Some Applications of Weierstrass's Elliptic Functions.

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In this paper it is proposed to exhibit the use of Weierstrass's Elliptic Functions, by showing their direct application to several geometrical and physical problems, and thus to give illustrations of the meaning of the analytical formulæ expressing the relations between these functions.

The formulæ are, in general, quoted without demonstration, and applied immediately to the problem requiring their use; the reader, however, who is desirous of following out the rigorous demonstration of these formulæ by the methods of pure mathematics, is recommended to consult Schwarz's *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen;* or Halphen's *Traité des Fonctions Elliptiques et de leurs Applications*, Paris, 1886.

I. *Confocal Cartesians, and Quartic Curves.*

1. Let $$z = \wp u,$$

where $$z = x+iy, \quad u = \tfrac{1}{2}(\xi+i\eta),$$

and $\wp u$ denotes Weierstrass's elliptic function of u, defined by the

equation $$u = \int_z^\infty \frac{dz}{\sqrt{(4z^3-g_2z-g_3)}};$$

so that $$z = \wp u,$$

and $$\left(\frac{dz}{du}\right)^2 = (\wp' u)^2$$
$$= 4z^3-g_2z-g_3$$
$$= 4(z-e_1)(z-e_2)(z-e_3), \text{ suppose.}$$

Then, if e_1, e_2, e_3 are all real, they will define the positions of the three foci F_1, F_2, F_3 on the axis of x of a system of confocal Cartesians, given by the equations

$$\xi = \text{const.}, \quad \text{and} \quad \eta = \text{const.},$$

from the relation $$x+iy = \wp\, \tfrac{1}{2}(\xi+i\eta);$$

and, from the properties of *conjugate functions*, it follows immediately that these confocal Cartesians intersect at right angles.

2. With the notation of the *sigma functions* explained by Schwarz or Halphen, we put

$$\wp u - e_1 = \left(\frac{\sigma_1 u}{\sigma u}\right)^2,$$

$$\wp u - e_2 = \left(\frac{\sigma_2 u}{\sigma u}\right)^2,$$

$$\wp u - e_3 = \left(\frac{\sigma_3 u}{\sigma u}\right)^2.$$

In the ordinary notation of elliptic functions,

$$\wp u - e_1 = (e_1 - e_3)\frac{\operatorname{cn}^2 \sqrt{(e_1 - e_3)}\, u}{\operatorname{sn}^2 \sqrt{(e_1 - e_3)}\, u}, \quad \text{or } (e_1 - e_3)\operatorname{cs}^2 \sqrt{(e_1 - e_3)}\, u,$$

$$\wp u - e_2 = (e_1 - e_3)\frac{\operatorname{dn}^2 \sqrt{(e_1 - e_3)}\, u}{\operatorname{sn}^2 \sqrt{(e_1 - e_3)}\, u}, \quad \text{or } (e_1 - e_3)\operatorname{ds}^2 \sqrt{(e_1 - e_3)}\, u,$$

$$\wp u - e_3 = (e_1 - e_3)\frac{1}{\operatorname{sn}^2 \sqrt{(e_1 - e_3)}\, u}, \quad \text{or } (e_1 - e_3)\operatorname{ns}^2 \sqrt{(e_1 - e_3)}\, u,$$

with
$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad k'^2 = \frac{e_1 - e_2}{e_1 - e_3},$$

supposing $e_1 > e_2 > e_3$.

3. Denoting by r_1, r_2, r_3 the distances of a point P whose coordinates are x, y from the three foci F_1, F_2, F_3, and denoting $\frac{1}{2}(\xi - i\eta)$ by v; then

$$r_1 = \frac{\sigma_1 u}{\sigma\, u}\frac{\sigma_1 v}{\sigma\, v}, \quad r_2 = \frac{\sigma_2 u}{\sigma\, u}\frac{\sigma_2 v}{\sigma\, v}, \quad r_3 = \frac{\sigma_3 u}{\sigma\, u}\frac{\sigma_3 v}{\sigma\, v};$$

and, expressed in a real form by means of formula [*D*][9], p. 51, of Schwarz's *Formeln*,

$$r_1 = \frac{\sigma_1 u}{\sigma\, u}\frac{\sigma_1 v}{\sigma\, v}$$

$$= \frac{\sigma_1 u\ \sigma_2 u\ \sigma_1 v\ \sigma_2 v}{\sigma\, u\ \sigma_2 u\ \sigma\, v\ \sigma_2 v}$$

$$= -(e_3 - e_1)\frac{\sigma_1 \xi\ \sigma_2 i\eta + \sigma_2 \xi\ \sigma_1 i\eta}{\sigma_3 \xi\ \sigma_1 i\eta - \sigma_1 \xi\ \sigma_3 i\eta} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i.)};$$

or, again,
$$r_1 = \frac{\sigma_1 u\ \sigma_3 u\ \sigma_1 v\ \sigma_3 v}{\sigma\, u\ \sigma_3 u\ \sigma\, v\ \sigma_3 v}$$

$$= -(e_1-e_2)\,\frac{\sigma_1\xi\ \sigma_3 i\eta + \sigma_3\xi\ \sigma_1 i\eta}{\sigma_1\xi\ \sigma_2 i\eta - \sigma_2\xi\ \sigma_1 i\eta} \quad \text{.................(ii.).}$$

Similarly

$$r_2 = -(e_1-e_2)\,\frac{\sigma_2\xi\ \sigma_3 i\eta + \sigma_3\xi\ \sigma_2 i\eta}{\sigma_1\xi\ \sigma_2 i\eta - \sigma_2\xi\ \sigma_1 i\eta} \quad \text{.................(iii.),}$$

$$= -(e_2-e_3)\,\frac{\sigma_1\xi\ \sigma_2 i\eta + \sigma_2\xi\ \sigma_1 i\eta}{\sigma_2\xi\ \sigma_3 i\eta - \sigma_3\xi\ \sigma_2 i\eta} \quad \text{.................(iv.),}$$

and

$$r_3 = -(e_2-e_3)\,\frac{\sigma_3\xi\ \sigma_1 i\eta + \sigma_1\xi\ \sigma_3 i\eta}{\sigma_2\xi\ \sigma_3 i\eta - \sigma_3\xi\ \sigma_2 i\eta} \quad \text{.................(v.),}$$

$$= -(e_3-e_1)\,\frac{\sigma_2\xi\ \sigma_3 i\eta + \sigma_3\xi\ \sigma_2 i\eta}{\sigma_3\xi\ \sigma_1 i\eta - \sigma_1\xi\ \sigma_3 i\eta} \quad \text{.................(vi.).}$$

4. Therefore, from (iv.) and (v.),

$$\left.\begin{aligned} r_2\sigma_3\,\xi - r_3\sigma_2\,\xi &= \quad (e_2-e_3)\,\sigma_1\xi \\ r_2\sigma_3 i\eta - r_3\sigma_2 i\eta &= -(e_2-e_3)\,\sigma_1 i\eta \end{aligned}\right\} \quad \text{.................(A),}$$

the vectorial equations of conjugate confocal Cartesians; and from the remaining equations, by cyclical interchange of suffixes, we obtain

$$\left.\begin{aligned} r_3\sigma_1\xi - r_1\,\sigma_3\xi &= \quad (e_3-e_1)\,\sigma_2\xi \\ r_3\sigma_1 i\eta - r_1\sigma_3 i\eta &= -(e_3-e_1)\,\sigma_2 i\eta \end{aligned}\right\} \quad \text{.................(B),}$$

and

$$\left.\begin{aligned} r_1\,\sigma_2\xi - r_2\,\sigma_1\xi &= \quad (e_1-e_2)\,\sigma_3\xi \\ r_1\sigma_2 i\eta - r_2\sigma_1 i\eta &= -(e_1-e_2)\,\sigma_3 i\eta \end{aligned}\right\} \quad \text{.................(C),}$$

also

$$\left.\begin{aligned} (e_2-e_3)\,r_1\sigma_1\,\xi + (e_3-e_1)\,r_2\sigma_2\,\xi + (e_1-e_2)\,r_3\sigma_3\,\xi &= 0 \\ (e_2-e_3)\,r_1\sigma_1 i\eta + (e_3-e_1)\,r_2\sigma_2 i\eta + (e_1-e_2)\,r_3\sigma_3 i\eta &= 0 \end{aligned}\right\} \quad \text{...(D);}$$

and (A), (B), (C), (D) are the vectorial equation of the same confocal Cartesians in a symmetrical form (Darboux, *Annales Scientifiques de l'École Normale Supérieure*, Tome IV., 1867).

5. To indicate the values of the *invariants* g_2 and g_3, the notation

$$z = \wp\,(u\,;\ g_2,\ g_3)$$

is sometimes employed; and then it follows, from considerations of homogeneity, that

$$\begin{aligned} \wp\,(mu\,;\ g_2,\ g_3) &= \frac{1}{m^2}\,\wp\,(u\,;\ m^4g_2,\ m^6g_3), \\ \sigma\,\ (mu\,;\ g_2,\ g_3) &= m\sigma\,\ (u\,;\ m^4g_2,\ m^6g_3), \\ \sigma_\lambda\,(mu\,;\ g_2,\ g_3) &= \quad \sigma_\lambda\,(u\,;\ m^4g_2,\ m^6g_3)\,; \end{aligned}$$

so that
$$\wp\,(i\eta\,;\ g_2, g_3) = -\wp\,(\eta\,;\ g_2, -g_3),$$
$$\sigma\,(i\eta\,;\ g_2, g_3) = i\sigma\,(\eta\,;\ g_2, -g_3),$$
$$\sigma_\lambda(i\eta\,;\ g_2, g_3) = \sigma_\lambda(\eta\,;\ g_2, -g_3)\,;$$

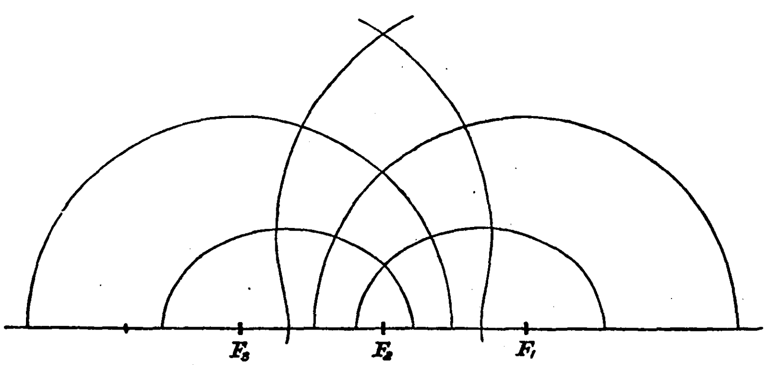
equivalent in Legendre and Jacobi's notation to a transformation to the complementary modulus.

If
$$\wp\omega_1 = e_1,\quad \wp\omega_2 = e_2,\quad \wp\omega_3 = e_3,$$

then ω_1, ω_2, ω_3 are called *half-periods* of the elliptic function $\wp u$; but of these only two are independent, as they are connected by the relation
$$\omega_1+\omega_2+\omega_3 = 0;$$
also
$$e_1 + e_2 + e_3 = 0.$$

Supposing $e_1 > e_2 > e_3$, then ω_1 is real, but ω_3 is imaginary; and
$$\frac{\omega_3}{\omega_1} = i\frac{K'}{K},$$

where K and K' denote Jacobi's periods.



6. Then, when $\xi = \frac{1}{2}\omega_1$, the corresponding Cartesian is a circle, centre F_1, and containing F_2, F_3 being the corresponding point to F_2; and when $i\eta = \frac{1}{2}\omega_3$, the corresponding Cartesian is a circle, centre F_3, and containing F_2, F_1 being the corresponding point to F_2.

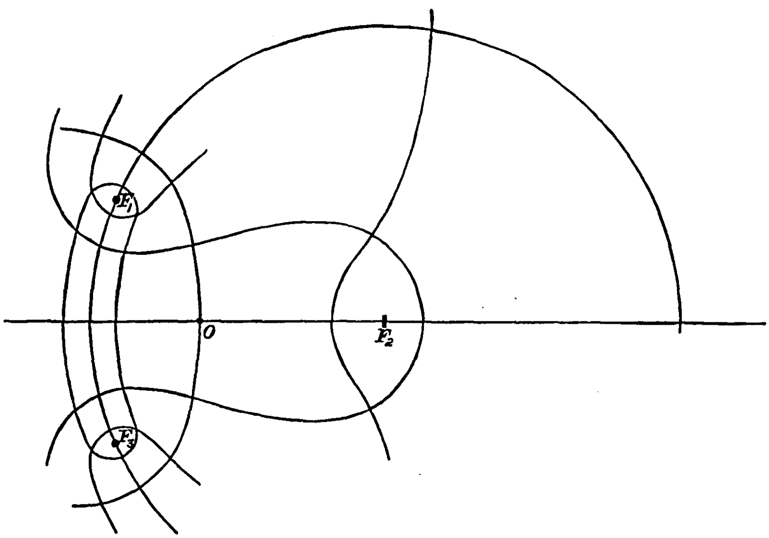
The two ovals of the same Cartesian are given by ξ and $\omega_1 - \xi$, or $i\eta$ and $\omega_3 - i\eta$.

7. If the *discriminant* $g_2^3 - 27g_3^2$ is negative, two of the quantities e_1, e_2, e_3 are imaginary; denoting them by e_1 and e_3, where $e_1 - e_3$ is positive imaginary, the corresponding foci form an isosceles triangle $F_1F_2F_3$, and the origin O is at the centre of gravity of the triangle, since

$$e_1 + e_2 + e_3 = 0.$$

Then $$x + iy = \wp \tfrac{1}{2}(\xi + i\eta)$$

denotes a series of orthogonal quartic curves, associated with Cartesians.



The values $\xi = \frac{1}{2}\omega_1$ or $i\eta = \frac{1}{2}\omega_3$ will each give $r_2 = F_1F_2$, so that the corresponding quartics double down into circular arcs of centre F_2, limited by F_1 and F_3.

(Holzmüller, *Einführung in die Theorie der isogonalen Verwandtschaften*).

If $g_2 = 0$, the triangle $F_1F_2F_3$ is equilateral.

8. Incidentally we notice the electrical application of these formulæ; namely, the electrification of an insulated cylinder whose cross section in one of these quartic curves is proportional to $(r_1r_2r_3)^{-\frac{1}{2}}$; and in particular, for a cross section the limited arc of a circle, the electrification is proportional to $(r_1r_3)^{-\frac{1}{2}}$, r_1 and r_3 denoting the distances of a point on the surface from the edges of the cylinder.

Then $\xi =$ const. and $\eta =$ const. will represent either the equi potential surfaces, or the orthogonal lines of force.

9. Consider the system given by

$$u = \int \frac{dz}{(1-z^3)^{\frac{2}{3}}}$$

(Siebeck, *Crelle*, 57 and 59, *Ueber eine Gattung von Curven vierten Grades, &c.*; Schwarz, *Crelle*, 77, *Ueber ebene algebraische Isothermen*).

Then
$$\frac{1+z}{1-z} = \wp'\left(\frac{u}{\sqrt[3]{3}};\ 0,\ \tfrac{1}{3}\right),$$

$$\frac{(1-z^3)^{\frac{1}{3}}}{1-z} = \sqrt{3}\,\wp\left(\frac{u}{\sqrt[3]{3}};\ 0,\ \tfrac{1}{3}\right);$$

or
$$\frac{1+z^3}{1-z^3} = \wp'(u;\ 0, -1),$$

$$\frac{z}{(1-z^3)^{\frac{2}{3}}} = \wp(u;\ 0, -1),$$

giving a system of sextic orthogonal curves.

II. *Reciprocants.*

10. Consider the Mixed Reciprocant

$$tc - 5ab = 0,$$

or
$$\frac{dy}{dx}\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2}\frac{d^3y}{dx^3} = 0,$$

given in Prof. Sylvester's Inaugural Lecture, Dec. 12, 1885, and published in *Nature*, Jan. 7, 1886.

Then
$$\frac{d^4y}{dx^4}\Big/\frac{d^3y}{dx^3} = 5\,\frac{d^2y}{dx^2}\Big/\frac{dy}{dx};$$

and, integrating,
$$\log\frac{d^3y}{dx^3} = 5\log\frac{dy}{dx} + \text{const.},$$

or
$$\frac{d^3y}{dx^3} = C\left(\frac{dy}{dx}\right)^5.$$

Multiplying by $\frac{d^2y}{dx^2}$, and integrating again,

$$\tfrac{1}{2}\left(\frac{d^2y}{dx^2}\right)^2 = \tfrac{1}{6}C\left(\frac{dy}{dx}\right)^6 + C;$$

or, changing the constants,

$$\left(\frac{dt}{dx}\right)^2 = \kappa t^6 + \lambda,$$

so that
$$x = \int \frac{dt}{\sqrt{(\kappa t^6+\lambda)}} + \mu,$$
$$y = \int \frac{t\,dt}{\sqrt{(\kappa t^6+\lambda)}} + \nu.$$

11. By a change of origin, we can make μ and ν vanish, and by orthogonal projection parallel to the axes we can reduce κ and λ to unity, so that we need only consider

$$x = \int \frac{dt}{\sqrt{(1+t^6)}}, \quad y = \int \frac{t\,dt}{\sqrt{(1+t^6)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{i.}),$$

or
$$x = \int \frac{dt}{\sqrt{(1-t^6)}}, \quad y = \int \frac{t\,dt}{\sqrt{(1-t^6)}} \quad \ldots\ldots\ldots\ldots\ldots(\text{ii.}),$$

or
$$x = \int \frac{dt}{\sqrt{(t^6-1)}}, \quad y = \int \frac{t\,dt}{\sqrt{(t^6-1)}} \quad \ldots\ldots\ldots\ldots\ldots(\text{iii.}).$$

12. From (i.), $x = -\int \frac{dt^{-2}}{\sqrt{(4t^{-6}+4)}}, \quad y = \int \frac{dt^2}{(4t^6+4)},$

so that
$$t^{-2} = \wp\,(x\,;\,0,\,-4),$$
$$t^2 = \wp\,(y\,;\,0,\,-4)\,;$$

and therefore
$$\wp x\,\wp y = 1,$$

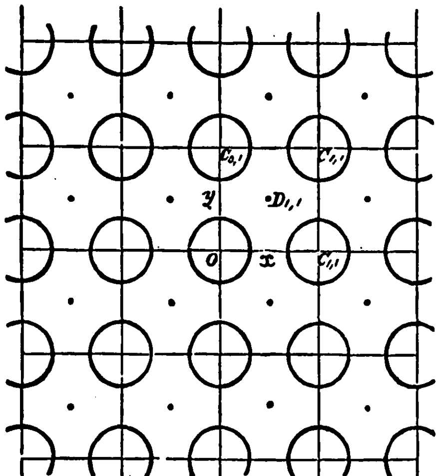
with $g_2 = 0$, $g_3 = -4$; representing, in figure (i.), a series of nearly circular curves round centres whose coordinates are $2m\omega_2$, $2m'\omega_2$; and a series of conjugate points at $(2m+1)\,\omega_2$, $(2m'+1)\,\omega_2$.

13. From (ii.),
$$x = -\int \frac{dt^{-2}}{\sqrt{(4t^{-6}-4)}}, \quad y = \int \frac{dt^2}{\sqrt{(-4t^6+4)}};$$

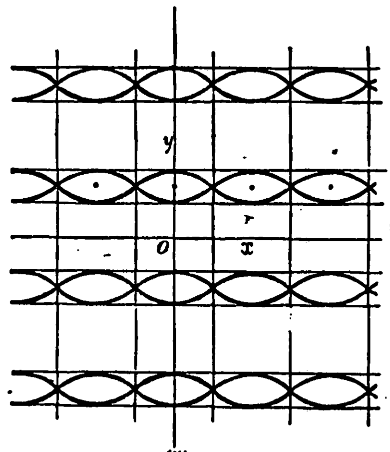
so that
$$t^{-2} = \wp\,(x\,;\,0,\,\,4),$$
$$t^2 = -\wp\,(y\,;\,0,\,-4)\,;$$

and therefore
$$\wp x\,\wp y = -1,$$

representing figure (ii.), and the figure of (iii.) is that of (ii.) turned through a right angle; having
$$t^{-2} = -\wp\,(x\,;\,0,\,-4),$$
$$t^2 = \wp\,(y\,;\,0,\,\,4).$$



(i.)



(ii.)

14. The intrinsic equation of (i.) is

$$s = \int \frac{\sqrt{(1+t^2)}\,dt}{\sqrt{(1+t^6)}}$$

$$= \int \frac{dt}{\sqrt{(1-t^2+t^4)}}$$

$$= \int \frac{d\psi}{\sqrt{(1-\frac{3}{4}\sin^2 2\psi)}}, \text{ if } t = \tan\psi,$$

the elliptic integral of the first kind, to modulus sin 60°; so that

$$2\psi = \text{am } 2s,$$

and

$$\sin 2\psi = \text{sn } 2s,$$

the intrinsic equation of the curve.

It is curious that the modular angle in the Cartesian equation of the curve is 15°, and in the intrinsic equation is 60°.

15. Similarly, in (ii.),

$$s = \int \frac{\sqrt{(1+t^2)}\,dt}{\sqrt{(1-t^6)}}$$

$$= \int \frac{d\psi}{\sqrt{(\cos^6\psi - \sin^6\psi)}}$$

$$= \int \frac{d\psi}{\sqrt{\{\cos 2\psi\,(1-\frac{1}{4}\sin^2 2\psi)\}}};$$

and, generally, $$s = \int \frac{\sqrt{(1+t^2)}\,dt}{\sqrt{(\kappa t^6+\lambda)}}$$
$$= \int \frac{d\psi}{\sqrt{(\kappa \cos^6\psi + \lambda \sin^6\psi)}},$$

which is expressible as the sum of two elliptic integrals of the first kind.

16. The Orthogonal Reciprocant

$$(t^2+1)\,c - 10abt + 15a^3 = 0,$$

obtained by integrating the above Mixed Reciprocant, has been integrated by Mr. Hammond (*Nature*, Jan. 7, 1886, p. 231) in the form

$$x = \int \frac{dt}{\sqrt{\{\kappa\,(1-15t^2+15t^4-t^6)+\lambda\,(6t-20t^3+6t^5)\}}} + \mu,$$
$$y = \int \frac{t\,dt}{\sqrt{\{\kappa\,(1-15t^2+15t^4-t^6)+\lambda\,(6t-20t^3+6t^5)\}}} + \nu;$$

changing his λ into 2λ; and then we see that

$$x+iy = \int \frac{(1+it)\,dt}{\sqrt{\{\tfrac{1}{2}\,(\kappa-i\lambda)(1+it)^6+\tfrac{1}{2}\,(\kappa+i\lambda)(1-it)^6\}}} + \mu + i\nu.$$

17. By a change of origin we can make μ and ν vanish, and by turning the axes through an angle $\frac{1}{6}\tan^{-1}\lambda/\kappa$ we can make λ vanish; so that

$$x+iy = \frac{1}{\sqrt{(\frac{1}{2}\kappa)}} \int \frac{\dfrac{dt}{(1+it)^2}}{\sqrt{\left\{\left(\dfrac{1-it}{1+it}\right)^6+1\right\}}}$$
$$= \frac{1}{\sqrt{(\frac{1}{2}\kappa)}} \int \frac{d\left(\dfrac{1-it}{1+it}\right)}{\sqrt{\left\{-\left(\dfrac{1-it}{1+it}\right)^6-1\right\}}}$$
$$= \frac{1}{\sqrt{(\frac{1}{2}\kappa)}} \int \frac{d\left(\dfrac{1+it}{1-it}\right)^2}{\sqrt{\left\{-4\left(\dfrac{1+it}{1-it}\right)^6-4\right\}}};$$

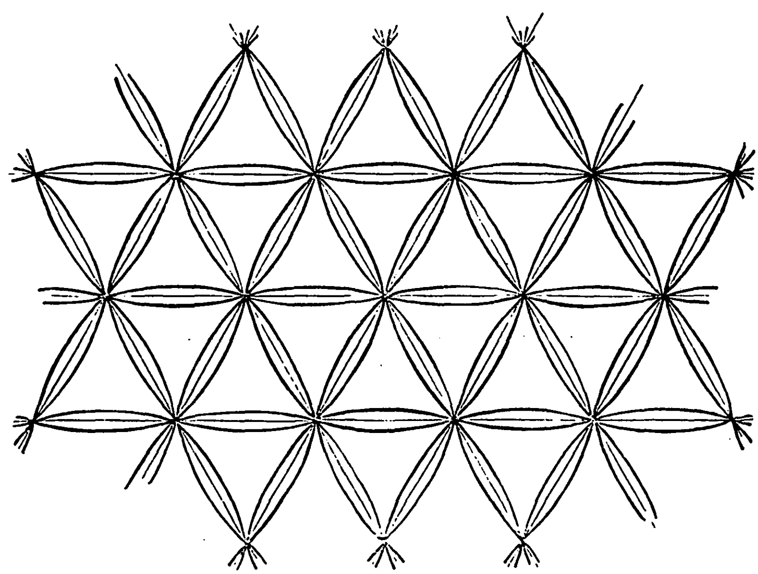
so that, replacing $\frac{1}{2}\kappa$ by unity, which may be done without loss of generality, $$\left(\frac{1+it}{1-it}\right)^2 = -\wp\,(x+iy;\ 0,\ 4).$$

18. Changing the sign of i,

$$\left(\frac{1-it}{1+it}\right)^2 = -\wp\,(x-iy;\ 0,\ 4),$$

so that $$\wp\,(x+iy)\,\wp\,(x-iy) = 1;$$

agreeing with Mr. Rogers's form (*Proc. Lond. Math. Soc.*, March, 1886), and giving a curve as in the annexed figure.



Also, with $g_2 = 0$, $g_3 = 4$,

$$\wp\,(x+iy)\,\wp\,(x-iy) = \frac{\wp^2 x\,\wp^2 iy + 4\,(\wp x + \wp iy)}{(\wp x - \wp iy)^2},$$

and the numerical values of $\wp x$ and $\wp iy$ are given in the Table, calculated by Mr. Hadcock, in *Proc. Lond. Math. Soc.*, Vol. XVII., p. 268.

We may also write the above relations as

$$\cos 4\psi + i \sin 4\psi = -\wp\,(x+iy),$$

$$\text{co } 4\psi - i \sin 4\psi = -\wp\,(x-iy).$$

19. When $g_2 = 0$, and ω denotes an imaginary cube root of unity,

$$\wp \omega u = \omega \wp u,$$

the simplest case of *Complex Multiplication* of Weierstrass's Elliptic Functions; so that our equation above (§ 18) may be written

$$\wp \omega (x+iy) \wp \omega^2 (x-iy) = 1,$$

showing that the coordinate axes may be turned through 120° without alteration of appearance, as indicated in the figure.

In Legendre's and Jacobi's notation, the equation of the curve to a different scale, with $8\sqrt{3\kappa} = 1$, and $\lambda = 0$ in § 16, may be written

$$k'^2 \operatorname{tn}^2 (x, k) = k^2 \operatorname{tn}^2 (y, k')$$

for the inclined branches; and

$$k^2 \operatorname{sn}^2 (x, k) = k'^2 \operatorname{sn}^2 (y, k')$$

for the horizontal branches (Sylvester, *American Journal of Mathematics*, Vol. VIII., p. 235), with $k = \sin 15°$, $k' = \sin 75°$; equivalent

to $$\operatorname{am} (x \pm K, k) = \operatorname{am} (y \pm K', k'),$$

or $$\operatorname{am} (x \pm iK', k) = \operatorname{am} (y \pm iK, k'),$$

where $$K'/K = \sqrt{3}.$$

20. This Reciprocant

$$(1+t^2)\, c - 10abt + 15a^3 = 0,$$

when expressed in the intrinsic form, has been shown by Captain MacMahon to become $\dfrac{d^3\psi}{ds^3} + 18 \left(\dfrac{d\psi}{ds}\right)^3 = 0.$

Integrating this equation,

$$\left(\frac{d^2\psi}{ds^2}\right)^2 = C - 9 \left(\frac{d\psi}{ds}\right)^4,$$

or, putting $\dfrac{d\psi}{ds} = \dfrac{1}{\rho} = q$; and denoting by m the maximum value of q,

$$\left(\frac{dq}{ds}\right)^2 = 9 (m^4 - q^4),$$

the solution of which is

$$\frac{d\psi}{ds} = q = m \operatorname{cn} \left(3\sqrt{2}\, ms, \frac{1}{\sqrt{2}}\right).$$

Therefore $$3\sqrt{2}\, \psi = \int \operatorname{cn} (3\sqrt{2}\, ms)\, d\, (3\sqrt{2}\, ms)$$

$$= \sqrt{2} \sin^{-1} \left\{ \frac{1}{\sqrt{2}} \operatorname{sn} (3\sqrt{2}\, ms) \right\},$$

or
$$\operatorname{sn}(3\sqrt{2}\,ms) = \sqrt{2}\sin 3\psi,$$
$$\sin 3\psi = \frac{1}{\sqrt{2}}\operatorname{sn}(3\sqrt{2}\,ms),$$
or
$$\cos 3\psi = \operatorname{dn}(3\sqrt{2}\,ms).$$

This is the intrinsic equation of a curve whose equation in Cartesian coordinates, using Jacobi's elliptic functions, is of one of the preceding forms of § 19, or

$$\operatorname{dn}(x, k)\operatorname{dn}(y, k') = k,$$

where $k = \sin 15°$, $k' = \sin 75°$, as mentioned in Mr. Hammond's paper (*Proc. Lond. Math. Soc.*, Vol. XVII., p. 130); and then another curious result is obtained, analogous to that of § 14, of a curve, whose Cartesian equation involves elliptic functions of modular angle 15°, having its arc expressed by an elliptic integral of the first kind of modular angle 45°; no simple transformation existing from one modulus to the other.

III. *Euler's Equations of Motion.*

21. These well-known equations, when there are no impressed forces, written in the form

$$\left.\begin{aligned} A\frac{dp}{dt}-(B-C)\,qr &= 0 \\ B\frac{dq}{dt}-(C-A)\,rp &= 0 \\ C\frac{dr}{dt}-(A-B)pq &= 0 \end{aligned}\right\},$$

are satisfied by
$$\left.\begin{aligned} Ap^2 &= -(B-C)(z-e_1) \\ Bq^2 &= -(C-A)(z-e_2) \\ Cr^2 &= -(A-B)(z-e_3) \end{aligned}\right\},$$

provided that

$$\frac{dz}{dt} = 2pqr,$$

or
$$\frac{dz^2}{dt^2} = 4p^2q^2r^2$$
$$= -4\frac{(B-C)(C-A)(A-B)}{ABC}(z-e_1)(z-e_2)(z-e_3)$$
$$= -\frac{(B-C)(C-A)(A-B)}{ABC}(4z^3-g_2z-g_3);$$

so that $$z = \wp u,$$

where $$\frac{du^2}{dt^2} = -\frac{(B-C)(C-A)(A-B)}{ABC} = M^2, \text{ suppose,}$$

since $(B-C)(C-A)(A-B)$ is negative; and then

$$u = Mt + \text{a constant.}$$

22. Then $$T = Ap^2+Bq^2+Cr^2$$
$$= (B-C)\,e_1 + (C-A)\,e_2 + (A-B)\,e_3;$$

and $$G^2 = A^2p^2+B^2q^2+C^2r^2$$
$$= A\,(B-C)\,e_1 + B\,(C-A)\,e_2 + C\,(A-B)\,e_3;$$

also $$0 = e_1 + e_2 + e_3;$$

so that $$e_1 = \frac{G^2\,(-2A+B+C) - T\,(2BC-CA-AB)}{3\,(B-C)(C-A)(A-B)},$$

$$e_2 = \frac{G^2\,(A-2B+C) - T\,(-BC+2CA-AB)}{3\,(B-C)(C-A)(A-B)},$$

$$e_3 = \frac{G^2\,(A+B-2C) - T\,(-BC-CA+2AB)}{3\,(B-C)(C-A)(A-B)};$$

and then $$g_2 = -4\,(e_2e_3+e_3e_1+e_1e_2), \quad g_3 = 4e_1e_2e_3.$$

Also $$e_2 - e_3 = \frac{AT-G^2}{(C-A)(A-B)},$$

$$e_3 - e_1 = \frac{BT-G^2}{(A-B)(B-C)},$$

$$e_1 - e_2 = \frac{CT-G^2}{(B-C)(C-A)};$$

and $$g_2 = \tfrac{2}{3}\left\{(e_2-e_3)^2+(e_3-e_1)^2+(e_1-e_2)^2\right\}.$$

Also the *discriminant*

$$D = g_2^3 - 27g_3^2$$
$$= 16\,(e_2-e_3)^2\,(e_3-e_1)^2\,(e_1-e_2)^2$$
$$= 16\,\frac{(AT-G^2)^2\,(BT-G^2)^2\,(CT-G^2)^2}{(B-C)^4\,(C-A)^4\,(A-B)^4}.$$

23. Supposing $A > B > C$; then

(i.) When the polhode encloses the axis A, $BT - G^2$ is negative, and then $e_2 - e_3$ is negative, $e_3 - e_1$ is negative, and $e_1 - e_2$ is positive; so that $e_1 > e_3 > e_2$, and $\wp u$ oscillates in value between e_2 and e_3, so that we have
$$u = Mt + \omega_2;$$

(ii.) when the polhode encloses the axis C, $BT - G^2$ is positive, and then $e_2 - e_3$ is negative, $e_3 - e_1$ is positive, and $e_1 - e_3$ is positive; so that $e_3 > e_1 > e_2$, and $\wp u$ oscillates in value between e_1 and e_2, so that
$$u = Mt + \omega_2,$$
as before.

IV. *The Spherical Pendulum and Top.*

24. In the spherical pendulum, of length l, the equations of motion may be written
$$\tfrac{1}{2} l^2 (\dot\theta^2 + \sin^2\theta\, \dot\psi^2) = g\,(b - l\cos\theta) \quad\text{.................(1)},$$
the *equation of energy;* and
$$l^2 \sin^2\theta\, \dot\psi = G \quad\text{............................(2)},$$
the *equation of conservation of angular momentum* about the vertical; using as coordinates, θ the polar distance on the sphere in circular measure from the highest point, and ψ the longitude.

Eliminating $\dot\psi$ between (1) and (2),
$$\tfrac{1}{2} l^2 \sin^2\theta\, \dot\theta^2 + \tfrac{1}{2}\frac{G^2}{l^2} = g\,(b - l\cos\theta)(1 - \cos^2\theta),$$
or
$$\tfrac{1}{2}\sin^2\theta\, \dot\theta^2 = \frac{g}{l}(\cos^2\theta - 1)\left(\cos\theta - \frac{b}{l}\right) - \tfrac{1}{2}\frac{G^2}{l^4} \quad\text{........(3)}.$$

25. Put $\cos\theta = 2z + \gamma$; then
$$2\dot z^2 = \frac{g}{l}\left\{(4z^2 + 4\gamma z + \gamma^2 - 1)\left(2z + \gamma - \frac{b}{l}\right) - \tfrac{1}{2}\frac{G^2}{gl^3}\right\}$$
$$= \frac{g}{l}\left\{8z^3 + 4\left(3\gamma - \frac{b}{l}\right)z^2 + \left(6\gamma^2 - 4\gamma\frac{b}{l} - 2\right)z + (\gamma^2 - 1)\left(\gamma - \frac{b}{l}\right) - \tfrac{1}{2}\frac{G^2}{gl^3}\right\};$$
and then, if
$$3\gamma = b/l,$$
$$\dot z^2 = \frac{g}{l}\left\{4z^3 - (3\gamma^2 + 1)z - \gamma(\gamma^2 - 1) - \frac{G^2}{4gl^3}\right\},$$

so that $$z = \wp(u;\ g_2, g_3),$$

and $$u = \sqrt{\frac{g}{l}}\, t + \text{a constant},$$

where $$g_2 = 3\gamma^2+1,\quad g_3 = \gamma(\gamma^2-1) - G^2/4gl^3;$$

and $$\cos\theta = 2\wp u + \gamma,$$

where $$\gamma = \tfrac{1}{3}b/l.$$

26. If $G = 0$, then the *discriminant*

$$D = (1-9\gamma^2)^2 \text{ (Salmon, \textit{Higher Algebra}, p. 171)},$$

and the solution of the simple circular pendulum is obtained.

Then (i.) when the pendulum oscillates, $\cos\theta$ ranges from -1 to b/l, and $\wp u$ ranges from e_3 to e_2, where

$$e_1 = -\tfrac{1}{2}(\gamma-1),\quad e_2 = \gamma,\quad e_3 = -\tfrac{1}{2}(\gamma+1),$$

and therefore $$u = \sqrt{\frac{g}{l}}\, t + \omega_3.$$

In small oscillations, $b = -l$, $\gamma = -\frac{1}{3}$, and $e_2 = e_3$.

(ii.) When the pendulum revolves, $\cos\theta$ ranges from -1 to $+1$, and $\wp u$ ranges from e_3 to e_2, where

$$e_1 = \gamma,\quad e_2 = -\tfrac{1}{2}(\gamma-1),\quad e_3 = -\tfrac{1}{2}(\gamma+1),$$

and therefore $$u = \sqrt{\frac{g}{l}}\, t + \omega_3,$$

as before.

In the separating case, $b = l$, $\gamma = \frac{1}{3}$, and $e_1 = e_2$.

27. Returning to the spherical pendulum,

$$\begin{aligned} l^2\dot{\psi} &= \frac{G}{\sin^2\theta} \\ &= \tfrac{1}{2}\frac{G}{1-\cos\theta} + \tfrac{1}{2}\frac{G}{1+\cos\theta} \\ &= \tfrac{1}{4}\frac{G}{-\frac{1}{2}(\gamma-1)-\wp u} + \tfrac{1}{4}\frac{G}{\wp u + \frac{1}{2}(\gamma+1)} \\ &= \tfrac{1}{4}\frac{G}{\wp a - \wp u} + \tfrac{1}{4}\frac{G}{\wp u - \wp b}, \end{aligned}$$

putting $$\wp a = -\tfrac{1}{2}(\gamma-1),\quad \wp b = -\tfrac{1}{2}(\gamma+1).$$

Then $\sin^2\frac{1}{2}\theta = \wp a - \wp u, \quad \cos^2\frac{1}{2}\theta = \wp u - \wp b,$

since $$\wp a - \wp b = 1,$$

and $$\tan^2\tfrac{1}{2}\theta = \frac{\wp a - \wp u}{\wp u - \wp b};$$

also $$\wp'^2 a = \wp'^2 b = -G^2/4gl^3;$$

so that $$\frac{d\psi}{du} = \tfrac{1}{2}\frac{i\wp' a}{\wp u - \wp a} + \tfrac{1}{2}\frac{i\wp' b}{\wp u - \wp b},$$

since, as we shall see in § 34, $\wp' a$ is positive imaginary, and $\wp' b$ is negative imaginary (Maggi, *Rendiconti*, *Reale Istituto Lombardo*, Serie II., Vol. XVII., Pisa, 1884).

28. In this case of the spherical pendulum, a and b are connected

by the relation $$\wp'^2 a = \wp'^2 b,$$

so that $$\wp' a = -\wp' b,$$

equivalent to $$\wp(a-b) + \wp a + \wp b = 0,$$

an equation discussed by Halphen in the *Journal de l'Ecole Polytechnique*, 54 Cahier, 1884: *Note sur l'Inversion des Intégrales Elliptiques.*

It may be noticed here that the solution of this equation, when the invariant $g_2 = 0$, is $a = \omega b$, where ω denotes a real or imaginary cube or sixth root of unity.

29. In the more general case of the motion of the Top, or solid of revolution, moving under gravity about a fixed point in its axis, the previous equations of motion for the spherical pendulum are but slightly modified; equation (1) being again applicable, and equation (2) must be changed to

$$A\sin^2\theta\,\dot\psi + Cn\cos\theta = G \quad \text{.......................(3)}$$

(*Quarterly Journal of Mathematics*, Vol. XV.: "On the Motion of a Top, and Allied Problems in Dynamics").

Again, eliminating $\dot\psi$ as before,

$$\tfrac{1}{2}\sin^2\theta\,\dot\theta^2 = \frac{g}{l}(\cos^2\theta - 1)\left(\cos\theta - \frac{b}{l}\right) - \tfrac{1}{2}\left(\frac{G - Cn\cos\theta}{A}\right)^2$$

$$= \frac{g}{l}(\cos\theta - d)(\cos\theta - \cos\alpha)(\cos\theta - \cos\beta),$$

suppose, θ being supposed to lie between α and β, so that $\alpha < \theta < \beta$.

30. Putting, as before, $\cos\theta = 2z+\gamma$,

then $$\dot{z}^2 = \frac{g}{l}(4z^3-g_2z-g_3),$$

provided that $$3\gamma = \frac{b}{l}+\tfrac{1}{2}\frac{l}{g}\frac{C^2n^2}{A^2};$$

so that $$z = \wp u,$$

where $$u = \sqrt{\frac{g}{l}}\,t + \text{a constant},$$

and $$g_2 = 3\gamma^2+1-GCnl/gA^2,$$
$$g_3 = \gamma(\gamma^2-1)+(G^2-2GCn\gamma+C^2n^2)\,l/4gA^2.$$

31. Then if, as before, for the spherical pendulum,

$$u = a \text{ when } \cos\theta = 1, \quad \text{so that } \wp a = -\tfrac{1}{2}(\gamma-1),$$
$$u = b \quad ,, \quad \cos\theta = -1, \quad ,, \quad \wp b = -\tfrac{1}{2}(\gamma+1),$$

then $$\sin^2\tfrac{1}{2}\theta = \wp a-\wp u, \quad \cos^2\tfrac{1}{2}\theta = \wp u-\wp b,$$

and $$\wp'^2 a = -\tfrac{1}{4}\frac{l}{g}\left(\frac{G-Cn}{A}\right)^2,$$
$$\wp'^2 b = -\tfrac{1}{4}\frac{l}{g}\left(\frac{G+Cn}{A}\right)^2.$$

From $$\frac{d\psi}{dt} = \frac{G-Cn\cos\theta}{A\sin^2\theta}$$
$$= \tfrac{1}{2}\frac{G-Cn}{A}\frac{1}{1-\cos\theta}+\tfrac{1}{2}\frac{G+Cn}{A}\frac{1}{1+\cos\theta}$$

we obtain, as before,

$$\frac{d\psi}{du} = \tfrac{1}{2}\frac{i\wp' a}{\wp u-\wp a}+\tfrac{1}{2}\frac{i\wp' b}{\wp u-\wp b} \quad \text{......................}(4).$$

32. Introducing at this stage σu, the *sigma function* of Weierstrass, as defined by $$\frac{d^2}{du^2}\log\sigma u = -\wp u,$$

and also the fundamental formula

$$\wp u-\wp v = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\;\sigma^2 v},$$

called by Schwarz the *pocket edition* of the elliptic functions; differentiating this formula logarithmically with respect to u and v,

$$\frac{\wp' u}{\wp u - \wp v} = \frac{\sigma'(u+v)}{\sigma(u+v)} + \frac{\sigma'(u-v)}{\sigma(u-v)} - 2\frac{\sigma' u}{\sigma u},$$

$$\frac{-\wp' v}{\wp u - \wp v} = \frac{\sigma'(u+v)}{\sigma(u+v)} - \frac{\sigma'(u-v)}{\sigma(u-v)} - 2\frac{\sigma' v}{\sigma v};$$

and integrating with respect to v and u, respectively,

$$\int \frac{\wp' u\, dv}{\wp u - \wp v} = \log\frac{\sigma(u+v)}{\sigma(u-v)} - 2v\frac{\sigma' u}{\sigma u},$$

$$\int \frac{\wp' v\, du}{\wp u - \wp v} = \log\frac{\sigma(u-v)}{\sigma(u+v)} + 2u\frac{\sigma' v}{\sigma v};$$

Weierstrass's form of the Third Elliptic Integral; so that

$$\int \frac{\wp' u\, dv + \wp' v\, du}{\wp u - \wp v} = 2u\frac{\sigma' v}{\sigma v} - 2v\frac{\sigma' u}{\sigma u},$$

corresponding to Jacobi's formula for the interchange of *argument* and *parameter* in the third elliptic integral.

33. Then equation (4) becomes

$$\frac{d\psi}{du} = \tfrac{1}{2}i\frac{\sigma'(u-a)}{\sigma(u-a)} - \tfrac{1}{2}i\frac{\sigma'(u+a)}{\sigma(u+a)} + i\frac{\sigma' a}{\sigma a}$$
$$+ \tfrac{1}{2}i\frac{\sigma'(u-b)}{\sigma(u-b)} - \tfrac{1}{2}i\frac{\sigma'(u+b)}{\sigma(u+b)} + i\frac{\sigma' b}{\sigma b};$$

and, integrating,

$$\psi = \tfrac{1}{2}i\log\frac{\sigma(u-a)\,\sigma(u-b)}{\sigma(u+a)\,\sigma(u+b)} + i\left(\frac{\sigma' a}{\sigma a} + \frac{\sigma' b}{\sigma b}\right)u \quad\text{.........}\quad (5).$$

34. The values ω_1, ω_2, ω_3 of u, and therefore the values e_1, e_2, e_3 of $\wp u$, correspond to the values d, $\cos\alpha$, $\cos\beta$ of $\cos\theta$; so that

$$\frac{1-d}{1+d} = \frac{\wp a - e_1}{e_1 - \wp b} = -\left(\frac{\sigma_1 a}{\sigma a}\right)^2 \Big/ \left(\frac{\sigma_1 b}{\sigma b}\right)^2,$$

$$\frac{1-\cos\alpha}{1+\cos\alpha} = \tan^2\tfrac{1}{2}\alpha = -\left(\frac{\sigma_2 a}{\sigma a}\right)^2 \Big/ \left(\frac{\sigma_2 b}{\sigma b}\right)^2,$$

$$\frac{1-\cos\beta}{1+\cos\beta} = \tan^2\tfrac{1}{2}\beta = -\left(\frac{\sigma_3 a}{\sigma a}\right)^2 \Big/ \left(\frac{\sigma_3 b}{\sigma b}\right)^2.$$

In order that $\wp u$ should oscillate in magnitude between e_2 and e_3, we must put

$$u = \sqrt{\frac{g}{l}}\,t + \omega_3.$$

Also, since

$$e_1 > \wp a > e_2,$$

therefore we can put $a = \omega_1 + r\omega_3$, when r is a proper fraction; and then $\wp' a$ is positive imaginary; and, since

$$e_3 > \wp b > -\infty,$$

therefore we can put $b = s\omega_3$, where s is a proper fraction; and then $\wp' b$ is negative imaginary.

35. When $G = 0$ or $Cn = 0$, then $\wp'^2 a = \wp'^2 b$, and the motion of the Top is directly comparable with that of a spherical pendulum.

When G and Cn are both zero, the Top oscillates in a vertical plane like a simple circular pendulum, and then $a = \omega_1$, $b = \omega_3$.

If $a = \omega_1$, then $G - Cn = 0$;

and if $b = \omega_3$, then $G + Cn = 0$.

If $a + b = \omega_1 + \omega_3$, then $G - Cn\cos\alpha = 0$,

and the trace of the axis on the unit sphere of reference has a series of cusps on the parallel of latitude $\theta = \alpha$.

36. According to the method of Hermite (*Sur quelques Applications des Fonctions Elliptiques*, 1885, p. 109), taking the axis of z vertically upwards, the equations of motion of the spherical pendulum are

$$\frac{d^2x}{dt^2} + N\frac{x}{l} = 0,$$

$$\frac{d^2y}{dt^2} + N\frac{y}{l} = 0,$$

$$\frac{d^2z}{dt^2} + N\frac{z}{l} = -g;$$

with

$$x^2 + y^2 + z^2 = l^2;$$

the two first integrals of which are

$$\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = g(c - z),$$

$$x\dot{y} - \dot{x}y = G.$$

Then, as before, $z = l\cos\theta = l(2\wp u + \gamma)$,

where

$$\gamma = \tfrac{1}{3}c/l.$$

Also $$x\ddot{x}+y\ddot{y}+z\ddot{z}+Nl = -gz,$$

or, since $$x\ddot{x}+y\ddot{y}+z\ddot{z}+\dot{x}^2+\dot{y}^2+\dot{z}^2 = 0,$$

$$Nl = -gz+\dot{x}^2+\dot{y}^2+\dot{z}^2$$
$$= -gz+2g(c-z) = g(2c-3z),$$

N representing the pressure per unit mass on the sphere.

37. Then $$\frac{d^2}{dt^2}(x+iy) = -\frac{N}{l}(x+iy) = \frac{g}{l^2}(3z-2c)(x+iy),$$

or $$\frac{d^2}{du^2}(x+iy) = (6\wp u-3\gamma)(x+iy),$$

Lamé's differential equation for $n = 2$; and the solution is

$$x+iy = 2il\frac{\sigma(u+a)\,\sigma(u+b)}{\sigma a\,\sigma b\,\sigma^2 u}\exp\left(-\frac{\sigma' a}{\sigma a}-\frac{\sigma' b}{\sigma b}\right)u,$$

or $$x-iy = 2il\frac{\sigma(u-a)\,\sigma(u-b)}{\sigma a\,\sigma b\,\sigma^2 u}\exp\left(\frac{\sigma' a}{\sigma a}+\frac{\sigma' b}{\sigma b}\right)u,$$

where $$\gamma = \wp(a-b) = -\wp a-\wp b.$$

This may also be obtained by combining the value of $e^{i\psi}$ from equation (5) with the result of § 27 or § 31,

$$\sin^2\theta = 4(\wp a-\wp u)(\wp u-\wp b);$$

and it is interesting to compare this result with Hermite's (p. 112)

$$x+iy = AD_u\frac{H'OH(u+\omega)}{\Theta\omega\,\Theta u}\exp\left(\lambda-\frac{\Theta'\omega}{\Theta\omega}\right)u,$$

where ω and λ are constants; the equivalence of the two forms being secured by putting $\omega = a+b$, and

$$\lambda = \wp'(a-b) = \wp' a = -\wp' b = \zeta(a+b)-\zeta a-\zeta b,$$

using Halphen's notation (Chapter v.) ζa for $\dfrac{\sigma' a}{\sigma a}$; also

$$\zeta(a-b) = \zeta a-\zeta b.$$

V. *The Trajectory for the Cubic Law of Resistance.*

38. In Volume XIV. of the *Proceedings of the Royal Artillery Institution* the trajectory of a projectile in a resisting medium, with a tangential resistance varying as the cube of the velocity, is investigated, and it is there shown that a great simplification is effected by the employment of Weierstrass's functions.

The equation of the trajectory referred to oblique axes, one the tangent at the point of infinite velocity, and the other vertical, is then

$$y = -3x\,\zeta b - \log\sigma(b-x) - \omega^2\log\sigma(b-\omega x) - \omega\,\log\sigma(b-\omega^2 x)\,;$$

and the time of flight is given by

$$t = - \qquad \log\sigma(b-x) - \omega\,\log\sigma(b-\omega x) - \omega^2\log\sigma(b-\omega^2 x)\,;$$

b denoting the value of x at the vertical asymptote.

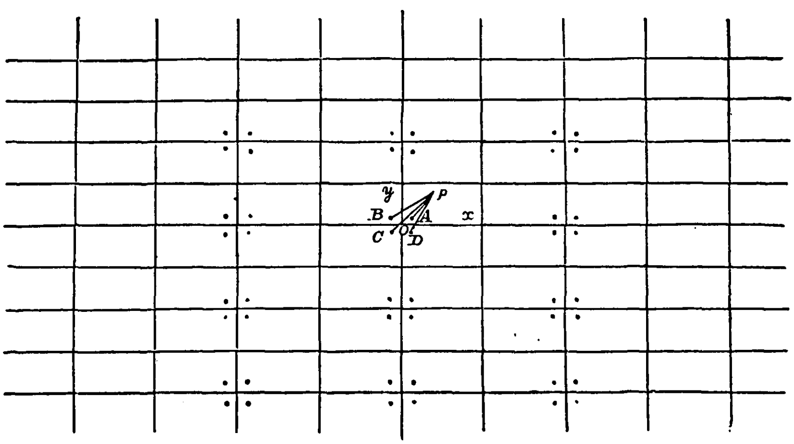
VI. *Uniplanar Electrical and Hydrodynamical Problems.*

39. Referring to the *Quarterly Journal of Mathematics*, Vols. XVII. and XVIII., "Solution by means of Elliptic Functions of some Problems in the Conduction of Heat and of Electricity," and "Functional Images in Cartesians," for the statement of the problems to be solved and of the notation employed; then, for a source of strength 2π at $z' = x' + iy'$, within the rectangle bounded by $x = 0$, $x = a$, $y = 0$, $y = b$,

$$\phi + i\psi = \log\sigma\alpha\;\sigma\beta\;\sigma\gamma\;\sigma\delta,$$

and for a vortex of circulation 2π at z',

$$\phi_1 + i\psi_1 = \log\frac{\sigma\alpha\;\sigma\gamma}{\sigma\beta\;\sigma\delta}.$$



Here

$$\alpha = x - x' + i\,.\,y - y',$$
$$\beta = x + x' + i\,.\,y - y',$$
$$\gamma = x + x' + i\,.\,y + y',$$
$$\delta = x - x' + i\,.\,y + y'\,;$$

so that α, β, γ, δ represent the four vectors AP, BP, CP, DP, proceeding to any point P from A at z', and the images B, C, D of A in the coordinate axes.

The remaining images form similar groups of four, round centres in the plane whose coordinates are $2ma$, $2m'b$, where m and m' are integers.

To be accurate, the sigma functions should have certain simple exponential factors, but these are cancelled by placing an equal and opposite source, *i.e.* a *sink*, inside the rectangle, and then the physical impossibility of zero flow across a boundary with a single source is removed; and by placing the source and sink at corners of the rectangle, we obtain the various results of the article in the *Quarterly Journal*, XVII., "Solution by means of Elliptic Functions, &c."; and now $\omega_1 = a$, $\omega_3 = ib$.

40. Transforming the coordinates by the use of conjugate functions, given by

$$x + iy = \mathrm{f}(\chi + i\rho),$$

then, for a right-angled quadrilateral figure bounded by

$$\chi = \chi_0, \quad \chi = \chi_1, \quad \rho = \rho_0, \quad \rho = \rho_1,$$

we must put the vectors

$$\begin{aligned} \alpha &= \chi - \chi' \qquad\quad + i\,.\,\rho - \rho', \\ \beta &= \chi + \chi' - 2\chi_0 + i\,.\,\rho - \rho', \\ \gamma &= \chi + \chi' - 2\chi_0 + i\,.\,\rho + \rho' - 2\rho_0, \\ \delta &= \chi - \chi' \qquad\quad + i\,.\,\rho + \rho' - 2\rho_0. \end{aligned}$$

Then, for a source at (χ', ρ'),

$$\phi + i\psi = \log \sigma\alpha\;\sigma\beta\;\sigma\gamma\;\sigma\delta;$$

and for a vortex or an electrified point,

$$\phi_1 + i\psi_1 = \log \frac{\sigma\alpha\;\sigma\gamma}{\sigma\beta\;\sigma\delta};$$

and here $$\omega_1 = \chi_1 - \chi_0, \quad \omega_3 = i\,.\,\rho_1 - \rho_0,$$

the periods of the Weierstrass functions.

41. For a doubly connected plane region, bounded by $\rho = \rho_0$ and $\rho = \rho_1$, we may put, as in "Functional Images in Cartesians,"

$$\phi + i\psi = \log \sigma(\chi - \chi' + i\,.\,\rho - \rho')\,\sigma(\chi - \chi' + i\,.\,\rho + \rho' - 2\rho_0),$$

$$\phi_1 + i\psi_1 = \log \frac{\sigma(\chi - \chi' + i\,.\,\rho - \rho')}{\sigma(\chi - \chi' + i\,.\,\rho + \rho' - 2\rho_0)},$$

supposing χ to increase by ω_1 in a complete circuit of the region.

It is easily verified in these expressions, from the formulæ given by

Schwarz, that ψ and ϕ_1 have constant values round the boundaries $\chi = \chi_0$, $\chi = \chi_1$, $\rho = \rho_0$, $\rho = \rho_1$, or can be made constant by the addition of simple expressions.

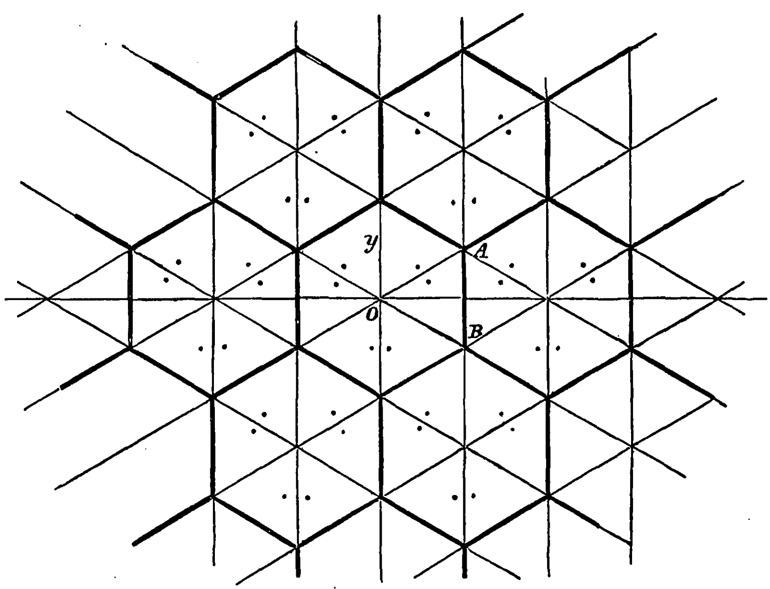
42. When a source or vortex is placed at z' inside an equilateral triangle OAB, then the vectors of the images are given by

$$\omega z', \ \omega^2 z', \text{ and } z'', \ \omega z'', \ \omega^2 z'',$$

where $z'' = -x'+iy'$, and $\omega^3 = 1$, so that ω denotes an imaginary cube root of unity; and similar groups of six images ranged round centres of hexagons forming a tesselated pavement, the coordinates of the centres being

$$2mh, \ 2m'h\sqrt{3}, \text{ and } (2m+1)h, \ (2m'+1)h\sqrt{3},$$

where h denotes the altitude of the equilateral triangle, and m and m' are integers.



Then, for a source inside an equilateral triangle, like OAB,

$$\begin{aligned}\phi+i\psi = \log \sigma\,(z-z')\,\sigma\,(z-\omega z')\,\sigma\,(z-\omega^2 z')\\ \sigma_2(z-z')\,\sigma_2(z-\omega z')\,\sigma_2(z-\omega^2 z')\\ \sigma\,(z-z'')\,\sigma\,(z-\omega z'')\,\sigma\,(z-\omega^2 z'')\\ \sigma_2(z-z'')\,\sigma_2(z-\omega z'')\,\sigma_2(z-\omega^2 z'');\end{aligned}$$

and for a vortex, or electrified point,

$$\phi_1 + i\psi_1 = \log \frac{\sigma\,(z-z')\;\sigma\,(z-\omega z')\;\sigma\,(z-\omega^2 z')}{\sigma\,(z-z'')\;\sigma\,(z-\omega z'')\;\sigma\,(z-\omega^2 z'')}$$

$$\frac{\sigma_3\,(z-z')\;\sigma_3\,(z-\omega z')\;\sigma_3\,(z-\omega^2 z')}{\sigma_3\,(z-z'')\;\sigma_3\,(z-\omega z'')\;\sigma_3\,(z-\omega^2 z'')};$$

and here $\omega_1 = h, \quad \omega_3 = ih\sqrt{3},$

so that $\dfrac{\omega_3}{\omega_1} = i\dfrac{K'}{K} = i\sqrt{3},$

and therefore the modular angle is 15°.

(O. Zimmerman, *Das Logarithmische Potential einer gleichseitig dreieckigen Platte*, Diss. Jena. 1880.)

VII. *Attractions.*

43. The well-known expression for the potential V of the homogeneous ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

of mass M at an external point x, y, z, viz.,

$$V = \tfrac{3}{4}M \int_\lambda^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda\,.\,b^2+\lambda\,.\,c^2+\lambda)}} \left(1 - \frac{x^2}{a^2+\lambda} - \frac{y^2}{b^2+\lambda} - \frac{z^2}{c^2+\lambda}\right),$$

where $a^2+\lambda$, $b^2+\lambda$, $c^2+\lambda$ are the squares of the semi-axes of the confocal ellipsoid passing through the point x, y, z, is reduced to Weierstrass's functions by putting

$$a^2+\lambda = \wp u - e_1, \quad b^2+\lambda = \wp u - e_2, \quad c^2+\lambda = \wp u - e_3,$$

supposing $a^2 < b^2 < c^2$, and therefore $e_1 > e_2 > e_3$.

Then $\wp u = \tfrac{1}{3}(a^2+b^2+c^2) + \lambda,$

since $e_1 + e_2 + e_3 = 0\,;$

and $e_1 = \tfrac{1}{3}(-2a^2+b^2+c^2)$, $e_2 = \tfrac{1}{3}(a^2-2b^2+c^2)$, $e_3 = \tfrac{1}{3}(a^2+b^2-2c^2)$;

so that

$$g_2 = -4\,(e_2e_3 + e_3e_1 + e_1e_2) = 2\,(e_1^2+e_2^2+e_3^2)$$
$$= \tfrac{2}{3}\left\{(b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2\right\};$$

and

$$D = g_2^3 - 27g_3^2 = (e_2-e_3)^2\,(e_3-e_1)^2\,(e_1-e_2)^2$$
$$= (b^2-c^2)^2\,(c^2-a^2)^2\,(a^2-b^2)^2.$$

Then

$$\int_\lambda^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda\,.\,b^2+\lambda\,.\,c^2+\lambda)}} = 2\,u,$$

and $$\frac{V}{\frac{3}{2}M} = \int_0^u \left(1 - \frac{x^2}{\wp u - e_1} - \frac{y^2}{\wp u - e_2} - \frac{z^2}{\wp u - e_3}\right) du;$$

also $$\int_0^u \frac{du}{\wp u - e_\lambda} = \frac{\zeta_\lambda u + e_\lambda u}{(e_\lambda - e_\mu)(e_\nu - e_\lambda)};$$

so that, if $a^2+\mu = \wp v - e_1$, $b^2+\mu = \wp v - e_2$, $c^2+\mu = \wp v - e_3$ are the squares of the semi-axes of the confocal hyperboloid of one sheet, and if

$$a^2+\nu = \wp w - e_1, \quad b^2+\nu = \wp w - e_2, \quad c^2+\nu = \wp w - e_3,$$

of the confocal hyperboloid of two sheets through the point (xyz), then

$$u = r\omega_1, \quad v = \omega_1 + s\omega_3, \quad w = t\omega_1 + \omega_3,$$

where r, s, t are proper fractions; and

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)} = \frac{\left(\frac{\sigma_1 u}{\sigma u}\right)^2 \left(\frac{\sigma_1 v}{\sigma v}\right)^2 \left(\frac{\sigma_1 w}{\sigma w}\right)^2}{(e_1-e_2)(e_1-e_3)},$$

with similar symmetrical expressions for y^2 and z^2.

On the Converse of Stereographic Projection and on Contangential and Coaxal Spherical Circles. By Mr. H. M. JEFFERY, F.R.S.

[*Read May 13th*, 1886.]

On Systems of Spherical Circles.

1. The first section is on a form of conical projection and introduces the equations and processes herein used. The second treats of systems of coaxal and contangential circles. Next, similitude and inversion are defined and illustrated. Lastly, the processes are applied to the solution of the problem of Contacts.

In developing the analogies to Plane Geometry, it is shown that theorems which are distinct in Planimetry are dual in Spherics; that those which relate to the magnitude of angles are identical in both Geometries; while theorems on arcs are modified when the radius of the sphere becomes infinite.

On the Converse of Stereographic Projection.

2. By stereographic projection, curves on a sphere are projected on an equatorial plane, whose pole is the pole of projection. The converse process is here considered; lines and curves on the equatorial