresponding complex multiplication formulas are given in the *Quar. Jour. of Math.*, Vol. XXII., p. 143, where Weierstrass's notation is employed.

Guided, however, by Hermite's *Équations Modulaires*, p. 44, we may in this Class 4°, employ a complex multiplier,

$$\frac{1}{M}=\tfrac{1}{4}(-\rho+\sqrt{\Delta}\,i),$$

where ρ is an odd integer, and then, following Mr. G. H. Stuart's method (*Quar. Journal of Math.*, Vol. XX., p. 38), we can express

$$y=\wp\frac{u}{M} \text{ in terms of } x=\wp u,$$

by means of an irrational formula; or, with Jacobi's notation, we can express

$$y=\operatorname{cn}\frac{u}{M} \text{ in terms of } x=\operatorname{cn} u;$$

which, in the simplest case of $\Delta=7$, and $\rho=1$, becomes

$$\frac{\sqrt{i}\,c-y}{\sqrt{i}\,c+y}=\sqrt[4]{(-ic)}\sqrt{\frac{1+x}{ic-x}};$$

where $$c=8+3\sqrt{7};$$

leading to the differential relation

$$\frac{dy}{\sqrt{(1-y^2\,.\,y^2+c^2)}}=\frac{\tfrac{1}{4}(-1+\sqrt{7}\,i)\,dx}{\sqrt{(1-x^2\,.\,x^2+c^2)}}$$

(*Proc. Camb. Phil. Soc.*, Vol. IV.); and generally, for any value of $\Delta = 8n-1$,

$$\frac{\sqrt{ic}-y}{\sqrt{ic}+y} = A\left(\frac{1+x}{ic-x}\right)^{\frac{1}{2}} \Pi \left\{\frac{\text{cn}\, 2s\omega+x}{\text{cn}\,(2s+1)\,\omega+x}\right\}^{\frac{1}{2}};$$

where $\omega = (K+iK')/n,$

connecting $x = \text{cn}\, u$, and $y = \text{cn}\, \frac{1}{4}(-1+\sqrt{\Delta}\, i)\, u$, and leading to the differential relation

$$\frac{dy}{\sqrt{(1-y^2 \,.\, y^2+c^2)}} = \frac{\frac{1}{4}(-1+\sqrt{\Delta}\, i)\, dx}{\sqrt{(1-x^2 \,.\, x^2+c^2)}},$$

where $c = \kappa'/\kappa$; or, in Weierstrass's notation,

$$\frac{y-\wp(\omega_2+\frac{1}{2}\omega_3)}{y-\wp\frac{1}{2}\omega_3} = \Pi\left[\frac{x-\wp\{\omega_2+(2s+1)\,\omega_1/n\}}{x-\wp(\omega_2+2s\omega_1/n)}\right]^{\frac{1}{2}},$$

connecting $x = \wp u$, and $y = \wp\frac{1}{4}(-1+\sqrt{\Delta}\, i)\, u$, and leading to the differential relation

$$\frac{dy}{\sqrt{(4y^3-g_2y-g_3)}} = \frac{\frac{1}{4}(-1+\sqrt{\Delta}\, i)\, dx}{\sqrt{(4x^3-g_2x-g_3)}}.$$

The modular functions required in the general case are then the n^{th} parts of multiples of the periods, where

$$n = \tfrac{1}{8}(\Delta+\rho^2),$$

an integer; and then $\Delta = 8n-\rho^2,$

thus giving the interpretation of the formula for class 4°, p. 44, of the *Équations Modulaires*, in which

$$u^2 = \frac{1-v^4}{2iv^2}, \quad u^8 = 1-x,$$

in the modular equation of the n^{th} degree, connecting Jacobi's u and v.

$\Delta = 7$. From Gutzlaff's modular equation of the 7th order,

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = 1,$$

we obtain, putting $\kappa = \lambda'$, $\kappa' = \lambda$,

$$2\sqrt[4]{\kappa\kappa'} = 1.$$

$\Delta = 15$. In this case Joubert (*Comptes Rendus*, t. 50) gives the value

$$\sqrt[4]{\kappa\kappa'} = \sin 18°,$$

for $K'/K = \sqrt{15}$;

while $\sqrt[4]{\kappa\kappa'} = \sin 54°,$

for $K'/K = \sqrt{(5 \div 3)}.$

These values of $\sqrt[4]{(\kappa\kappa')}$ can also be obtained from Fiedler's modular equation for $n = 15$.

$\Delta = 23$. Putting $\lambda = \kappa'$, $\lambda' = \kappa$ in Mr. Russell's modular equation of the 23rd order,

$$P^3 - 4R = 0,$$

or

$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} + \sqrt[3]{4}\sqrt[12]{\kappa\lambda\kappa'\lambda'} - 1 = 0,$$

then

$$2\sqrt[4]{\kappa\kappa'} + \sqrt[3]{4}\sqrt[6]{\kappa\kappa'} - 1 = 0,$$

or

$$x^3 + x^2 - 1 = 0,$$

where

$$x^{12} = 16\kappa\kappa';$$

the real root of this cubic being given by

$$\frac{1}{x} = \sqrt[3]{\left(\frac{3\sqrt{3}+\sqrt{23}}{6\sqrt{3}}\right)} + \sqrt[3]{\left(\frac{3\sqrt{3}-\sqrt{23}}{6\sqrt{3}}\right)}.$$

$\Delta = 31$. Putting $\lambda = \kappa'$, $\lambda' = \kappa$ in Mr. Russell's modular equation of the 31st order,

$$(P^2 - 4Q)^2 - 4PR = 0,$$

then

$$P = 2\sqrt[4]{\kappa\kappa'} + 1, \quad Q = \sqrt{\kappa\kappa'} + 2\sqrt[4]{\kappa\kappa'}, \quad R = \sqrt{\kappa\kappa'};$$

and

$$P = x^3 + 1, \quad Q = \tfrac{1}{4}x^6 + x^3, \quad R = \tfrac{1}{4}x^6,$$

if

$$x^{12} = 16\kappa\kappa';$$

so that

$$x^9 - 3x^6 + 4x^3 - 1 = 0,$$

$$(x^3 - 1)^3 = -x^3,$$

$$x^3 - 1 = -x,$$

$$x^3 + x - 1 = 0,$$

a cubic for x, the real root being

$$x = \sqrt[3]{\left(\frac{31+3\sqrt{3}}{6\sqrt{3}}\right)} - \sqrt[3]{\left(\frac{31-3\sqrt{3}}{6\sqrt{3}}\right)}.$$

$\Delta = 39 = 3 \times 13$. The equation for $x = 2\sqrt[4]{\kappa\kappa'}$ is given by Joubert in the *Comptes Rendus*, t. 50, in the form

$$x^4 + 2x^3 + 4x^2 + 3x - 1 = 0,$$

so that

$$(x^2 + x + \tfrac{3}{2})^2 = \tfrac{13}{4},$$

or

$$2x = -1 + \sqrt{(2\sqrt{13} - 5)} = -1 + \{\tfrac{1}{2}(\sqrt{13} - 1)\}^{\frac{3}{2}}.$$

$\Delta = 47$. The modular equation of the 47^{th} order has been given by Hurwitz in the *Math. Ann.*, Vol. XVII., p. 69, in the form

$$\{2(\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}-1)-\sqrt[3]{4}\sqrt[12]{(\kappa\lambda\,\kappa'\lambda')}\}^2$$
$$= 8(\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+1)-7\sqrt[3]{16}\sqrt[6]{(\kappa\lambda\,\kappa'\lambda')};$$

and Russell's form, given in the *Proc. Lond. Math. Soc.*, Vol. XIX., p. 111, is

$$(P^2-4Q)^3-4PR(7P^2+24Q)-128R^2=0.$$

Hence, if $K'/K=\sqrt{47}$, we have, putting

$$\lambda=\kappa', \quad \lambda'=\kappa,$$

$$(4\sqrt[4]{\kappa\kappa'}-\sqrt[3]{4}\sqrt[6]{\kappa\kappa'}-2)^2 = 16\sqrt{\kappa\kappa'}-7\sqrt[3]{16}\sqrt[3]{\kappa\kappa'}+8;$$

or, if $16\kappa\kappa'=x^{12}$,

$$(2x^3-x^2-2)^2=4x^6-7x^4+8,$$

or

$$x^5-2x^4+2x^3-x^2+1=0.$$

$$16\,\kappa\kappa'=y^6,$$

then

$$x=\sqrt{y},$$

and the quintic for y is

$$y^5+3y^2+2y-1=0,$$

a *Hauptgleichung* (Klein, *Icosaëder*) which has been solved by Prof. G. Paxton Young, the solution being given in the *American Journal of Mathematics*, Vol. X., p. 108.

The quintic has only one real root, which is

$$u_1+u_2+u_3+u_4,$$

where
$$u_1^5=\tfrac{13}{500}(15+7\sqrt{5})+\tfrac{1}{625}\sqrt{\{\tfrac{47}{8}(21125+9439\sqrt{5})\}},$$
$$u_2^5=\tfrac{13}{500}(15-7\sqrt{5})-\tfrac{1}{625}\sqrt{\{\tfrac{47}{8}(21125-9439\sqrt{5})\}},$$
$$u_3^5=\tfrac{13}{500}(15-7\sqrt{5})+\tfrac{1}{625}\sqrt{\{\tfrac{47}{8}(21125-9439\sqrt{5})\}},$$
$$u_4^5=\tfrac{13}{500}(15+7\sqrt{5})-\tfrac{1}{625}\sqrt{\{\tfrac{47}{8}(21125+9439\sqrt{5})\}};$$

the other imaginary roots of the quintic being of the form

$$\epsilon u_1+\epsilon^2 u_2+e^3 u_3+\epsilon^4 u_4,$$

where ϵ is an imaginary fifth root of unity.

We might have employed the modular equation of the 47^{th} order, given by Hurwitz, in the *Math. Ann.*, xvii., p. 69, of the form

$$\{2(\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}-1)-\sqrt[3]{4}\sqrt[12]{\kappa\lambda\kappa'\lambda'}\}^2$$
$$=8(\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+1)-7\sqrt[3]{16}\sqrt[6]{\kappa\lambda\kappa'\lambda'}.$$

Putting $\lambda=\kappa',\quad \lambda'=\kappa,\quad \text{or}\quad \kappa'\lambda'=\kappa\lambda=\kappa\kappa',$

in this equation, then $K'/K=\sqrt{47}$;

and $(4\sqrt[4]{\kappa\kappa'}-\sqrt[3]{4}\sqrt[6]{\kappa\kappa'}-2)^2=16\sqrt{\kappa\kappa'}-7\sqrt[3]{16}\kappa\kappa'+8$;

or, if $16\kappa\kappa'=x^{12}$,

$$(2x^3-x^2-2)^2=4x^6-7x^4+8,$$

or

$$x^5-2x^4+2x^3-x^2+1=0;$$

and if $x=\sqrt{y}$,

$$y^5+3y^2+2y-1=0,$$

as before.

$\Delta=55=5\times 11$. Combining Schröter's or Russell's modular equations of the 5^{th} and 11^{th} orders,

$$\kappa'\lambda+\kappa\lambda'+2\sqrt[3]{(4\kappa\lambda\kappa'\lambda')}=1,$$
$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+2\sqrt[6]{(4\kappa\lambda\kappa'\lambda')}=1;$$

putting $4\kappa\lambda\kappa'\lambda'=x^{12}$,

then, from the equation of the 5^{th} order,

$$\kappa'\lambda+\kappa\lambda'=1-2x^4,$$
$$\kappa\lambda+\kappa'\lambda'=2x^2-x^6;$$
$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=\sqrt{2}\,x$$
$$=1-2x^2,$$

from the equation of the 11^{th} order; so that

$$2x^2+\sqrt{2}\,x-1=0,$$
$$x=\frac{\sqrt{5}-1}{2\sqrt{2}}.$$

Then, as before, in $\Delta=115$,

$$\kappa\kappa'+\lambda\lambda'=x^2(2-5x^4+2x^8),$$
$$\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'}=\sqrt{2}\,x\,(1-x^4)$$
$$=\frac{7-\sqrt{5}}{8}$$
$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'}=\frac{\sqrt{5}\sqrt{(10\sqrt{5}-18)}}{8}.$$

$\Delta = 63 = 3^2 \times 7$. Performing the cubic transformation on the modulus for

$$K'/K = \sqrt{7},$$

we shall obtain the required modulus.

Putting
$$x = 2\sqrt[4]{\kappa\kappa'},$$

the equation is written by Joubert (*Comptes Rendus*, t. 50)

$$(x^2-x+5)^2-21\,(x-1)^2 = 0,$$

whence x can readily be determined.

For $\Delta = 71$ and 79, consult Dr. E. W. Fiedler, *Ueber eine besondere Classe irrationaler Modulargleichungen der elliptischen Functionen;* Zürich, 1885.

Putting
$$2\sqrt[4]{\kappa\kappa'} = x,$$

then in Fiedler's notation

$$Z_1 = x \mp 1,$$
$$Z_2 = \tfrac{1}{4}x^2 \mp x, \quad Z_2' = \pm 2x+1,$$
$$Z_3 = \mp \tfrac{1}{4}x^2;$$

and for $\Delta = 71$, Fiedler's equation for x is

$$(x-1)^9+x^2\left\{(2x+1)^3+9\,(x+1)^2\,(2x+1)^2+21\,(x+1)^4\,(2x+1) + 12\,(2x+1)^6\right\} - (x-1)^3\,x^4\left\{6\,(2x+1)+7\,(x-1)^2\right\} + x^6 = 0;$$

while for $\Delta = 79$, the equation for x is

$$(-2x+1)^5-(x+1)\,x^2\left\{(x+1)^6+10\,(x+1)^4\,(-2x+1) + 28\,(x+1)^2\,(-2x+1)^2+21\,(-2x+1)^3\right\} - x^4\left\{7\,(x+1)^4+26\,(x+1)^2\,(-2x+1)+24\,(-2x+1)^2\right\} + 8\,(x+1)\,x^6 = 0;$$

equations of the 9th and 8th degree respectively.

$\Delta = 87 = 3 \times 29$ can be solved by the combination of the modular equations of the 3rd and 29th order, but this last modular equation has not yet been calculated by Mr. Russell or others in a convenient form, the form given by Schröter being unsuitable for our purposes.

$\Delta = 95 = 5 \times 19$.

Putting, as before
$$4\kappa\lambda\kappa'\lambda' = x^{12},$$

and
$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = \sqrt{2}\,x,$$

from the modular equation of the 5^{th} order, then, in Mr. Russell's notation,

$$P = \sqrt{2}\,x - 1,$$
$$Q = \tfrac{1}{2}x^6 - \sqrt{2}\,x,$$
$$R = -\tfrac{1}{2}x^6,$$

which, substituted in his modular equation of the 19^{th} order, will give an equation of the 12^{th} degree for x.

Class C.

$$\Delta \equiv 1, \text{ mod. } 4.$$

This is Hermite's Class 1° (*Équations Modulaires*, p. 44), and the absolutely simplest numerical invariant, according to Hermite, is

$$a = -\frac{(x+1)^4}{x(x-1)^2},$$

which, on replacing x by $1-\kappa^{-2}$, becomes

$$a = \frac{(1-4\kappa^2\kappa'^2)^2}{\kappa^2\kappa'^2}, \quad a+16 = \frac{(1+4\kappa^2\kappa'^2)^2}{\kappa^2\kappa'^2};$$

so that, putting $\beta = \tfrac{1}{2}\sqrt{a}, \quad \gamma = \tfrac{1}{2}\sqrt{(a+16)}$,

$$\beta = \frac{1}{2\kappa\kappa'} - 2\kappa\kappa', \quad \gamma = \frac{1}{2\kappa\kappa'} + 2\kappa\kappa';$$

and, according to Hermite, β or γ are in a great many cases integers.

With the complex multiplier

$$\frac{1}{M} = \tfrac{1}{2}(-\rho + \sqrt{\Delta}\, i),$$

where ρ is an odd integer, we can connect

$$y = \operatorname{cn}\frac{u}{M} \text{ with } x = \operatorname{cn} u,$$

by means of an irrational relation in Mr. G. H. Stuart's manner, as explained in the *Quarterly Journal of Mathematics*, Vol. XXII., p. 147; and then the modular functions involved in these relations are functions of the n^{th} part of multiple of the periods, where

$$n = \tfrac{1}{2}(\Delta + \rho^2),$$

an integer; so that $\Delta = 2n - \rho^2$,

as in Hermite's formulas (*Équations Modulaires*, p. 44).

Or, in Weierstrass's notation, we can connect

$$x = \wp u \text{ and } y = \wp \tfrac{1}{2}(-1+\sqrt{\Delta}\, i),$$

where $\Delta = 4n+1, \quad m = 2n+1,$

by means of the relation

$$\frac{y-\wp\frac{1}{2}\omega_2'}{y-\wp\frac{1}{2}\omega_2} = \left(\frac{x-e_1}{x-e_3}\right)^2 \Pi \frac{x-\wp\{\omega_2-(4r+1)\,\omega_3/m\}}{x-\wp\,(4r+1)\,\omega_3/m},$$

a transformation of the order $n+\frac{1}{2} = \frac{1}{2}m$.

$\Delta = 1$. Then $\kappa' = \kappa$, and $\kappa' = \kappa = \sin 45°$;

also $J = 1, \quad J-1 = 0,$ so that $g_2 = 0.$

Then $\beta = 0, \quad \alpha = 0, \quad 2\kappa\kappa' = 1.$

$\Delta = 5$. Here $\beta = 2^2$, $\alpha = 2^6$; obtained from Russell's modular equation of the 5th order, with $\lambda = \kappa'$, $\lambda' = \kappa$. Then

$$2\kappa\kappa' + 2\sqrt[3]{4\kappa^2\kappa'^2} - 1 = 0,$$

or, putting $\sqrt[3]{2\kappa\kappa'} = x,$

$$x^3+2x^2-1 = 0, \quad (x+1)(x^2+x-1) = 0, \quad x = \tfrac{1}{2}(\sqrt{5}-1),$$

$$2\kappa\kappa' = \sqrt{5}-2 = \left(\frac{\sqrt{5}-1}{2}\right)^3 \text{ (Abel).}$$

$\Delta = 9 = 3^2$. Here

$$\alpha = 2^8 \times 3 \text{ (Kronecker)},$$

$$\beta = 8\sqrt{3}, \quad \gamma = 14,$$

or, putting $2\kappa\kappa' = z,$

$$z^2-14z-1 = 0, \quad z = 7-4\sqrt{3} = (2-\sqrt{3})^2 = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4.$$

$\Delta = 13$. Here $\beta = 36$, $\alpha = 2^6 \times 3^4$ (Kronecker);

$$2\kappa\kappa' = 5\sqrt{13}-18 = \left(\frac{\sqrt{13}-3}{2}\right)^3.$$

$\Delta = 17$. Taking Mr. Russell's modular equation for the 17th order, and putting $\lambda = \kappa'$, $\lambda' = \kappa$, there results the equation in $z = 2\kappa\kappa'$,

$$(z-1)(z^2-36z-1)^2\,(z^4-80z^3-98z^2-80z+1) = 0.$$

The factor $z-1 = 0$ corresponds to $\Delta = 1$, and the factor

$$z^2-36z-1 = 0 \text{ to } \Delta = 13; \text{ so that}$$

$$z^4-80z^3-98z^2-80z+1 = 0;$$

$$\gamma^2-80\gamma-100=0,$$

$$\gamma=10\sqrt{17}+40,$$

$$\beta=\frac{1}{z}-z=4\sqrt{(206+50\sqrt{17})},$$

$$a=4\beta^2=2^7(25\sqrt{17}+103).$$

$\Delta=21=3\times 7$. Then (Kronecker)

$$a=2^2\times 3^2(\sqrt{3}+1)^8,$$

$$2\kappa\kappa'=\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^8\left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^2,\ \text{for } K'/K=\sqrt{21},$$

$$2\lambda\lambda'=\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^8\left(\frac{3+\sqrt{7}}{2}\right)^2,\ \text{for } \Lambda'/\Lambda=\sqrt{7}\div 3.$$

$\Delta=25=5^2$. Then

$$\beta=2^4\times 3^2\times\sqrt{5},\quad \gamma=322=2\times 7\times 23;$$

$$2\kappa\kappa'=\left(\frac{\sqrt{5}-1}{2}\right)^{12}.$$

$\Delta=29$. The form of the $(\kappa\lambda, \kappa'\lambda')$ modular equation, according to Mr. Russell, will be

$$P^{15}+R(AP^{12}+BP^{10}Q+CP^8Q^2+DP^6Q^3+EP^4Q^4+FP^2Q^5+G^0Q^6)$$
$$+R^2(HP^9+JP^7Q+KP^5Q^2+LP^3Q^3+MPQ^4)$$
$$+R^3(NP^6+OP^4Q+SP^2Q^2+TQ^3)$$
$$+R^4(UP^3+VPQ)+WR^5=0,$$

connecting $\quad P=x+y-1,\quad Q=xy-x-y,\quad R=-xy,$

where $\quad x=\kappa\lambda,\quad y=\kappa'\lambda';$

and A, B, C, ... U, V, W are numerical coefficients, the values of which have not yet been determined.

Putting $\lambda=\kappa'$, $\lambda'=\kappa$, and then $z=2\kappa\kappa'$; $P=z-1$, $Q=\frac{1}{4}z^2-z$, $R=-\frac{1}{4}z^2$; and, by analogy with the preceding cases, the resulting equation will have a factor $z+1$, and other factors corresponding to previous values of Δ.

Pending the determination of the numerical coefficients in Russell's modular equation of the 29th order, let us determine the numerical values of the modular functions for $\Delta=29$ in Hermite's manner for his Class 1°, by means of the modular equation for $n=19$.

Then the corresponding values of Δ are given by

$$2n-\rho^2 = 37,\ 29,\ 13;$$

of which the solutions for $\Delta = 37$ and 13 are simple and well known.

Putting, in Hermite's notation,

$$\kappa\lambda = u^4v^4 = w^4,$$

then, since
$$\kappa^2 = u^8 = x,$$

and
$$u^4 = \frac{v^4-1}{v^4+1},$$

$$\kappa = \frac{\lambda-1}{\lambda+1},\quad \lambda = \frac{1+\kappa}{1-\kappa};$$

$$w^4 = \kappa\lambda = \sqrt{x}\,\frac{1+\sqrt{x}}{1-\sqrt{x}};$$

$$\kappa'\lambda' = 2i\sqrt[4]{x}\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}} = 2i\sqrt{\kappa\lambda}.$$

Then, in Russell's notation, with

$$\kappa\lambda = w^4,\quad \kappa'\lambda' = 2iw^2,\quad \epsilon^4 = -1,$$

$$P = w\,(t+\epsilon\sqrt{2}),$$

$$Q = w^3\,(\epsilon\sqrt{2}\,t-1),$$

$$R = -\epsilon\sqrt{2}\,w^3;$$

where
$$t = w-\frac{1}{w} = \sqrt[4]{\kappa\lambda}-\frac{1}{\sqrt[4]{\kappa\lambda}}.$$

Then
$$t^2+2 = w^2+w^{-2}$$

$$= \frac{1+x}{\sqrt[4]{x}\sqrt{(1-x)}} = \epsilon\sqrt[4]{a} = (1+i)\sqrt{\beta},\quad \epsilon^4 = -1.$$

With Russell's notation, for $n = 19$,

$$P = w^2+\epsilon\sqrt{2}\,w-1 \quad = \quad w\,(t+1+i),$$

$$Q = \epsilon\sqrt{2}\,w^3-w^2-\epsilon\sqrt{2}\,w = \quad w^2\,(t+it-1),$$

$$R = \quad -\epsilon\sqrt{2}\,w^3 \quad = -w^3\,(1+i);$$

and substituting in

$$P^5-112P^2R+256QR = 0,$$

we obtain

$$(t+1+i)^5+112\,(1+i)\,(t+1+i)^2-256\,(1+i)\,(t+i\ -1) = 0,$$

or

$$t^6+5t^4+92t^3-20t+28+i\,(5t^4+20t^3+132t^2-64t+476)=0.$$

This equation has the factor

$$t^2+8t+16-18i=0,$$

giving
$$t=-4\pm 3\sqrt{2}\,\epsilon$$
$$=-1+3i,\ \text{or}\ -7-3i;$$

and therefore
$$\beta=6^3\quad\text{or}\ 42^2,$$

corresponding to
$$\Delta=13\quad\text{or}\ 37.$$

The remaining cubic factor is

$$t^3-(3-5i)\,t^2+(8-2i)\,t-14+14i=0;$$

so that

$$t^6+(32+26i)\,t^4+(116+192i)\,t^2+392i=0;$$

and since
$$t^2=(1+i)\,\sqrt{\beta-2},$$

therefore
$$\beta^{\frac{3}{2}}+26\beta+44\beta^{\frac{1}{2}}+56=0,$$

a cubic equation with discriminant $32^2\times 29\div 27$; then

$$\beta^3-588\beta^2-976\beta-3136=0;$$

giving
$$\beta=\frac{1}{2\kappa\kappa'}-2\kappa\kappa';$$

and putting $z=2\kappa\kappa'$, the equation for z is

$$z^6+588z^5-979z^4+1960z^3+979z^2+588z-1=0.$$

The knowledge of this factor will be of great assistance in the determination of the numerical factors $A, B, C, \ldots U, V, W$ in the modular equation of the 29th order.

$\Delta=33=3\times 11$. Here

$$a=2^4\times 3\,(300+52\sqrt{33})^2,$$

$$2\kappa\kappa'=\left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6.$$

These values are obtained by the combination of Schröter's or Russell's modular equation of the 11th order,

$$\sqrt{\kappa\lambda'}+\sqrt{\kappa'\lambda}+2\sqrt[6]{(4\kappa\lambda\kappa'\lambda')}=1 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

with the equation of the 3rd order,

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Putting $$4\kappa\lambda\,\kappa'\lambda' = x^6,$$

then $$\sqrt{\kappa\lambda'} + \sqrt{\kappa'\lambda} = 1-2x,$$

$$\kappa\lambda' + \kappa'\lambda = 1-4x+4x^2-x^3,$$

and $$\kappa\lambda + \kappa'\lambda' = 1-x^3,$$

$$(\kappa'\lambda'-\kappa\lambda)^2 = 1-2x^3\,;$$

so that $$1-2x^3+(1-4x+4x^2-x^3)^2 = (\kappa'\lambda'-\kappa\lambda)^2+(\kappa\lambda'+\kappa'\lambda)^2 = 1,$$

or $$(1-4x+4x^2-x^3)^2 = 2x^3,$$

or $$x^6-8x^5+24x^4-36x^3+24x^2-8x+1 = 0,$$

a reciprocal sextic, having the factors

$$(x^2-4x+1)(x^4-4x^3+7x^2-4x+1) = 0.$$

Putting $$x+\frac{1}{x} = y,$$

then $$y^3-8y^2+21y-20 = 0,$$

$$(y-4)(y^2-4y+5) = 0\,;$$

so that $y = 4,\ 2\pm i$.

Taking the real root $$x+\frac{1}{x} = 4,$$

$$x = 2-\sqrt{3} = \tfrac{1}{2}(\sqrt{3}-1)^2\,;$$

then $$\kappa\lambda'+\kappa'\lambda = \sqrt{2}x^{\frac{3}{2}} = \tfrac{1}{2}(\sqrt{3}-1)^3,$$

$$\kappa\lambda+\kappa'\lambda' = 1-x^3 = \tfrac{5}{2}(\sqrt{3}-1)^3,$$

$$(\kappa'+\kappa)(\lambda'+\lambda) = 3(\sqrt{3}-1)^3,$$

$$(\kappa'-\kappa)(\lambda'-\lambda) = 2(\sqrt{3}-1)^3\,;$$

$$(1+2\kappa\kappa')(1+2\lambda\lambda') = 9(\sqrt{3}-1)^6,$$

$$(1-2\kappa\kappa')(1-2\lambda\lambda') = 4(\sqrt{3}-1)^6,$$

$$4(\kappa\kappa'+\lambda\lambda') = 5(\sqrt{3}-1)^6,$$

$$64\,\kappa\lambda\,\kappa'\lambda' = \tfrac{1}{4}(\sqrt{3}-1)^{12},$$

$$16(\kappa\kappa'-\lambda\lambda')^2 = \tfrac{99}{4}(\sqrt{3}-1)^{12},$$

$$4(\lambda\lambda'-\kappa\kappa') = \tfrac{3}{2}\sqrt{11}(\sqrt{3}-1)^6,$$

$$8\kappa\kappa' = \tfrac{1}{2}(10-3\sqrt{11})(\sqrt{3}-1)^6,$$

$$2\kappa\kappa' = \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \text{ for } K'/K = \sqrt{33}\,;$$

$$2\lambda\lambda' = \left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^3 \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6 \text{ for } \Lambda'/\Lambda = \sqrt{(11 \div 3)}.$$

$$\beta = \tfrac{1}{2}\sqrt{\alpha} = \frac{1}{2\kappa\kappa'} - 2\kappa\kappa'$$

$$= (10+3\sqrt{11})\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6 - (10-3\sqrt{11})\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6$$

$$= 156\sqrt{11} + 300\sqrt{3}$$

$$= 4\sqrt{3}\,(75+13\sqrt{33});$$

$$\alpha = 2^4 \times 3\,(75+13\sqrt{33})^2;$$

$$\gamma = \frac{1}{2\kappa\kappa'} + 2\kappa\kappa'$$

$$= 520 + 90\sqrt{33}$$

$$= 10\,(52+9\sqrt{33});$$

$$\alpha + 16 = 2^4 \times 5^2\,(52+9\sqrt{33})^2.$$

Similarly for $\lambda\lambda'$ and the values of α', corresponding to

$$\Lambda'/\Lambda = \sqrt{(11 \div 3)},$$

$$\alpha' = 2^4 \times 3\,(75-13\sqrt{33})^2,$$

$$\alpha' + 16 = 2^4 \times 5^2\,(52-9\sqrt{33}).$$

$\Delta = 37$. Here $\alpha = 2^6 \times 3^4 \times 7^4,$

$$\beta = 2^3 \times 3^2 \times 7^2, \qquad \gamma = 2 \times 5 \times 29\sqrt{37};$$

obtained from Hermite's *Théorie des Équations Modulaires*, Note, p. 50; also by Kronecker, *Berlin Sitz.* 1862; then

$$2\kappa\kappa' = (\sqrt{37}-6)^3, \quad \frac{1}{2\kappa\kappa'} = (\sqrt{37}+6)^3.$$

$\Delta = 41$. Not yet solved; but a quartic equation for α, β, or γ must be expected, as $p = 4$.

$\Delta = 45 = 3^2 \times 5$. Here, by means of the cubic transformation in $\Delta = 5$,

$$\alpha = 2^6\,(17+10\sqrt{3})^4, \quad \alpha + 16 = 80\,(527+304\sqrt{3})^2,$$

$$\beta = 2^3\,(17+10\sqrt{3})^2, \quad \gamma = 2\sqrt{5}\,(527+304\sqrt{3});$$

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^{12}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^4, \quad K'/K = \sqrt{(45)};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}-1}{2}\right)^{12}\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^4, \quad \Lambda'/\Lambda = \sqrt{(9 \div 5)}.$$

$\Delta = 49 = 7^2$. Here, by the 7^{ic} transformation on $\Delta = 1$,

$$a = 2^5 \times 3^4 (3+\sqrt{7})^6 \sqrt{7},$$

Then $$\sqrt[4]{\kappa},\ \sqrt[4]{\kappa'} = \sqrt[8]{2},$$

$$\kappa\kappa' = \left(\frac{\sqrt{7}+1-\sqrt{2}\,\sqrt[4]{7}}{2\sqrt{2}}\right)^{12}$$

(Kronecker, *Berlin Sitz.* 1862 : G. H. Stuart, *Quart. Journal o Math.*, Vol. xx.).

$\Delta = 53$, a prime number not yet solved. Here $p = 3$, so that a cubic for a must be expected.

$\Delta = 57 = 3 \times 19$. We combine the modular equations of the 3rd and 19th order, and put

$$4\kappa\lambda\,\kappa'\lambda' = y^4.$$

Then from the equation of the 3rd order

$$\kappa'\lambda+\kappa\lambda' = 1-y^2,$$

$$\kappa\lambda+\kappa'\lambda' = \sqrt{2}\,y,$$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = \sqrt{(y^2+\sqrt{2}\,y)}.$$

With Russell's notation, we have

$$P = \sqrt{(y^2+\sqrt{2}\,y)}-1,$$

$$Q = \tfrac{1}{2}y^2-\sqrt{(y^2+\sqrt{2}\,y)},$$

$$R = -\tfrac{1}{2}y^2;$$

and then, substituting in his modular equation of the 19th order, we obtain an equation for y, which is reciprocal when rationalised; and then putting

$$y-\frac{1}{y} = \sqrt{2}\,v,$$

we obtain $$v^5+5v^4-46v^3+706v^2-611v+169 = 0,$$

or $$(v+13)(v^4-8v^3+58v^2-48v+13) = 0,$$

or $$(v+13)\left\{(v^2-4v+3)^2+(6v+2)^2\right\} = 0.$$

Taking $$v = -13,$$

$$\frac{1}{y}-y = 13\sqrt{2},$$

$$\frac{1}{y}+y = 3\sqrt{38};$$

$$\kappa\kappa'+\lambda\lambda' = \sqrt{2}\,y\,(1-y^2) = -\,2y^3v$$
$$= 26y^3\,;$$
$$2\sqrt{\kappa\lambda\,\kappa'\lambda'} = y^3\,;$$
$$\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = 3\sqrt{3}\,y,$$
$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'} = 5y.$$

But
$$2y = \sqrt{2}\,(3\sqrt{19}-13),$$

so that
$$2\kappa\kappa' = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6\left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^3,\quad K'/K = \sqrt{57},$$
$$2\lambda\lambda' = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6\left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^3,\quad \Lambda'/\Lambda = \sqrt{(19\div 3.)}$$

$\Delta = 61$. A prime number, not yet solved, but depending in Hermite's manner on $n = 31$.

$\Delta = 65 = 5\times 13$. Combine the modular equations of the 5th and 13th orders; then, if we put

$$4\kappa\lambda\kappa'\lambda' = x^6,$$

we obtain from the equation of the 5th order,

$$\kappa'\lambda+\kappa\lambda' = 1-2x^2,$$
$$\kappa\lambda+\kappa'\lambda' = 2x-x^3\,;$$

and in the equation of the 13th order, with Russell's notation,

$$P = -\,1+2x-x^3,$$
$$Q = -\,2x+x^3-\tfrac{1}{4}x^6,$$
$$R = -\,\tfrac{1}{4}x^6.$$

Substituting these values of P, Q, and R in the modular equation of the 13th order, and dividing out the factors $x+1$, $x^2\pm x+1$, we are left with a reciprocal equation for x, which, on putting

$$\frac{1}{x}+x = y,$$

becomes
$$y^2-5y-10 = 0,$$

so that
$$y = \tfrac{1}{2}\,(\sqrt{65}+5)\,;$$

whence $\kappa\kappa'$ and $\lambda\lambda'$ can be determined.

$\Delta = 69 = 3\times 23$. Combine Schröter's, Hurwitz's, or Russell's modular equation of the 23rd order,

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+\sqrt[12]{(256\kappa\lambda\kappa'\lambda')} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

with Jacobi's equation of the 3rd order,

$$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'}=1 \quad\text{.............................. (2).}$$

Putting $\quad 4\kappa\lambda\kappa'\lambda'=x^{12},$

then $\quad \sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}=1-\sqrt{2}\,x,$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=1-2\sqrt{2}\,x+2x^2-\sqrt{2}\,x^3,$$

$$\kappa\lambda+\kappa'\lambda'=(1-2\sqrt{2}\,x+2x^2-\sqrt{2}\,x^3)^2-x,$$

and $\quad \kappa'\lambda+\kappa\lambda'=1-x^6,$

$$(\kappa'\lambda-\kappa\lambda')^2=1-2x^6,$$

therefore $\quad \kappa\lambda+\kappa'\lambda'=\sqrt{2}\,x^3,$

or $\quad (1-2\sqrt{2}\,x+2x^2-\sqrt{2}\,x^3)^2-x^6=\sqrt{2}\,x^3,$

or $\quad x^6-4\sqrt{2}\,x^5+12x^4-10\sqrt{2}\,x^3+12x^2-4\sqrt{2}\,x+1=\sqrt{2}\,x^3,$

a reciprocal sextic for x.

Put $\quad \dfrac{1}{x}+x=y,$

then $\quad y^3-4\sqrt{2}\,y^2+9y=3\sqrt{2}.$

The equation

$$y^3-4\sqrt{2}\,y^2+9y-3\sqrt{2}=0$$

has the factors

$$(y-\sqrt{2})\,(y^2-3\sqrt{2}\,y+3)=0,$$

and therefore the roots $\sqrt{2},\ \dfrac{3\pm\sqrt{3}}{\sqrt{2}}.$

For x to be real, y must be greater than 2, and therefore we must

put $\quad \dfrac{1}{x}+x=\dfrac{3+\sqrt{3}}{\sqrt{2}}.$

Then $\quad \kappa\lambda+\kappa'\lambda'=\sqrt{2}\,x^3,$

$$\kappa'\lambda+\kappa\lambda'=1-x^6;$$

and, multiplying these equations together,

$$\kappa\kappa'+\lambda\lambda'=\sqrt{2}\,x^3\,(1-x^6),$$

also $\quad 2\sqrt{\kappa\lambda\kappa'\lambda'}=x^6,$

so that $$\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'}=x^3\sqrt{\left\{\sqrt{2}\left(\frac{1}{x^3}-x^3\right)+1\right\}},$$

$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'}=x^3\sqrt{\left\{\sqrt{2}\left(\frac{1}{x^3}-x^3\right)-1\right\}}.$$

$\Delta=73$. A prime number not yet solved; but depending in Hermite's manner on $n=37$. Since $p=2$, we must expect a quadratic for α.

$\Delta=77=7\times 11$. Combine the modular equation of the 11^{th} order,

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+2\sqrt[6]{(4\kappa\lambda\kappa'\lambda')}=1,$$

with Gutzlaff's equation, of the 7^{th} order,

$$\sqrt[4]{\kappa'\lambda}+\sqrt[4]{\kappa\lambda'}=1,$$

by putting $$4\kappa\lambda\kappa'\lambda'=x^{12}.$$

Then $$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=1-2x^2,$$

$$\kappa\lambda+\kappa'\lambda'=1-4x^2+4x^4-x^6,$$

and $$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'}=1-\sqrt{2}\,x^3,$$

$$\kappa'\lambda+\kappa\lambda'=1-2\sqrt{2}\,x^3+x^6.$$

But $$(\kappa\lambda+\kappa'\lambda')^2+(\kappa'\lambda+\kappa\lambda')^2=1+4\kappa\lambda\kappa'\lambda',$$

and therefore

$$(1-4x^2+4x^4-x^6)^2+(1-2\sqrt{2}\,x^3+x^6)=1+x^{12},$$

or $$(1-2\sqrt{2}\,x^3+x^6)^2=1+x^{12}-(1-4x^2+4x^4-x^6)^2$$

$$=8x^2-24x^4+34x^6-24x^8+8x^{10}$$

$$=2x^2(2-3x^2+2x^4)^2.$$

$$1-2\sqrt{2}\,x^3+x^6=2\sqrt{2}\,x-3\sqrt{2}\,x^3+2\sqrt{2}\,x^5,$$

or $$x^6-2\sqrt{2}\,x^5+\sqrt{2}\,x^3-2\sqrt{2}\,x+1=0,$$

a reciprocal sextic in x, which, putting

$$\frac{1}{x}+x=y,$$

becomes $$y^3-2\sqrt{2}\,y^2-3y+5\sqrt{2}=0,$$

$$(y-\sqrt{2})(y^2-\sqrt{2}\,y-5)=0,$$

having roots $\sqrt{2}$, $\dfrac{1 \pm \sqrt{11}}{\sqrt{2}}$, of which the root greater than 2 must be chosen.

Then $$\kappa\lambda + \kappa'\lambda' = 1 - 4x^2 + 4x^4 - x^6,$$

$$\kappa'\lambda + \kappa\lambda' = 2\sqrt{2}\,x - 3\sqrt{2}\,x^3 + 2\sqrt{2}\,x^5,$$

$$\kappa\kappa' + \lambda\lambda' = \sqrt{2}\,x^6 \left\{ \frac{1}{x^3} - x^3 - 4\left(\frac{1}{x} - x\right) \right\} \left\{ 2\left(\frac{1}{x^3} + x^3\right) - 3 \right\},$$

$$2\sqrt{\kappa\lambda\,\kappa'\lambda'} = x^6;$$

whence $\sqrt{\kappa\kappa'}$ and $\sqrt{\lambda\lambda'}$; and then $2\kappa\kappa'$ and $2\lambda\lambda'$.

$\Delta = 81 = 3^4$. Obtained from $\Delta = 3$.

$\Delta = 85 = 5 \times 17$. Taking the modular equation of the 5th order, and putting $$4\kappa\lambda\,\kappa'\lambda' = x^6,$$

then $$x = \tfrac{1}{2}\left(\sqrt{85} - 9\right)$$

is a value implied from the approximate solutions given by Professor H. J. S. Smith, in the *Report on the Theory of Numbers*, 1865, *to the British Association*, p. 374.

Then $$\kappa'\lambda + \kappa\lambda' = 1 - 2x^2,$$

$$\kappa\lambda + \kappa'\lambda' = 2x - x^3;$$

so that, by multiplication,

$$\kappa\kappa' + \lambda\lambda' = x(2 - x^2)(1 - 2x^2)$$

$$= x^3\left(2x^{-2} - 5 + 2x^2\right)$$

$$= 161\,x^3;$$

$$-\kappa\kappa' + \lambda\lambda' = 72\sqrt{5}\,x^3;$$

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^{12} x^3;$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}+1}{2}\right)^{12} x^3;$$

$$2\kappa\kappa' = \left(\frac{\sqrt{5}-1}{2}\right)^{12} \left(\frac{\sqrt{85}-9}{2}\right)^3, \quad K'/K = \sqrt{85},$$

$$2\lambda\lambda' = \left(\frac{\sqrt{5}+1}{2}\right)^{12} \left(\frac{\sqrt{85}-9}{2}\right)^3, \quad \Lambda'/\Lambda = \sqrt{(17 \div 5)}.$$

$\Delta = 89$. A prime number, not yet solved.

$\Delta = 93 = 3\times 31$. From the modular equation of the 3rd order,

$$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'}=1,$$

putting

$$\sqrt[4]{4\,\kappa\lambda\,\kappa'\lambda'}=x,$$

$$\kappa'\lambda+\kappa\lambda'=1-2x^2,$$

$$(\kappa\lambda+\kappa'\lambda')^2=1+4\kappa\lambda\kappa'\lambda'-(\kappa'\lambda+\kappa\lambda')^2$$

$$=1+4x^4-(1-2x^2)^2=4x^2,$$

$$\kappa\lambda+\kappa'\lambda'=2x;$$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=\sqrt{(2x+2x^2)}$$

$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}=\sqrt{\{\sqrt{(2x+2x^2)}+2x\}}.$$

Then, in Russell's notation,

$$P=1+\sqrt{\{\sqrt{(2x+2x^2)}+2x\}},$$

$$Q=x+\sqrt{\{\sqrt{(2x+2x^2)}+2x\}},$$

$$R=x,$$

$$P^2-4Q=1-2x+\sqrt{(2x+2x^2)}-2\sqrt{\{\sqrt{(2x+2x^2)}+2x\}};$$

and the modular equation of the 31st order,

$$(P^2-4Q)^2-4PR=0,$$

becomes

$$\left[1-2x+\sqrt{(2x+2x^2)}-2\sqrt{\{\sqrt{(2x+2x^2)}+2x\}}\right]^2$$
$$-4x-4x\sqrt{\{\sqrt{(2x+2x^2)}+2x\}}=0;$$

and, rationalising,

$$1+2x+6x^2+2(3-2x)\sqrt{(2x+2x^2)}$$
$$=4\{1-x+\sqrt{(2x+2x^2)}\}\sqrt{\{\sqrt{(2x+2x^2)}+2x\}},$$

or

$$1-20x-8x^2-72x^3+68x^4=4(1+12x-18x^2+12x^3)\sqrt{(2x+2x^2)},$$

or

$$1-72x-416x^2-4048x^3+12680x^4-8096x^5-1664x^6-576x^7+16x^8=0,$$

which, on putting $\sqrt{2}\,x = y$, becomes

$$1-36\sqrt{2}\,y-208y^2-1012\sqrt{2}\,y^3+3170y^4$$
$$-1012\sqrt{2}\,y^5-208y^6-36\sqrt{2}\,y^7+y^8=0,$$

a reciprocal equation for y.

Now, put $$y+\frac{1}{y}=\sqrt{2}\,v,$$

then $$y^2+\frac{1}{y^2}=2v^2-2,$$

$$y^3+\frac{1}{y^3}=2\sqrt{2}\,v^3-3\sqrt{2}\,v,$$

$$y^4+\frac{1}{y^4}=4v^4-8v^2+2;$$

so that

$$v^4-36v^3-106v^2-452v+897=0;$$

$$(v-39)(v^3+3v^2+11v-23)=0.$$

Taking $v=39$, then

$$\frac{1}{y}+y=39\sqrt{2},$$

$$\frac{1}{y}-y=7\sqrt{2}\sqrt{31},$$

$$2y=\sqrt{2}\,(39-7\sqrt{31}),$$

$$2x=39-7\sqrt{31};$$

$$\kappa\kappa'+\lambda\lambda'=2x\,(1-2x^2)$$

$$=2xy\left(\frac{1}{y}-y\right)$$

$$=14\sqrt{31}\,y^2,$$

$$2\sqrt{\kappa\lambda\,\kappa'\lambda'}= \qquad y^2;$$

$$-\kappa\kappa'+\lambda\lambda'=45\sqrt{3}\,y^2,$$

$$2\kappa\kappa'=(14\sqrt{31}-45\sqrt{3})\,y^2$$

$$=\left(\frac{\sqrt{31}-3\sqrt{3}}{2}\right)^3\left(\frac{39-7\sqrt{31}}{\sqrt{2}}\right)^2,$$

for $$K'/K = \sqrt{93};$$

$$2\lambda\lambda' = \left(\frac{\sqrt{31}+3\sqrt{3}}{2}\right)^3 \left(\frac{39-7\sqrt{31}}{\sqrt{2}}\right)^2.$$

for $$\Lambda'/\Lambda = \sqrt{(31 \div 3)}.$$

$\Delta = 97$. Guided by the approximate numerical values given by Kronecker, and quoted by Smith in the 1865 *Report on the Theory of Numbers*, p. 374, we infer that

$$a = 33210\sqrt{97}+327078;$$

a being given by a quadratic equation, since $p = 2$ for the determinant -97.

$\Delta = 101$, a prime number. Since $p = 7$ for the determinant -101, we must expect an irreducible equation of the 7^{th} degree for the determination of a.

$\Delta = 105 = 3 \times 5 \times 7$, a number composed of three prime factors, the earliest number of the kind to be encountered.

The solution of the modular equation in this case has been given by Kronecker in the *Berlin Sitzungsberichte*, 1862, but the method by which the solution was obtained is very briefly indicated, and the results contain numerous misprints.

We shall obtain the solution by combining Gutzlaff's equation,

$$\sqrt[4]{\kappa'\lambda} + \sqrt[4]{\kappa\lambda'} = 1,$$

with Fiedler's modular equation of the 15^{th} order,

$$Z_1 Z_{2'} + 4Z_3 = 0,$$

where
$$Z_1 = \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} + 1,$$

$$Z_2 = \sqrt[4]{\kappa\lambda\kappa'\lambda'} + \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'},$$

$$Z_3 = +\sqrt[4]{\kappa\lambda\kappa'\lambda'},$$

and
$$Z_{2'} = Z_1^2 - 4Z_2.$$

Write x for $\sqrt[4]{\kappa\lambda\kappa'\lambda'}$ and w for $\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'}$; then

$$Z_1 = w+1, \quad Z_2 = w+x, \quad Z_3 = x,$$

$$Z_{2'} = (w+1)^2 - 4(w+x),$$

$$= (w-1)^2 - 4x,$$

and Fiedler's equation becomes

$$(w+1)(w-1)^2-4x\,(w+1)+4x=0,$$

or $$(w+1)(w-1)^2=4wx,$$

or $$w^3-w^2-w+1=4wx.$$

Now, from Gutzlaff's equation,

$$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'}=1-2x,$$
$$\kappa'\lambda+\kappa\lambda'=1-4x+2x^2,$$

and therefore $$(\kappa\lambda+\kappa'\lambda')^2=1+4x^4-(1-4x+2x^2)^2$$
$$=8x-20x^2+16x^3.$$

But $$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}=w,$$
$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=w^2-2x,$$
$$\kappa\lambda+\kappa'\lambda'=w^4-4w^2x+2x^2$$
$$=w^3+w^2-w+2x^2=2w^2+4wx+2x^2-1$$
$$=2\,(w+x)^2-1.$$

Therefore, putting $w+x=z$,

$$2z^2-1=\sqrt{(16x^3-20x^2+8x)},$$

and $$z^3-(3x+1)\,z^2+(3x^2-2x-1)\,z-x^3+3x^2+x+1=0,$$

and z is to be eliminated between these two equations.

Putting $$2z^2-1=t,\quad z^2=\tfrac{1}{2}\,(t+1),$$
then, since

$$\{z^3+(3x^2-2x-1)\,z\}^2-\{(3x+1)\,z^2-x^3+3x^2+x+1\}^2=0,$$

or $$z^6-(3x^2+10x+3)\,z^4+(3x^4+4x^3+10x^2+12x+3)\,z^2$$
$$-(x^3-3x^2-x-1)^2=0,$$

therefore

$$\left(\frac{t+1}{2}\right)^3-(3x^2+10x+3)\left(\frac{t+1}{2}\right)^2+(3x^4+4x^3+10x^2+12x+3)\,\frac{t+1}{2}$$
$$-(x^3-3x^2-x-1)^2=0,$$

or $$t^3-(6x^2+20x+3)\,t^2+(12x^4+16x^3+28x^2+8x+3)\,t$$
$$-(8x^6-48x^5+44x^4+16x^3+22x^2-12x+1)=0,$$

or
$$t\,(t^2+12x^4+16x^3+28x^2+8x+3)$$
$$\equiv \sqrt{(16x^3-20x^2+8x)(12x^4+32x^3+8x^2+16x+3)}$$
$$= 8x^6+48x^5+244x^4-288x^3+122x^2+12x+1,$$

a reciprocal equation in $\sqrt{2}\,x$.

Putting $\sqrt{2}\,x = y$, and squaring,

$$(4\sqrt{2}\,y^3-10y^2+4\sqrt{2}\,y)(3y^4+8\sqrt{2}\,y^3+4y^2+8\sqrt{2}\,y+3)^2$$
$$= (y^6+6\sqrt{2}\,y^5+61y^4-72\sqrt{2}\,y^3+61y^2+6\sqrt{2}\,y+1)^2;$$

and putting
$$y+\frac{1}{y} = \sqrt{2}\,v,$$
$$y^2+\frac{1}{y^2} = 2v^2-2, \quad y^3+\frac{1}{y^3} = 2\sqrt{2}\,v^3-3\sqrt{2}\,v,$$
$$(8v-10)(6v^2+16v-2)^2 = 2\,(2v^3+12v^2+58v-84)^2,$$
or
$$(4v-5)(3v^2+8v-1)^2 = (v^3+6v^2+29v-42)^2,$$
or
$$v^6-24v^5-53v^4+272v^3+691v^2-2520v+1769 = 0.$$

A quadratic factor of this equation, $v^2-28v+61$, was discovered by calculating the approximate numerical values of x in a manner to be explained subsequently.

The remaining quartic factor of the sextic

$$v^4+4v^3-2v^2-28v+29 = (v^2+2v-5)^2+4\,(v-1)^2 = 0,$$

has only imaginary roots.

Taking v as determined by the quadratic

$$v^2-28v+61 = 0,$$
then
$$v = 14+3\sqrt{15}.$$

The reciprocal 12^{ic} for y, expanded at full length, would be

$$y^{12}-24\sqrt{2}\,y^{11}-100y^{10}+424\sqrt{2}\,y^9+2355y^8-8688\sqrt{2}\,y^7$$
$$+19064y^6-8688\sqrt{2}\,y^5\ldots\ldots-24\sqrt{2}\,y+1 = 0,$$

which has the reciprocal factor,

$$y^4-28\sqrt{2}\,y^3+124y^2-28\sqrt{2}\,y+1,$$

obtained from
$$v^2-28v+61.$$

Now
$$\sqrt[4]{\kappa'\lambda}+\sqrt[4]{\kappa\lambda'}=1,$$
$$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'}=1-\sqrt{2}\,y,$$
$$\kappa'\lambda+\quad\kappa\lambda'=1-2\sqrt{2}\,y+y^2,$$
and
$$\kappa\lambda+\quad\kappa'\lambda'=\sqrt{(4\sqrt{2}\,y-10y^2+4\sqrt{2}\,y^3)}.$$
Now, if
$$y^4=4\kappa\lambda\kappa'\lambda',$$
$$\frac{1}{y}+y=\sqrt{2}\,v,$$
where
$$v=14+3\sqrt{15},$$
$$\frac{1}{y}-y=\sqrt{(2v^2-4)}$$
$$=\sqrt{(658+168\sqrt{15})}$$
$$=\sqrt{2}\,(3\sqrt{21}+2\sqrt{35}).$$
Therefore
$$\frac{2}{y}=\sqrt{2}\,(14+3\sqrt{15}+3\sqrt{21}+2\sqrt{35})$$
$$=\sqrt{2}\,(2\sqrt{7}+3\sqrt{3})(\sqrt{7}+\sqrt{5}),$$
$$\frac{1}{y}=\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}},$$
$$y=\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}.$$

Now
$$\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}=1,$$
$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}=1-\sqrt{2}\,y,$$
$$\kappa\lambda+\quad\kappa'\lambda'=1-2\sqrt{2}\,y+y^2,$$
$$\kappa'\lambda+\quad\kappa\lambda'=\sqrt{\{1+y^4-(1-2\sqrt{2}\,y+y^2)^2\}}$$
$$=\sqrt{(4\sqrt{2}\,y-10y^2+4\sqrt{2}\,y^3)}.$$
Therefore
$$\kappa\kappa'+\lambda\lambda'=y^2\left(\frac{1}{y}+y-2\sqrt{2}\right)\sqrt{\left\{4\sqrt{2}\left(\frac{1}{y}+y\right)-10\right\}}$$
$$=y^2\,(\sqrt{2}v-2\sqrt{2})\sqrt{(8v-10)}$$
$$=2y^2\,(v-2)\sqrt{(4v-5)}$$
$$=2y^2\,(12+3\sqrt{15})(6+\sqrt{15})$$
$$=y^2\,(234+60\sqrt{15}),$$

and $2\sqrt{\kappa\lambda\kappa'\lambda'} = y^2$;

therefore
$$\sqrt{\kappa\kappa'} + \sqrt{\lambda\lambda'} = y\sqrt{(235+60\sqrt{15})}$$
$$= y(3\sqrt{15}+10),$$
$$-\sqrt{\kappa\kappa'} + \sqrt{\lambda\lambda'} = y\sqrt{(233+60\sqrt{15})}$$
$$= y(5\sqrt{5}+6\sqrt{3}).$$

Therefore
$$2\sqrt{\kappa\kappa'} = y(3\sqrt{15}+10-5\sqrt{5}-6\sqrt{3})$$
$$= y(3\sqrt{3}-5)(\sqrt{5}-2)$$
$$= 4y\left(\frac{\sqrt{3}-1}{2}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^3,$$
$$2\sqrt{\lambda\lambda'} = y\left(\frac{\sqrt{3}+1}{2}\right)^3\left(\frac{\sqrt{5}+1}{2}\right)^3;$$

or
$$\sqrt{2\kappa\kappa'} = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^3\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}},$$

when
$$K'/K = \sqrt{(105)};$$

and
$$\sqrt{2\lambda\lambda'} = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^3\left(\frac{\sqrt{5}+1}{2}\right)^3\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}},$$

when
$$K'/K = \sqrt{(15\div 7)}.$$

Therefore
$$\frac{1}{\sqrt{2\kappa\kappa'}} = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^3\left(\frac{\sqrt{5}+1}{2}\right)^3\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}};$$

and therefore
$$\beta = \tfrac{1}{2}\sqrt{\alpha} = \frac{1}{2\kappa\kappa'} - 2\kappa\kappa'$$
$$= \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6\left(\frac{\sqrt{5}+1}{2}\right)^6\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6\left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)^2$$
$$-\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6\left(\frac{\sqrt{5}-1}{2}\right)^6\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6\left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^2;$$

and
$$\beta' = \tfrac{1}{2}\sqrt{\alpha'} = \frac{1}{2\lambda\lambda'} - 2\lambda\lambda'$$
$$= \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^6\left(\frac{\sqrt{5}-1}{2}\right)^6\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6\left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)^2$$
$$-\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^6\left(\frac{\sqrt{5}+1}{2}\right)^6\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^6\left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}\right)^2;$$

so that β and β' are the roots of a quadratic equation.

If we had taken the other root of the quadratic

$$v^2-28v+61=0,$$

$$v=14-3\sqrt{15},$$

we should find $\frac{2}{y}=\sqrt{2}\,(14-3\sqrt{15}+3\sqrt{21}-2\sqrt{35})$

$$=\sqrt{2}\,(2\sqrt{7}+3\sqrt{3})(\sqrt{7}-\sqrt{5}),$$

$$\frac{1}{y}=\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3\frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}},$$

$$y=\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}.$$

Also $\kappa\kappa'+\lambda\lambda'=y^3\,(234-60\sqrt{15}),$

and therefore $\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'}=y\,(3\sqrt{15}-10),$

$$-\sqrt{\kappa\kappa'}+\sqrt{\lambda\lambda'}=y\,(5\sqrt{5}-6\sqrt{3}),$$

and $\sqrt{2\kappa\kappa'}=\frac{y}{\sqrt{2}}\,(3\sqrt{15}-10-5\sqrt{5}+6\sqrt{3})$

$$=(3\sqrt{3}-5)(\sqrt{5}+2)\,\frac{y}{\sqrt{2}}$$

$$=\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3\left(\frac{\sqrt{5}+1}{2}\right)^3\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}},$$

when $K'/K=\sqrt{(35\div 3)}$;

and $\sqrt{2\lambda\lambda'}=\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^3\left(\frac{\sqrt{5}-1}{2}\right)^3\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}},$

when $\Lambda'/\Lambda=\sqrt{(21\div 5)}.$

According to Kronecker, *Berlin Sitz.*, 1862,

$$2\kappa\kappa'=(2\gamma-3\alpha)^2\,(5+9\alpha+16\beta+4\gamma+7\beta\gamma+12\alpha\beta+3\alpha\beta\gamma),$$

where α, β, γ denote $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, respectively.

The factor $2\gamma-3\alpha=\left(\frac{\gamma-\alpha}{2}\right)^3,$

but the remaining factor cannot be made to agree with

$$(2+\alpha)^3\left(\frac{\beta-1}{2}\right)^6(6+\beta\gamma),$$

the result obtained above.

The approximate numerical values of x and y were obtained from the formulas

$$\sqrt[4]{\kappa\kappa'} = \sqrt{2}\, q^{\frac{1}{8}}, \quad \sqrt[4]{2\kappa\kappa'} = 2^{\frac{3}{4}} q^{\frac{1}{8}}.$$

Now, if $K'/K = \sqrt{105}$,

$$\begin{aligned}
\log 105 &= 2{\cdot}0211893, \\
\log K'/K = \log \sqrt{105} &= 1{\cdot}01059465, \\
\log \pi \log \epsilon &= {\cdot}1349342, \\
\log \log \left(\frac{1}{q}\right) &= 1{\cdot}14552885, \\
\log \left(\frac{1}{q}\right) &= 13{\cdot}9806984, \\
\log \left(\frac{1}{q}\right)^{\frac{1}{8}} &= 1{\cdot}7475873, \\
\log 2^{\frac{3}{4}} &= {\cdot}2257725, \\
\log \frac{1}{\sqrt[4]{2\kappa\kappa'}} &= 1{\cdot}5218148, \\
\log \sqrt[4]{2\kappa\kappa'} &= \bar{2}{\cdot}4781852.
\end{aligned}$$

Again, if $\Lambda'/\Lambda = \sqrt{(15 \div 7)}$,

$$\begin{aligned}
\log 15 &= 1{\cdot}1760913, \\
\log 7 &= {\cdot}8450980, \\
\log \tfrac{15}{7} &= {\cdot}3309933, \\
\log \Lambda'/\Lambda = \log \sqrt{\tfrac{15}{7}} &= {\cdot}16549665, \\
\log \pi \log \epsilon &= {\cdot}1349342, \\
\log \log \left(\frac{1}{q}\right) &= {\cdot}30043085, \\
\log \left(\frac{1}{q}\right) &= 1{\cdot}9972425, \\
\log \left(\frac{1}{q}\right)^{8} &= {\cdot}2496553, \\
\log 2^{\frac{3}{4}} &= {\cdot}2257725, \\
\log \frac{1}{\sqrt[4]{2\lambda\lambda'}} &= {\cdot}0238828, \\
\log \sqrt[4]{2\lambda\lambda'} &= \bar{1}{\cdot}9761172.
\end{aligned}$$

Combining these transformations when the modular equations of the 7th and 15th order are employed,

$$\log\frac{1}{y} = 1{\cdot}5456976, \qquad \frac{1}{y} = 35{\cdot}131633,$$

$$\log y = \bar{2}{\cdot}4543024, \qquad y = {\cdot}028464,$$

$$\frac{1}{y}+y = 35{\cdot}161097,$$

$$\log\sqrt{2}\,v = 1{\cdot}5460624,$$

$$\log\sqrt{2} = {\cdot}1505150,$$

$$\log v = 1{\cdot}3955474, \qquad v = 24{\cdot}862.$$

In a similar manner, if $K'/K = \sqrt{(25\div 3)}$,

$$\log\sqrt[4]{2\kappa\kappa'} = \bar{1}{\cdot}64324355,$$

and if $\Lambda'/\Lambda = \sqrt{(21\div 5)}$,

$$\log\sqrt[4]{2\lambda\lambda'} = \bar{1}{\cdot}87623125,$$

$$\log y' = \bar{1}{\cdot}5194748,$$

$$\log\frac{1}{y'} = {\cdot}4805252,$$

$$y' = {\cdot}33073,$$

$$\frac{1}{y'} = 3{\cdot}02360,$$

$$\sqrt{2}\,v' = 3{\cdot}35433,$$

$$\log\sqrt{2}\,v' = {\cdot}5256058,$$

$$\log\sqrt{2} = {\cdot}1505150,$$

$$\log v' = {\cdot}3750908, \qquad v' = 2{\cdot}37187,$$

$$\log v = 1{\cdot}3955474,$$

$$\log vv' = 1{\cdot}7706382, \qquad vv' = 58{\cdot}971,$$

and

$$v+v' = 24{\cdot}862$$
$$2{\cdot}372$$
$$= 27{\cdot}234;$$

indicating, as the approximations are rather rough for the ratios of the periods

$$K'/K = \sqrt{(15\div 7)}, \quad \sqrt{(21\div 5)}, \quad \text{and} \quad \sqrt{(35\div 3)},$$

the true values of $v+v'$ and vv', namely

$$v+v' = 28 \quad \text{and} \quad vv' = 61.$$

Taking, however, the values from Legendre's tables, we find that

$$K'/K = \surd(15\div 7), \quad 2\kappa\kappa' \approx \sin 45° \, 30';$$

$$K'/K = \surd(21\div 5), \quad 2\kappa\kappa' \approx \sin 18° \, 40';$$

$$K'/K = \surd(35\div 3), \quad 2\kappa\kappa' \approx \sin 2° \, 8'.$$

If we had combined Jacobi's modular equation of the 3rd order,

$$\sqrt{\kappa'\lambda}+\sqrt{\kappa\lambda'} = 1,$$

with Fiedler's form of the 35th order,

$$Z_1'^3 - 8Z'Z_2' + 8Z_3' - 4Z_0^{(4)} = 0,$$

where $Z_1' = \sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}-1,$

$$Z' = \sqrt{\kappa\lambda\kappa'\lambda'}-\sqrt{\kappa\lambda}-\sqrt{\kappa'\lambda'},$$

$$Z_3' = -\sqrt{\kappa\lambda\kappa'\lambda'},$$

and $Z_0^{(4)} = -(\kappa'\lambda+\kappa\lambda')\sqrt[4]{\kappa\lambda\kappa'\lambda'}+(\kappa'-\lambda')\sqrt[4]{\kappa'\lambda'}-(\kappa-\lambda)\sqrt[4]{\kappa\lambda};$

then, putting

$$\kappa\lambda\kappa'\lambda' = x^4,$$

$$\kappa'\lambda+\kappa\lambda' = 1-2x^2,$$

$$(\kappa\lambda+\kappa'\lambda')^2 = 1+4x^4-(1-2x^2)^2$$

$$= 4x^2,$$

$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = \surd(2x+2x^2);$$

so that $Z_1' = \surd(2x+2x^2)-1,$

$$Z_2' = x^2-\surd(2x+2x^2),$$

$$Z_3' = -x^2,$$

$$Z_0^{(4)} = -(x-2x^3)+(\kappa'-\lambda')\sqrt[4]{\kappa'\lambda'}-(\kappa-\lambda)\sqrt[4]{\kappa\lambda}.$$

Then

$$\{\surd(2x+2x^2)-1\}^3-8\{\surd(2x+2x^2)-1\}\{x^2-\surd(2x+2x^2)\}$$
$$+4x-8x^2-8x^3$$
$$= 4\{(\kappa'-\lambda')\sqrt{\kappa'\lambda'}-(\kappa-\lambda)\sqrt[4]{\kappa\lambda}\}$$
$$= 4\surd(1-2x)\surd\{(1-2x+2x^2)\surd(2x+2x^2)+2x-4x^3\}.$$

Squaring and rearranging,

$$1-10x+250x^2+448x^3-172x^4-136x^5+136x^6$$
$$=(6+80x+128x^2-16x^3+152x^4-96x^5)\sqrt{(2x+2x^3)};$$

and squaring again, and reducing,

$$1-92x-1392x^2-21896x^3-3252x^4+155296x^5+82976x^6$$
$$-310592x^7-13008x^8+175168x^9-22272x^{10}+2944x^{11}+64x^{12}=0,$$

a reciprocal equation in $\sqrt{2}\,x=y$, so that

$$y^{12}+46\sqrt{2}\,y^{11}-696y^{10}+5474\sqrt{2}\,y^9-813y^8-19412\sqrt{2}\,y^7+6032y^6$$
$$+19412\sqrt{2}\,y^5-813y^4-5474\sqrt{2}\,y^3-696y^2-46\sqrt{2}\,y+1=0.$$

Putting, as before, $y+\dfrac{1}{y}=\sqrt{2}\,v,$

then $v^6-351v^4+495v^2+783+46\sqrt{(v^2-2)}(v^4+58v^2-135)=0.$

But, putting

$$\frac{1}{y}-y=\sqrt{2}\,u,$$
$$\frac{1}{y^2}+y^2=2u^2+2,$$
$$\frac{1}{y^3}-y^3=\sqrt{2}\,(2u^3+3u),$$
$$\frac{1}{y^4}+y^4=4u^4+8u^2+2,$$
$$\frac{1}{y^5}-y^5=\sqrt{2}\,(4u^5+10u^3+5u),$$
$$\frac{1}{y^6}+y^6=8u^6+24u^4+18u^2+2;$$

then $u^6-46u^5-345u^4-2852u^3-897u^2+690u+377=0,$

having a quadratic factor $u^2-54u+29,$

inferred as before from the approximate numerical values of the moduli.

This sextic then splits into the factors

$$(u^2-54u+29)(u^4+8u^3+58u^2+48u+13)=0,$$

and the quartic factor

$$=(u^2+4u+3)^2+(6u+2)^2,$$

and therefore gives imaginary roots only.

But if $\frac{1}{y} - y = \sqrt{2}\, u$, and $u = 27 + 10\sqrt{7}$, from

$$u^2 - 54u + 29 = 0;$$

$$\frac{1}{y} + y = \sqrt{(2u^2+4)}$$
$$= \sqrt{(2862 + 1080\sqrt{7})}$$
$$= \sqrt{2}\,(6\sqrt{21} + 15\sqrt{3});$$
$$\frac{2}{y} = \sqrt{2}\,(27 + 10\sqrt{7} + 6\sqrt{21} + 15\sqrt{3})$$
$$= \sqrt{2}\,(3\sqrt{3}+5)(2\sqrt{7}+3\sqrt{3}),$$
$$\frac{1}{y} = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^3 \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3,$$
$$y = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3.$$

But $$\sqrt{\kappa\lambda} + \sqrt{\kappa'\lambda'} = 1,$$
$$\kappa\lambda + \kappa'\lambda' = 1 - y^2,$$
$$\kappa'\lambda + \kappa\lambda' = \sqrt{\{1 + y^2 - (1-y^2)^2\}} = \sqrt{2}\, y,$$
$$\kappa\kappa' + \lambda\lambda' = \sqrt{2}\, y\,(1-y^2) = 2y^2 u,$$
$$2\sqrt{\kappa\lambda\kappa'\lambda'} = y^2;$$

therefore $$\sqrt{\kappa\kappa'} + \sqrt{\lambda\lambda'} = y\,\sqrt{(2u+1)}$$
$$= y\,\sqrt{(55 + 20\sqrt{7})}$$
$$= y\,(\sqrt{35} + 2\sqrt{5});$$
$$-\sqrt{\kappa\kappa'} + \sqrt{\lambda\lambda'} = y\,\sqrt{(2u-1)}$$
$$= y\,\sqrt{(53 + 20\sqrt{7})}$$
$$= y\,(2\sqrt{7} + 5);$$
$$2\sqrt{\kappa\kappa'} = y\,(\sqrt{35} + 2\sqrt{5} - 2\sqrt{7} - 5)$$
$$= y\,(\sqrt{5}-2)(\sqrt{7}-\sqrt{5});$$
$$\sqrt{2\kappa\kappa'} = y\left(\frac{\sqrt{5}-1}{2}\right)^3 \frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}}$$
$$= \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3 \left(\frac{\sqrt{5}-1}{2}\right)^3 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3 \frac{\sqrt{7}-\sqrt{5}}{\sqrt{2}},$$

where $K'/K = \sqrt{105}$, as before; and

$$\sqrt{2\lambda\lambda'} = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3 \left(\frac{\sqrt{5}+1}{2}\right)^3 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3 \frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}},$$

when $\Lambda'/\Lambda = \sqrt{(35 \div 3)}$, as before.

$\Delta = 161 = 7 \times 23$. Combine Schröter's or Russell's modular equation of the 23^{rd} order,

$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} + \sqrt[3]{4}\,\sqrt[12]{(\kappa\lambda\kappa'\lambda')} = 1,$$

with Gutzlaff's of the 7^{th},

$$\sqrt[4]{\kappa'\lambda} + \sqrt[4]{\kappa\lambda'} = 1,$$

by putting $4\kappa\lambda\,\kappa'\lambda' = x^{12}$.

Then, for the equation of the 23^{rd} order,

$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} = 1 - \sqrt{2}\,x,$$

$$\sqrt{\kappa\lambda} + \sqrt{\kappa'\lambda'} = 1 - 2\sqrt{2}\,x + 2x^2 - \sqrt{2}\,x^3,$$

$$\begin{aligned}\kappa\lambda + \kappa'\lambda' &= (1 - 2\sqrt{2}\,x + 2x^2 - \sqrt{2}\,x^3)^2 - x^6 \\ &= 1 - 4\sqrt{2}\,x + 12x^2 - 10\sqrt{2}\,x^3 + 12x^4 - 4\sqrt{2}\,x^5 + x^6,\end{aligned}$$

a reciprocal expression.

Again, from the equation of the 7^{th} order,

$$\sqrt{\kappa'\lambda} + \sqrt{\kappa\lambda'} = 1 - \sqrt{2}\,x^3,$$

$$\kappa'\lambda + \kappa\lambda' = 1 - 2\sqrt{2}\,x^3 + x^6.$$

But
$$(\kappa\lambda + \kappa'\lambda')^2 + (\kappa'\lambda + \kappa\lambda')^2 = 1 + 4\kappa\lambda\,\kappa'\lambda';$$

therefore

$$(1 - 4\sqrt{2}\,x + 12x^2 - 10\sqrt{2}\,x^3 + 12x^4 - 4\sqrt{2}\,x^5 + x^6)^2 + (1 - 2\sqrt{2}\,x^3 + x^6)^2 = 1 + x^{12},$$

a reciprocal equation of the 12^{th} degree for x.

Put
$$\frac{1}{x} + x = \sqrt{2}\,y,$$

then
$$4y^6 - 32y^5 + 100y^4 - 160y^3 + 113y^2 - 24y + 9 = 0,$$

a sextic equation for y, of which a quadratic factor can be discovered by calculating as before the approximate numerical values of a pair of the roots.

$\Delta = 193$. This is a prime number, for which, according to Gauss, $p = 2$; so that a should be of the form $M\sqrt{193}+N$, when M and N are integers; and the values of M and N can be determined by approximate numerical calculation.

Class D.

$\Delta \equiv 2$, mod. 4.

This is the same as Hermite's class 2° (*Équations modulaires*, p. 44).

To solve the modular equation according to Hermite, we put

$$u^4 = -\frac{v^4-1}{v^4+1}, \quad u^8 = x;$$

then
$$\kappa' = u^4 = \frac{1-\lambda}{1+\lambda},$$

$$\kappa' = \frac{2\sqrt{\lambda}}{1+\lambda};$$

so that
$$\kappa'\lambda' = \frac{2\sqrt{\lambda}}{1+\lambda}\sqrt{(1-\lambda^2)} = 2\sqrt{\kappa\lambda},$$

equivalent to the modular queation of the quadric transformation.

Then, if we put

$$\beta = \frac{1}{\sqrt{\kappa\lambda}} - \sqrt{\kappa\lambda} = \frac{1}{u^2v^2} - u^2v^2,$$

$$\beta = \frac{1}{\sqrt{x}}\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}} - \sqrt{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

$$= \frac{1+x}{\sqrt{x}\sqrt{(1-x)}}$$

$$= \sqrt[4]{(-a)},$$

where a denotes Hermite's absolute invariant, given by

$$a = -\frac{(1+x)^4}{x(1-x)^2},$$

and then
$$x = \kappa^2.$$

If we put
$$v/u = e^{\phi}, \quad u/v = e^{-\phi},$$

then, if $w = uv$,

$$v^2 = we^{i\phi}, \quad u^2 = we^{-i\phi};$$

so that $$v^{2n}+u^{2n} = 2w^n \cosh n\phi,$$
$$v^{2n}-u^{2n} = 2w^n \sinh n\phi;$$

and, since $$u^4+v^4 = 1-u^4v^4,$$

therefore $$2w^2 \cosh 2\phi = 1-w^4,$$

or $$\beta = \frac{1}{w^2}-w^2 = 2\cosh 2\phi,$$
$$\beta+2 = 4\cosh^2\phi, \quad \beta-2 = 4\sinh^2\phi;$$

so that $$\kappa+\lambda = 2\sqrt{\kappa\lambda}\cosh 2\phi,$$
$$-\kappa+\lambda = 2\sqrt{\kappa\lambda}\sinh 2\phi.$$

In the following numerical illustrations we shall find it convenient to put
$$\sqrt{(\beta^2+4)} = \frac{1}{\sqrt{\kappa\lambda}}+\sqrt{\kappa\lambda} = \gamma,$$

and then $$\sqrt{\kappa\lambda} = w^2 = \tfrac{1}{2}(\gamma-\beta).$$

$\Delta = 2.$ $\quad \kappa = \sqrt{2}-1, \quad \alpha = -2^4, \quad \beta = 2, \quad \gamma = 2\sqrt{2};$
$$\cosh\phi = 1, \quad \phi = 0, \quad v = u, \quad \kappa = \lambda.$$

$\Delta = 6.$ Putting $\kappa'\lambda' = 2\sqrt{\kappa\lambda}$ in the modular equation of the 3rd order,
$$\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'} = 1,$$

then $$\sqrt{\kappa\lambda}+\sqrt{2}\sqrt[4]{\kappa\lambda} = 1,$$

or $$\frac{1}{\sqrt[4]{\kappa\lambda}}-\sqrt[4]{\kappa\lambda} = \sqrt{2},$$
$$\gamma = \frac{1}{\sqrt{\kappa\lambda}}+\sqrt{\kappa\lambda} = 4,$$
$$\beta = \frac{1}{\sqrt{\kappa\lambda}}-\sqrt{\kappa\lambda} = 2\sqrt{3};$$
$$\cosh 2\phi = \sqrt{3}, \quad \sinh 2\phi = \sqrt{2}, \quad w^2 = 2-\sqrt{3};$$
$$\alpha = -2^4\times 3^3,$$
agreeing with Hermite's result.

Solving this equation, we shall find
$$\kappa = (\sqrt{3}-\sqrt{2})(2-\sqrt{3}), \quad \text{for} \quad K'/K = \sqrt{6};$$
$$\lambda = (\sqrt{3}+\sqrt{2})(2-\sqrt{3}), \quad \text{for} \quad \Lambda'/\Lambda = \sqrt{(2\div 3)}$$
(Legendre, *Fonctions elliptiques*).

$\Delta = 10$. Put $\kappa'\lambda' = 2\sqrt{\kappa\lambda}$ in the modular equation of the 5th order,

$$\kappa\lambda + \kappa'\lambda' + 2\sqrt[3]{4\kappa\lambda\kappa'\lambda'} = 1;$$

then
$$\kappa\lambda + 2\sqrt{\kappa\lambda} + 4\sqrt{\kappa\lambda} = 1;$$

or
$$\beta = \frac{1}{\sqrt{\kappa\lambda}} - \sqrt{\kappa\lambda} = 6 = 2\times 3;$$

$$\gamma = 2\sqrt{10}, \quad w^3 = \sqrt{10} - 3;$$

$$a = -2^4 \times 3^4.$$

Solving these equations for κ and λ, we shall find

$$\kappa = (\sqrt{2}-1)^2(\sqrt{10}-3), \quad \text{for} \quad K'/K = \sqrt{10};$$

$$\lambda = (\sqrt{2}+1)^2(\sqrt{10}-3), \quad \text{for} \quad \Lambda'/\Lambda = \sqrt{(2 \div 5)}.$$

$\Delta = 14$. From the modular equation of the 7th order,

$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} = 1,$$

we obtain
$$\sqrt[4]{\kappa\lambda} + \sqrt[4]{2}\,\sqrt[8]{\kappa\lambda} = 1,$$

or
$$\frac{1}{\sqrt[8]{\kappa\lambda}} - \sqrt[8]{\kappa\lambda} = \sqrt[4]{2},$$

$$\frac{1}{\sqrt[4]{\kappa\lambda}} + \sqrt[4]{\kappa\lambda} = 2 + \sqrt{2};$$

$$\gamma = \frac{1}{\sqrt{\kappa\lambda}} + \sqrt{\kappa\lambda} = 4(\sqrt{2}+1),$$

$$\beta = \frac{1}{\sqrt{\kappa\lambda}} - \sqrt{\kappa\lambda} = 2\sqrt{(8\sqrt{2}+11)},$$

$$a = -2^4(8\sqrt{2}+11)^2.$$

Then
$$\sqrt{\kappa\lambda} = 2\sqrt{2} + 2 - \sqrt{(8\sqrt{2}+11)},$$

$$\cosh 2\phi = \sqrt{(8\sqrt{2}+11)}, \quad \sinh 2\phi = \sqrt{(8\sqrt{2}+10)},$$

$$\kappa = \{2\sqrt{2}+2-\sqrt{(8\sqrt{2}+11)}\}\{\sqrt{(8\sqrt{2}+11)} - \sqrt{(8\sqrt{2}+10)}\},$$

for
$$K'/K = \sqrt{14};$$

$$\lambda = \{2\sqrt{2}+2-\sqrt{(8\sqrt{2}+11)}\}\{\sqrt{(8\sqrt{2}+11)} + \sqrt{(8\sqrt{2}+10)}\},$$

for
$$\Lambda'/\Lambda = \sqrt{(2 \div 7)}.$$

(G. H. Stuart, "Complex Multiplication of Elliptic Functions," *Quar. Jour. of Math.*, xx., p. 54).

$\Delta = 18$. Then $$a = -2^4 \times 7^4,$$
obtained from Hermite's *Équations Modulaires*, p. 51.

Then $$\cosh 2\phi = 7, \quad \sinh 2\phi = 4\sqrt{3}, \quad \cosh\phi = 2;$$

$$\sqrt{\kappa\lambda} = w^2 = 5\sqrt{2}-7 = (\sqrt{2}-1)^3;$$

$$\kappa = (\sqrt{2}-1)^3(2-\sqrt{3})^2, \qquad K'/K = 3\sqrt{2};$$

$$\lambda = (\sqrt{2}-1)^3(2+\sqrt{3})^2, \qquad \Lambda'/\Lambda = \sqrt{2} \div 3.$$

$\Delta = 22$. From the equation of the 11^{th} order, we obtain

$$\sqrt{\kappa\lambda} + \sqrt{2}\sqrt[4]{\kappa\lambda} + 2\sqrt{2}\sqrt[4]{\kappa\lambda} = 1,$$

or $$\frac{1}{\sqrt[4]{\kappa\lambda}} - \sqrt[4]{\kappa\lambda} = 3\sqrt{2},$$

$$\gamma = \frac{1}{\sqrt{\kappa\lambda}} + \sqrt{\kappa\lambda} = 20;$$

$$\beta = \frac{1}{\sqrt{\kappa\lambda}} - \sqrt{\kappa\lambda} = 6\sqrt{11};$$

$$\cosh\phi = 3\sqrt{11}, \quad \sinh\phi = 7\sqrt{2};$$

$$a = -\beta^4 = -2^4 \times 3^4 \times 11^2.$$

Then $$\sqrt{\kappa\lambda} = w^2 = 10-3\sqrt{11} = \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2;$$

$$\kappa = (3\sqrt{11}-7\sqrt{2})(10-3\sqrt{11}), \quad \text{for} \quad K'/K = \sqrt{22};$$

$$\lambda = (3\sqrt{11}+7\sqrt{2})(10-3\sqrt{11}), \quad \text{for} \quad \Lambda'/\Lambda = \sqrt{(2 \div 11)}.$$

$\Delta = 26$. Taking Mr. Russell's form of the modular equation of the 13^{th} order,

$$P^7 - 2^5R(105P^4 - 2^7 \times 11P^2Q + 2^{12}Q^2) + 2^{10}PR^2 = 0,$$

where $$P = \kappa\lambda + \kappa'\lambda' - 1,$$
$$Q = \kappa\lambda\kappa'\lambda' - \kappa\lambda - \kappa'\lambda',$$
$$R = -\kappa\lambda\kappa'\lambda';$$

and putting $$\kappa\lambda = x^2, \quad \kappa'\lambda' = 2x,$$

then $$P = x^2 + 2x - 1 = -x(\beta-2),$$
$$Q = 2x^3 - x^2 - 2x = -x^2(2\beta+1),$$
$$R = -2x^3,$$

putting $$\beta = \frac{1}{x} - x;$$

and then substituting, and dividing out by z^7,

$$(\beta-2)^7 - 64\{105(\beta-2)^4 + 2^7 \times 11(\beta-2)^2(2\beta+1) + 2^{12}(2\beta+1)^2\} + 2^{18}(\beta-2) = 0.$$

Putting $$\beta - 2 = 4t,$$

then $$t = \sinh^2\phi,$$

and $$t^7 - 105t^4 - 704t^3 - 1464t^2 - 1216t - 400 = 0,$$

or $$(t^2+t+4)(t^2+4t+25)(t^3-5t^2-8t-4) = 0.$$

Taking the cubic $$t^3 - 5t^2 - 8t - 4 = 0,$$

the other factors giving imaginary roots, then if $t = y^2$, the equation in y becomes

$$y^3 + 3y^2 + 2y + 2 = 0,$$

or $$(y+1)^3 = y-1,$$

where $$y = \sinh\phi.$$

Putting $y+1 = v$, then $$y - 1 = v - 2,$$

and $$v^3 - v + 2 = 0;$$

which, compared with $$4v^3 - g_2 v - g_3 = 0,$$

has $$g_2 = 4, \quad g_3 = -8,$$

$$g_2^3 - 27g_3^2 = -64 \times 26,$$

so that the cubic has only one real root.

The absolute invariant of this cubic

$$J = \frac{g_2^3}{g_2^3 - 27g_3^2} = -\frac{1}{26};$$

so that, putting cosech $3a = \sqrt{26}$, then

$$v = \frac{2\cosh a}{\sqrt{3}}$$

(*Proc. Lond. Math. Soc.*, Vol. XVII., p. 263).

$\Delta = 30$. Taking Fiedler's modular equation of the 15th order,

$$P^3 - 4PQ + 4R = 0,$$

where $$P = \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} + 1,$$
$$Q = \sqrt[4]{\kappa\lambda\kappa'\lambda'} + \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'},$$
$$R = \sqrt[4]{\kappa\lambda\kappa'\lambda'};$$

and putting $$\kappa\lambda = x^8, \quad \kappa'\lambda' = 2\sqrt{\kappa\lambda} = 2x^4,$$

then $$P = x^2 + \sqrt[4]{2}\,x + 1 = x\,(t + \sqrt[4]{2}),$$
$$Q = \sqrt[4]{2}\,x^3 + x^2 + \sqrt[4]{2}\,x = x^2\,(\sqrt[4]{2}\,t + 1),$$
$$R = \sqrt[4]{2}\,x^3,$$

putting $$t = x + \frac{1}{x};$$

and then $$(t + \sqrt[4]{2})^3 - 4\,(t + \sqrt[4]{2})(\sqrt[4]{2}\,t + 1) + 4\sqrt[4]{2} = 0,$$
$$t^3 - \sqrt[4]{2}\,t^2 - \sqrt{2}\,(2\sqrt{2} + 1)\,t + \sqrt{2}\,\sqrt[4]{2} = 0,$$

factorizing into
$$\{t - \sqrt[4]{2}\,(\sqrt{2} + 1)\}\,(t^2 + \sqrt{2}\,\sqrt[4]{2}\,t - 2 + \sqrt{2}) = 0,$$

so that, if $$t = \sqrt[4]{2}\,(\sqrt{2} + 1),$$
$$\beta = \frac{1}{x^4} - x^4 = 2\sqrt{3}\,(4\sqrt{2} + 5),$$
$$\cosh 2\phi = \sqrt{3}\,(4\sqrt{2} + 5),$$
$$\sinh 2\phi = \sqrt{10}\,(3 + 2\sqrt{2}),$$
$$e^{2\phi} = \cosh 2\phi + \sinh 2\phi$$
$$= (\sqrt{6} + \sqrt{5})(4 + \sqrt{15}),$$
$$e^{-2\phi} = (\sqrt{6} - \sqrt{5})(4 - \sqrt{15});$$
$$\gamma = \frac{1}{x^4} + x^4 = 20 + 12\sqrt{2},$$
$$\sqrt{\kappa\lambda} = w^2 = \tfrac{1}{2}\,(\gamma - \beta)$$
$$= (2 - \sqrt{3})(5 - 2\sqrt{6}).$$

Therefore
$$\kappa = w^2 e^{-2\phi} = (2 - \sqrt{3})(5 - 2\sqrt{6})(\sqrt{6} - \sqrt{5})(4 - \sqrt{15}),$$

for $$K'/K = \sqrt{30};$$
$$\lambda = w^2 e^{2\phi} = (2 - \sqrt{3})(5 - 2\sqrt{6})(\sqrt{6} + \sqrt{5})(4 + \sqrt{15}),$$

for $$\Lambda'/\Lambda = \sqrt{(2 \div 15)}.$$

Similarly, $\kappa = (2 + \sqrt{3})(5 - 2\sqrt{6})(\sqrt{6} - \sqrt{5})(4 + \sqrt{15}),$

for $$K'/K = \sqrt{(6 \div 5)};$$

$$\lambda = (2+\sqrt{3})(5-2\sqrt{6})(\sqrt{6}+\sqrt{5})(4-\sqrt{15}),$$

for $$\Lambda'/\Lambda = \sqrt{(10 \div 3)}.$$

$\Delta = 34$. Taking Mr. Russell's form of the modular equation of the 17th order, and putting

$$\kappa\lambda = x^2, \qquad \kappa'\lambda' = 2x,$$

then $$P = x^2+2x-1 \;= -x(\beta-2),$$

$$Q = 2x^3-x^2-2x = -x^2(2\beta+1),$$

$$R = -2x^3,$$

and the modular equation

$$P^9+2^6R(-287P^6+2^5\times 261P^4Q-2^{12}\times 151P^2Q^2+2^{17}Q^3)$$
$$+2^{10}R^2(7309P^3-2^8\times 117PQ)+2^{21}\times 3^8R^3 = 0$$

becomes, on putting $\beta-2 = 4t$, and dividing out x^9,

$$t^9-574t^6-8352t^5-35940t^4-63859t^3-58464t^2-29040t-6272 = 0,$$

which can be factorized into

$$(t+1)(t+4)^2(t^2-11t-8)\left\{(t^2+t-5)^2+24(2t+1)^2\right\}.$$

Taking the quadratic $\quad t^2-11t-8 = 0,$

then $$t = \tfrac{1}{2}(11+3\sqrt{17});$$

and $$\beta = 4t+2 = 6(\sqrt{17}+4).$$

If we had taken Sohncke's modular equation of the 17th order (*Crelle*, XVI.),

$$(v-u)^{18}-16uv(1-u^8)(1-v^8)\left\{17uv(v-u)^6-(v^4-u^4)^2+16(1-u^4v^4)^2\right\}$$
$$= 0,$$

connecting $$u = \sqrt[4]{\kappa} \quad\text{and}\quad v = \sqrt[4]{\lambda};$$

and put $$v/u = e^{\phi},$$

then, if $$\kappa'\lambda' = 2\sqrt{\kappa\lambda} = 2u^2v^2,$$

$$(1-u^8)(1-v^8) = 4u^4v^4,$$

$$(1-u^4v^4)^2 = (v^4+u^4)^2 = 4u^4v^4\cosh^2 2\phi.$$

Putting $$s = \cosh\phi-1 = 2\sinh^2\tfrac{1}{2}\phi,$$

the equation becomes

$$2^9s^9-64(17\times 2^3s^3-4\sinh^2 2\phi+64\cosh^2 2\phi) = 0,$$

or, since $$\cosh\phi = s+1, \quad \cosh 2\phi = 2s^2+4s+1,$$
$$\sinh 2\phi = 2\,(s+1)\,\sqrt{(s^2+2s)},$$
$$s^9-17s^3+2\,(s+1)^2\,(s^2+2s)-8\,(2s^2+4s+1)^2 = 0,$$
or $$s^9-30s^4-137s^3-150s^2-60s-8 = 0.$$

But, since $$\beta = 2\cosh 2\phi, \quad t = \sinh^2\phi,$$
this equation in s will be found not in agreement with the previous equation for t.

There is consequently a misprint in Sohncke's equation; it should be
$$(v-u)^{18}-16uv\,(1-u^8)(1-v^8)\,\{17uv\,(v-u)^6-(v^4-u^4)^2+16\,(1+u^4v^4)^2\} = 0,$$
and now the equation for s becomes
$$2^9s^9-64\,\{17\times 2^3s^3-4\sinh^2 2\phi+64\,(\cosh^2 2\phi+1)\} = 0,$$
or $$s^9-17s^3+2\,(s+1)^2\,(s^2+2s)-8\,(2s^2+4s+1)-8 = 0,$$
or $$s^9-30s^4-137s^3-150s^2-60s-16 = 0\,;$$
factorizing into
$$(s+1)(s^2-s-4)(s^2+2s+4)\,\{(s^2-s-1)^2+6s^2\} = 0,$$
the factor $$s^2-s-4 = 0,$$
giving $$s = \tfrac{1}{2}\,(\sqrt{17}+1),$$
the required solution; and thus
$$\cosh\phi = \tfrac{1}{2}\,(\sqrt{17}+3),$$
$$\cosh 2\phi = 3\,(\sqrt{17}+4).$$

$\Delta = 38$. Taking Fiedler's or Russell's form of the modular equation of the 19th order,
$$P^5-112P^2R+256QR = 0,$$
where $$P = \sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}-1,$$
$$Q = \sqrt{\kappa\lambda\kappa'\lambda'}-\sqrt{\kappa\lambda}-\sqrt{\kappa'\lambda'},$$
$$R = -\sqrt{\kappa\lambda\kappa'\lambda'}\,;$$
and putting $$\kappa\lambda = x^4, \quad \kappa'\lambda' = 2x^2\,;$$
also $$x-\frac{1}{x} = t,$$

then
$$P = x^2+\sqrt{2}\,x-1 = x(t+\sqrt{2}),$$
$$Q = \sqrt{2}\,x^3-x^2-\sqrt{2}\,x = x^2(\sqrt{2}\,t-1),$$
$$R = -\sqrt{2}\,x^3;$$
and therefore
$$(t+\sqrt{2})^5+16\sqrt{2}\,\{7\,(t+\sqrt{2})^2-16\,(\sqrt{2}\,t-1)\} = 0;$$
or, putting
$$t+\sqrt{2} = \sqrt{2}\,y,$$
$$4\sqrt{2}\,y^5+224\sqrt{2}\,y^2-256\sqrt{2}\,(2y-3) = 0,$$
$$y^5+56y^2-64\,(2y-3) = 0;$$
or, putting
$$y = 2v,$$
$$v^5+7v^2-8v+6 = 0,$$
a *Hauptgleichung* quintic for v, having only one real root between 2 and -3.

This equation can be factorized into
$$(v^2-v+3)(v^3+v^2-2v+2) = 0;$$
the factor
$$v^2-v+3 = 0$$
giving
$$v = \tfrac{1}{2}(1+i\sqrt{11});$$
while
$$v^3+v^2-2v+2 = 0$$
has only one real root
$$v = -\tfrac{1}{3}\left[1+\sqrt[3]{\{37+3\sqrt{(114)}\}}+\sqrt[3]{\{37-3\sqrt{(114)}\}}\right].$$
Otherwise, the equation in t factorizes into
$$(t^2+22)(t^3+5\sqrt{2}\,t^2-2t+22\sqrt{2}) = 0;$$
in which equation
$$t = -\beta.$$

$\Delta = 42$. Taking Fiedler's form of the modular equation of the 21st order,
$$Z_1''-2Z_0^{(0,2)} = 0,$$
where
$$Z_1'' = \kappa\lambda+\kappa'\lambda'-1,$$
$$Z_0^{(0,2)} = -(\sqrt{\kappa\lambda'}+\sqrt{\kappa'\lambda})\sqrt[4]{\kappa\lambda\kappa'\lambda'}+(\sqrt{\kappa'}-\sqrt{\lambda'})\sqrt[4]{\kappa'\lambda'}$$
$$-(\sqrt{\kappa}-\sqrt{\lambda})\sqrt[4]{\kappa\lambda};$$
and putting
$$\kappa\lambda = w^4,\quad \kappa'\lambda' = 2w^2,\quad \beta = \frac{1}{w^2}-w^2;$$
an equation can be found for the determination of w and β, and thence of κ and λ.

$\Delta = 46$. Taking Fiedler's or Russell's form of the modular equation of the 23rd order,

$$P^3 - 4R = 0,$$

where
$$P = \sqrt[4]{\kappa\lambda} + \sqrt[4]{\kappa'\lambda'} - 1,$$
$$R = -\sqrt[4]{\kappa\lambda\kappa'\lambda'},$$

and putting
$$\kappa\lambda = x^8, \quad \kappa'\lambda' = 2x^4,$$

then
$$P = x^2 + \sqrt[4]{2}\,x - 1 = -x\,(t - \sqrt[4]{2}),$$
$$R = -\sqrt[4]{2}\,x^3,$$

putting
$$t = \frac{1}{x} - x\,;$$

and then
$$(t - \sqrt[4]{2})^3 - 4\sqrt[4]{2} = 0,$$
$$t - \sqrt[4]{2} - \sqrt{2}\,\sqrt[4]{2} = 0\,;$$
$$\frac{1}{x} - x = \sqrt[4]{2}\,(\sqrt{2}+1),$$
$$\frac{1}{x^2} + x^2 = 3\sqrt{2}\,(\sqrt{2}+1),$$
$$\frac{1}{x^2} - x^2 = \sqrt{(50+36\sqrt{2})},$$
$$\beta = \frac{1}{x^4} - x^4 = (\sqrt{2}+1)\sqrt{(4\sqrt{2}+3)},$$
$$= 3\sqrt{2}\sqrt{(294+208\sqrt{2})}$$
$$= 6\sqrt{(147+104\sqrt{2})},$$
$$\gamma = 52 + 36\sqrt{2} = 4\,(13+9\sqrt{2}).$$

$\Delta = 58$. According to Hermite (*Équations Modulaires*, p. 51), this is a determinant Δ for which the number a is integral, and by approximate numerical calculation from the formula

$$16a \approx -e^{\pi\sqrt{\Delta}} + 104,$$

we find $\quad a = -2^4 \times 3^8 \times 11^4, \quad \beta = 198, \quad \gamma = 26\sqrt{58}.$

If we put
$$\kappa\lambda = x^4, \quad \kappa'\lambda' = 2x^2,$$

and then
$$t = \frac{1}{x} - x,$$

and substitute in Mr. Russell's modular equation of the 29th order, we shall obtain an equation of the 15th order for t, having a factor corresponding to the value $\beta = 198$, and thus affording an independent verification of the numerical coefficients in this modular equation.

$\Delta = 62$. Taking Schröter's, Fiedler's, or Russell's modular equation of the 31^{st} order,

$$(P^2-4Q)^2-4PR = 0,$$

and putting $\kappa\lambda = x^8, \quad \kappa'\lambda' = 2x^4,$

then

$$P = \sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}+1 = x\,(t+\sqrt[4]{2}),$$

$$Q = \sqrt[4]{\kappa\lambda\kappa'\lambda'}+\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'} = x^2\,(\sqrt[4]{2}\,t+1),$$

$$R = \sqrt[4]{\kappa\lambda\kappa'\lambda'} = \sqrt[4]{2}\,x^3,$$

where

$$t = \frac{1}{x}+x,$$

then

$$\{(t+\sqrt[4]{2})^2-4\sqrt[4]{2}\,t-4\}^2-4\sqrt[4]{2}\,(t+\sqrt[4]{2}) = 0,$$

factorizing into

$$\{t^2-(2-\sqrt{2})\sqrt[4]{2}\,t-6+3\sqrt{2}\}\{t^2-(2+\sqrt{2})\sqrt[4]{2}\,t-2+\sqrt{2}\} = 0,$$

of which the second factor will give the required numerical results.

$\Delta = 78$. Taking Fiedler's form of the modular equation of the 39^{th} order, and putting

$$\kappa\lambda = x^8, \quad \kappa'\lambda' = 2x^4, \quad t = x-\frac{1}{x},$$

then

$$Z_1 = x^2+\sqrt[4]{2}\,x-1 = x\,(t+\sqrt[4]{2}),$$

$$Z_2 = \sqrt[4]{2}\,x^3-x^2-\sqrt[4]{2}\,x = x^2\,(\sqrt[4]{2}\,t-1),$$

$$Z_3 = -\sqrt[4]{2}\,x^3,$$

$$Z_{2'} = x^2\,\{t-\sqrt[4]{2})^2+4\};$$

and then the modular equation becomes, dividing out x^7,

$$(t+\sqrt[4]{2})^6\,\{(t-\sqrt[4]{2})^2+4\}+4\sqrt[4]{2}\,\{(t-\sqrt[4]{2})^2+4\}^2$$
$$+20\sqrt[4]{2}\,(t+\sqrt[4]{2})^2\,\{(t-\sqrt[4]{2})^2+4\}-8\sqrt[4]{2}\,(t+\sqrt[4]{2})^4$$
$$-144\sqrt{2}\,(t+\sqrt[4]{2}) = 0,$$

an equation of the 7^{th} degree for t.

$\Delta = 94$. Taking Fiedler's or Russell's modular equation of the 47^{th} order, and putting

$$\sqrt[4]{\kappa\lambda} = x^2, \quad \sqrt[4]{\kappa'\lambda'} = \sqrt[4]{2}\,x, \quad \frac{1}{x}+x = t,$$

then $$P = x^2+\sqrt[4]{2}\,x+1 = x\,(t+\sqrt[4]{2}),$$
$$Q = \sqrt[4]{2}\,x^3+x^2+\sqrt[4]{2}\,x = x^2\,(\sqrt[4]{2}\,t+1),$$
$$R = \sqrt[4]{2}\,x^3,$$

and $$P^2-4Q = x^2\,(t^2-2\sqrt[4]{2}\,t+\sqrt{2}-4)$$
$$= x^2\left\{(t-\sqrt[4]{2})^2-4\right\}.$$

Then Russell's modular equation of the 47th order (*Proc. Lond Math. Soc.*, Nov., 1887) becomes, dividing out x^6,

$$\left\{(t-\sqrt[4]{2})^2-4\right\}^3-28\sqrt[4]{2}\,(t+\sqrt[4]{2})^3-96\sqrt[4]{2}\,(t+\sqrt[4]{2})(\sqrt[4]{2}\,t+1)$$
$$-128\sqrt{2} = 0,$$

a sextic equation for t.

But, if we take Hurwitz's modular equation of the 47th order (*Math. Ann.*, XIV.),

$$\left\{2\,(\sqrt[4]{\kappa\lambda}+\sqrt[4]{\kappa'\lambda'}-1)-\sqrt[3]{4}\,\sqrt[12]{\kappa\lambda\kappa'\lambda'}\right\}^2$$
$$= 8\,(\sqrt{\kappa\lambda}+\sqrt{\kappa'\lambda'}+1)-7\sqrt[3]{16}\,\sqrt[6]{\kappa\lambda\kappa'\lambda'},$$

and put $$\kappa\lambda = x^8, \qquad \kappa'\lambda' = 2x^4,$$

then $$(2x^2+2\sqrt[4]{2}\,x-2-\sqrt{2}\,\sqrt[4]{2}\,x)^2$$
$$= 8x^4+8\sqrt{2}\,x^2+8-14\sqrt{2}\,x^2$$
$$= 8x^4-6\sqrt{2}\,x^2+8,$$

or $$\left\{2x^2+\sqrt[4]{2}\,(2-\sqrt{2})\,x-2\right\}^2 = 8x^4-6\sqrt{2}\,x^2+8,$$

$$4x^4+4\sqrt[4]{2}\,(2-\sqrt{2})\,x^3+\sqrt{2}\,(6-4\sqrt{2})\,x^2-8x^2$$
$$-4\sqrt[4]{2}\,(2-\sqrt{2})\,x+4 = 8x^4-6\sqrt{2}\,x^2+8,$$

or $$4x^4-4\sqrt[4]{2}\,(2-\sqrt{2})\,x^3-4\sqrt{2}\,(3-2\sqrt{2})\,x^2+4\sqrt[4]{2}\,(2-\sqrt{2})\,x+y = 0,$$

$$x^4-\sqrt[4]{2}\,(2-\sqrt{2})\,x^3-\sqrt{2}\,(3-2\sqrt{2})\,x^2+\sqrt[4]{2}\,(2-\sqrt{2})\,x+1 = 0.$$

Putting $$\frac{1}{x}-x = v,$$

this equation becomes

$$v^2-\sqrt[4]{2}\,(2-\sqrt{2})\,v-3\sqrt{2}+6 = 0,$$

$$\left(v-\frac{\sqrt{2}-1}{\sqrt[4]{2}}\right)^2 = \frac{9-8\sqrt{2}}{\sqrt{2}}.$$

But putting $$\frac{1}{x}+x = t,$$

then $t^2-2+\sqrt[4]{2}\,(2-\sqrt{2})\sqrt{(t^2-4)}-\sqrt{2}\,(3-2\sqrt{2})=0,$

$$t^2-3\sqrt{2}+2+\sqrt[4]{2}\,(2-\sqrt{2})\sqrt{(t^2-4)}=0,$$

$$t^4-(6\sqrt{2}-4)\,t^2+22-12\sqrt{2}=(6\sqrt{2}-8)(t^2-4),$$

$$t^4-(12\sqrt{2}-12)\,t^2-10+12\sqrt{2}=0,$$

a quadratic for t^2, from which the equations for β and γ can be derived.

Appendix.

[The current number of the *Acta Mathematica*, XI. 4, contains an article by H. Weber, "Zur Theorie der elliptischen Functionen" (zweite Abhandlung), which gives a number of numerical results for the modular functions in Complex Multiplication, agreeing in many respects with the results given in this paper. By the aid of Weber's calculations, it is possible in some instances to add to and simplify some of the cases considered above, examples of which are given herewith, as well as developments of cases not treated completely before.

CLASS A.

This is the case of $\Delta\equiv 3$, mod. 8; it was convenient to put

$$\alpha=\frac{(1-t^8)^3}{t^8}=\frac{(1-256s^{24})^3}{256s^{24}},$$

so that $$t^8=256s^{24}=16\kappa^2\kappa'^2;$$

and then $$s\approx q^{\frac{1}{12}}.$$

With this notation, then, for $\Delta=35$ (Weber, *Acta Math.*, XI., p. 388), $$2s^3-(\sqrt{5}+1)(s^2-s)-1=0.$$

$\Delta=51$: $$t^3+t^2+(\sqrt{17}+4)\,t-1=0 \quad \text{[p. 385].}$$

$\Delta=91$: $$2s^3+(\sqrt{13}+1)\,s^2+2s-1=0 \quad \text{[p. 385].}$$

$\Delta=99$: $$t^3+(\sqrt{33}+4)\,t^2+(13+2\sqrt{33})\,t-1=0 \quad \text{[p. 384],}$$

leading to the value of α, namely

$$\alpha=2^6.(4591804316+799330532\sqrt{33}) \quad \text{[p. 384].}$$

CLASS C.

$\Delta=17$. Weber's equation gives

$$\frac{1}{\sqrt[6]{(2\kappa\kappa')}}+\sqrt[6]{(2\kappa\kappa')}=\tfrac{1}{2}(\sqrt{17}+1).$$

$\Delta = 29$. Weber's equation for

$$x = \frac{1}{\sqrt[3]{(2\kappa\kappa')}}$$

is
$$2x^3 - 9x^2 - 8x - 5 = \sqrt{29}\,(x+1)^2;$$
which when rationalised becomes the reciprocal sextic

$$x^6 - 9x^5 + 5x^4 - 2x^3 - 5x^2 - 9x - 1 = 0.$$

The corresponding equation in $z = \frac{1}{x^3}$ agrees with the one given in this paper, p. 328,

$$z^6 + 588z^5 - 979z^4 + 1960z^3 + 979z^2 + 588z - 1 = 0,$$

which therefore can be reduced to the cubic

$$z^3 + 294z^2 + 155z + 70 = \sqrt{29}\,(55z^2 + 28z + 13).$$

$\Delta = 41$. The equation for

$$z = x + \frac{1}{x},$$

where
$$\frac{1}{x} = \sqrt[6]{(2\kappa\kappa')}$$

is, according to Weber [*Acta Math.*, XI., p. 388],

$$z^3 - \tfrac{1}{2}(\sqrt{41}+5)\,z + \tfrac{1}{2}(7+\sqrt{41}) = 0;$$

so that z is the root of the biquadratic

$$z^4 - 5z^3 + 3z^2 + 3z + 2 = 0;$$

and then the equation for $\gamma = \frac{1}{2\kappa\kappa'} + 2\kappa\kappa'$

is easily calculated, and also the equations for α and β.

$\Delta = 73$. Since $p = 2$ for this number, as well as for $\Delta = 17$, we can anticipate that α, β, γ will each be of the form $M\sqrt{73}+N$; and, in fact, by approximate numerical calculation, we shall find

$$\frac{1}{2\kappa\kappa'} + 2\kappa\kappa' = \gamma = 4930\sqrt{73} + 42120,$$

equivalent to
$$\frac{1}{\sqrt[6]{(2\kappa\kappa')}} + \sqrt[6]{(2\kappa\kappa')} = \tfrac{1}{2}(\sqrt{73}+5).$$

$\Delta = 193$. This is another number for which $p = 2$, according to

Gauss (*Werke*, t. II., p. 288); and from the approximate values of $\frac{1}{\sqrt[6]{(2\kappa\kappa')}}$, namely, $\frac{1}{2}\sqrt{2}\,e^{\frac{1}{12}\pi\sqrt{\Delta}}$, we find

$$\frac{1}{\sqrt[6]{(2\kappa\kappa')}}+\sqrt[6]{(2\kappa\kappa')}=\sqrt{193}+13,$$

whence α, β, γ can be inferred.

Similarly, for $\Delta=97$, we shall find

$$\frac{1}{\sqrt[6]{(2\kappa\kappa')}}+\sqrt[6]{(2\kappa\kappa')}=\tfrac{1}{2}\left(\sqrt{97}+9\right).$$

leading to $$\gamma=33210\sqrt{97}+327080,$$

instead of the value given above, p. 338.

These values lead to the approximate equations

$$e^{\frac{1}{6}\pi\sqrt{97}}\approx 9\sqrt{97}+85,$$

$$e^{\frac{1}{6}\pi\sqrt{193}}\approx 52\sqrt{193}+720.$$

10th Oct., 1888.]

Thursday, May 10*th*, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

The following communications were made:—

Some Theorems on Parallel Straight Lines, together with some attempts to prove Euclid's Twelfth Axiom: J. Cook Wilson, M.A.

On Cyclicants or Ternary Reciprocants, and allied Functions E. B. Elliott, M.A.

On the Flexure and the Vibrations of a Curved Bar: Prof. H. Lamb, F.R.S.

On the Figures formed by the Intercepts of a System of Straight Lines in a Plane, and on Analogous Relations in Space of Three Dimensions: S. Roberts, F.R.S.

Lamé's Differential Equation; and Stability of Orbits: Prof. Greenhill, F.R.S.

The following presents were received:—

Cabinet Likeness of Dr. Glaisher, F.R.S., for the Society's Album.

"Proceedings of the Royal Society," Vol. XLIII., No. 264.

"Educational Times," for May.

"Proceedings of the Physical Society of London," Vol. IX., Part II., April, 1888.

"Mathematics from the 'Educational Times,'" Vol. XLVIII.

"A Treatise on Hydrodynamics," by A. B. Basset, M.A., Vol. I., 8vo; Cambridge and London, 1888.

"Bulletin de la Société Mathématique de France," Tome XVI., Nos. 2 and 3.

"Beiblätter zu den Annalen der Physik und Chemie," Band XII., Stück 4; Leipzig, 1888.

"Journal für die reine und angewandte Mathematik," Band CIII., Heft I.; Berlin, 1888.

"Acta Mathematica," XI., 2.

"Rendiconti del Circolo Matematico di Palermo," Fasc. 1 and 2, Tomo II.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VIII., No. 3; Coimbra, 1887.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahr. XXXII., Heft IV.; Zürich, 1887.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. IV., Fasc. 1; Gennaio, 1888, Roma.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 55 and 56.

"The Earth's Polar Floods in Perihelion," by G. T. Carruthers (Subathu, India, March, 1888, 5 pp.).

Purchased: "Year-book of the Scientific and Learned Societies of Great Britain and Ireland" (Fifth Annual Issue; London, Charles Griffin & Co., 1888).

On the Flexure and the Vibrations of a Curved Bar.

By Professor HORACE LAMB, M.A., F.R.S.

[*Read May 10th*, 1888.]

The flexure of a curved bar has been treated in a general manner by Kirchhoff, Clebsch, and Thomson and Tait, but the special applications which have been made of the theory are very few. In this paper I propose to discuss the flexure in its own plane of a uniform bar whose axis forms in the unstrained state an arc of a circle. After establishing the general equations and the terminal conditions, some simple statical problems are solved, and I then proceed to discuss the vibrations of a "free-free" bar, with special reference to the case where the total curvature is slight. This latter problem is interesting as bearing on some observations by Chladni, referred to by Tyndall in his book on "Sound," Chap. iv.

Taking the centre of the circle as origin, and denoting the radius