the left-hand side of the last equation can be easily shown to be equal to $8\pi f'(a)$. Multiplying by $\dot{a}$ and integrating, we obtain

$$\left(2+\frac{1}{a^3}\right)\dot{a}^2+8\pi\left\{\left(\frac{\rho_0}{a_0}\right)^2 a-f(a)\right\} = \text{const.} \quad \ldots\ldots\ldots(24),$$

which is the equation of energy.

Equations (22), (23), and (24) are the equations obtained by Dirichlet.

Solution of the Cubic and Quartic Equations by means of Weierstrass's Elliptic Functions. *By* A. G. GREENHILL.

[*Read May 13th*, 1886.]

A. *Solution of the Cubic Equation.*

1. The solution of the cubic equation, when presented in the form

$$4x^3-Sx-T=0,$$

by means of the trigonometrical circular functions, is well known; for, putting $x=ny$, then

$$4y^3-\frac{S}{n^2}y-\frac{T}{n^3}=0,$$

and, comparing this equation with

$$4\cos^3 a-3\cos a-\cos 3a=0,$$

we can put $\qquad y=\cos a, \quad \text{and} \quad x=n\cos a,$

provided that $\qquad n^2=\tfrac{1}{3}S, \quad \text{and} \quad \cos 3a=\dfrac{T}{n^3};$

the other two roots being $n\cos(a\pm\frac{2}{3}\pi)$.

Denoting the *discriminant* S^3-27T^2 by Δ, and the *absolute invariant* $\dfrac{S^3}{\Delta}$ by J, according to Klein, then

$$\cos^2 3a=\frac{T^2}{n^6}=\frac{27T^2}{S^3}=\frac{J-1}{J},$$

$$\sin^2 3a=\frac{1}{J}, \quad \text{or} \quad \operatorname{cosec}^2 3a=J.$$

2. Changing the sign of x, the cubic equation becomes

$$4x^3-Sx+T=0,$$

which may be compared with

$$4\sin^3 a-3\sin a+\sin 3a=0,$$

so that $$x=n\sin a,\ \text{ or }\ n\sin(a\pm\tfrac{2}{3}\pi),$$

provided that $$n^2=\tfrac{1}{3}S,\ \text{ and }\ \sin 3a=\frac{T}{n^3};$$

and then $$\sin^2 3a=\frac{T^2}{n^6}=\frac{27T^2}{S^3}=\frac{J-1}{J},$$

$$\cos^2 3a=\frac{1}{J},\ \text{ or }\ \sec^2 3a=J.$$

In these two cases it is assumed that

$$\Delta=S^3-27T^2$$

is positive, so that all three roots of the cubic equation are real.

3. But, if Δ is negative, two of the roots of the cubic are imaginary; and, if S is positive, the equation

$$4x^3-Sx-T=0$$

must be compared with

$$4\cosh^3 a-3\cosh a-\cosh 3a=0,$$

and then the roots of the equation are

$$n\cosh a\ \text{ and }\ n\cosh(a\pm\tfrac{2}{3}\pi i),$$

provided that $$n^2=\tfrac{1}{3}S,\ \text{ and }\ \cosh 3a=\frac{T}{n^3};$$

so that $$\cosh^2 3a=\frac{T^2}{n^6}=\frac{27T^2}{S^3}=\frac{J-1}{J},$$

$$\sinh^2 3a=-\frac{1}{J},\ \text{ or }\ \operatorname{cosech}^2 3a=-J;$$

and these are real, because J is negative.

4. If Δ is negative, and S is negative, then, changing the sign of S, the cubic equation $$4x^3+Sx-T=0$$

must be compared with

$$4\sinh^3 a + 3\sinh a - \sinh 3a = 0,$$

and then the roots of the equation are

$$n\sinh a \text{ and } n\sinh(a \pm \tfrac{2}{3}\pi i),$$

provided that $n^2 = \frac{1}{3}S$, and $\sinh 3a = \dfrac{T}{n^3}$;

and then
$$\sinh^2 3a = \frac{T^2}{n^6} = \frac{27T^2}{S^3},$$

$$\cosh^2 3a = \frac{S^3+27T^2}{S^3} = \frac{1}{J}, \text{ or } \operatorname{sech}^2 3a = J.$$

Similarly, changing the sign of x, the roots of the cubic equation

$$4x^3 - Sx + T = 0$$

will be $-n\cosh a$ and $-n\cosh(a \pm \frac{2}{3}\pi i)$,

where $\operatorname{cosech}^2 3a = -J$,

J being negative; and the roots of the cubic equation

$$4x^3 + Sx + T = 0$$

will be $-n\sinh a$ and $-n\sinh(a \pm \frac{2}{3}\pi i)$,

where $\operatorname{sech}^2 3a = J$.

According to this method, the solution of the cubic, when only one root is real, depends on the values of the *hyperbolic* functions, which have the inconvenience of an infinite period, and so cannot conveniently be tabulated.

5. In the preceding cubics the second term of the equation has been removed; but, if we consider their reciprocal equations, we shall have a cubic equation of the form

$$z^3 + az^2 - 4b = 0,$$

a cubic equation with the *third* term removed, equivalent to the preceding equation $4x^3 - Sx - T = 0$,

if
$$z = \frac{1}{x},\quad a = \frac{S}{T},\quad b = \frac{1}{T};$$

and the roots of this new cubic in z will be all real, or one real and two imaginary as before, according as

$$\Delta = S^3 - 27T^2 \text{ is positive or negative,}$$

$$= \frac{a^3-27b}{b^3} \quad \text{,,} \quad \text{,,}$$

We may always suppose T, and therefore b, is positive; for, if negative, changing the sign of x and z would make them positive in the equations; the roots of the new equation

$$z^3 + az^2 - 4b = 0$$

will therefore be all real, or one real and two imaginary, as $a^3 - 27b$ is positive or negative, on the supposition that T and b are positive.

6. Now, consider two variable quantities s and t, connected by the

relation $$t = s - \frac{g_3}{s^2};$$

then $$\frac{dt}{ds} = 1 + \frac{2g_3}{s^3},$$

and $$4t^3 + h_3 = (4s^3 - g_3)\left(1 + \frac{2g_3}{s^3}\right)^2,$$

if $$h_3 = 27g_3;$$

so that $$\frac{dt}{\sqrt{(4t^3+h_3)}} = \frac{ds}{\sqrt{(4s^3-g_3)}},$$

and $$\int_t^\infty \frac{dt}{\sqrt{(4t^3+h_3)}} = \int_s^\infty \frac{ds}{\sqrt{(4s^3-g_3)}} = u \text{ suppose.}$$

But, according to the definitions of Weierstrass, the absolutely simplest elliptic function, denoted by $\mathrm{p}u$, of a variable quantity u, is

defined by $$u = \int_s^\infty \frac{ds}{\sqrt{(4s^3 - g_2 s - g_3)}},$$

and $$s = \mathrm{p}\,u,$$

so that $$\frac{ds}{du} = \mathrm{p}'u = -\sqrt{(4s^3 - g_2 s - g_3)}.$$

When it is desirable to indicate the quantities g_2 and g_3, called the *invariants*, then the notation

$$s = \mathrm{p}\,(u;\ g_2, g_3)$$

is employed; so that we may now write

$$s = \mathrm{p}\,(u;\ 0,\ g_3),$$
$$t = \mathrm{p}\,(u;\ 0,\ -h_3).$$

7. By means of the fundamental relation

$$\mathrm{p}\,(u;\ g_2, g_3) = m^2\mathrm{p}\left(mu;\ \frac{g_2}{m^4},\ \frac{g_3}{m^6}\right),$$

we find, putting $m^6 = -27,\ m^2 = -3,\ m = i\sqrt{3}$, that

$$t = \mathrm{p}\,(u;\ 0,\ -h_3) = -3\mathrm{p}\,(iu\sqrt{3};\ 0,\ g_3),$$

since $$h_3 = 27g_3;$$

and therefore, omitting the indication of g_3,

$$-3\mathrm{p}\,(iu\sqrt{3}) = \mathrm{p}u - \frac{g_3}{\mathrm{p}^2u},$$

or $$\mathrm{p}^3u + 3\mathrm{p}\,(iu\sqrt{3})\,\mathrm{p}^2u - g_3 = 0;$$

and, comparing this with the equation

$$z^3 + az^2 - 4b = 0,$$

we have $$z = \mathrm{p}u,$$

provided that $$a = 3\mathrm{p}\,(iu\sqrt{3}),$$
$$g_3 = 4b,\quad g_2 = 0.$$

8. In order to tabulate the function $\mathrm{p}u$, we must select some particular value of g_3; we shall find it convenient to put $g_3 = 4$, and then,

if $s = \mathrm{p}u$, $$u = \int_s^\infty \frac{ds}{\sqrt{(4s^3-4)}},$$

or $$2u = \int_s^\infty \frac{ds}{\sqrt{(s^3-1)}};$$

and then, in the notation of Legendre and Jacobi,

$$s = \mathrm{p}u = 1 + \sqrt{3}\,\frac{1+\mathrm{cn}\,2u\sqrt[4]{3}}{1-\mathrm{cn}\,2u\sqrt[4]{3}},\ \text{for}\ k = \sin 15^\circ.$$

9. In the general notation of Weierstrass, the roots of the equation

$$4s^3 - g_2s - g_3 = 0$$

are denoted by e_1, e_2, e_3; so that

$$4s^3 - g_2s - g_3 = 4\,(s-e_1)(s-e_2)(s-e_3);$$

and, if ω_1, ω_2, ω_3 denote corresponding values of u, so that

$$\mathrm{p}\,\omega_1 = e_1, \quad \mathrm{p}\,\omega_2 = e_2, \quad \mathrm{p}\,\omega_3 = e_3;$$

then ω_1, ω_2, ω_3 are called the *periods* of the elliptic functions; but they are connected by the relation

$$\omega_1 + \omega_2 + \omega_3 = 0,$$

also
$$e_1 + e_2 + e_3 = 0.$$

In our case of $g_2 = 0$, two of the quantities e_1, e_2, e_3 are imaginary; e_2 is then taken to be real, and, with $g_3 = 4$, we have

$$e_1 = \omega, \quad e_2 = 1, \quad e_3 = \omega^2;$$

ω and ω^2 denoting the imaginary cube roots of unity, such that

$$\omega - \omega^2 = i\sqrt{3}.$$

Then, since $s = 1$ when $u = \omega_2$, and $2u\sqrt[4]{3} = 2K$, in Jacobi's notation; therefore $K = \omega_2\sqrt[4]{3}$.

Also $\omega_3 - \omega_1$ is positive imaginary, and is denoted by ω_2' by Schwarz, and then $iK' = \omega_2'\sqrt[4]{3}$; so that

$$\frac{\omega_2'}{\omega_2} = i\,\frac{K'}{K} = i\sqrt{3}.$$

10. Then, as u decreases from ω_2 to 0, $\mathrm{p}u$ will pass through all real values from 1 to ∞; and as iu increases from 0 to $\omega_2 i$, or $iu\sqrt{3}$ from 0 to ω_2', $\mathrm{p}\,(iu\sqrt{3})$ will pass through all real values from $-\infty$ through 0 to $+1$; as exhibited in the following Table, kindly calculated for me by Mr. A. G. Hadcock, Inspector of Ordnance Machinery, Royal Artillery, in which the periods ω_2 and ω_2' have each been divided into 180 equal parts, and the corresponding values of $\mathrm{p}\,u$ tabulated in the same horizontal line.

Then $\omega_2 = 1{\cdot}2143$, and $\mathrm{p}\,\frac{r\omega_2}{180}$ can be calculated from the formula

$$\mathrm{p}u = \frac{1}{u^2} - \frac{u^4}{7} + \frac{u^{10}}{7^2 . 13} \cdots\cdots.$$

Also, denoting $\mathrm{p}\,\frac{r\omega_2}{180}$ by s and $\mathrm{p}\,\frac{r\omega_2'}{180}$ by S, then

$$S = -\frac{1}{3}\left(s - \frac{4}{s^2}\right),$$

whence $\mathrm{p}\,\frac{r\omega_2'}{180}$ can be calculated when $\mathrm{p}\,\frac{r\omega_2}{180}$ is known.

	$s = \mathrm{p}\,\frac{r\omega_2}{180}$	$S = \mathrm{p}\,\frac{r\omega_2'}{180}$		$s = \mathrm{p}\,\frac{r\omega_2}{180}$	$S = \mathrm{p}\,\frac{r\omega_2'}{180}$
$r =$ 0	+Infinity	−Infinity			
1	+21972·6	−7324·200	$r =$ 46	+ 10·3851	−3·44934
2	5494·39	1831·463	47	9·94817	3·30258
3	2331·61	777·2033	48	9·53820	3·16474
4	1373·21	457·7366	49	9·15299	3·03507
5	878·805	292·9350	50	8·79071	2·91297
6	599·074	199·6913	51	8·44971	2·79790
7	448·413	149·4710	52	8·12796	2·68914
8	343·303	114·4343	53	7·82443	2·59636
9	271·277	90·4256	54	7·53768	2·48909
10	219·726	73·2430	55	7·26630	2·39685
11	181·586	60·5286	56	7·00940	2·30932
12	152·593	50·8643	57	6·76574	2·22611
13	130·016	43·3385	58	6·53496	2·14710
14	112·103	37·3675	59	6·31568	2·07179
15	97·6523	32·55062	60	6·10721	2·00000
16	85·8317	28·61038	61	5·90906	1·93149
17	76·0292	25·34283	62	5·72044	1·86606
18	67·8150	22·60471	63	5·54062	1·80344
19	60·8671	20·28867	64	5·36929	1·74351
20	54·9316	18·31009	65	5·20585	1·68608
21	49·8237	16·60736	66	5·04976	1·63096
22	45·3966	15·13155	67	4·90066	1·57803
23	41·5363	13·84466	68	4·75815	1·52716
24	38·1466	12·71461	69	4·62180	1·47818
25	35·1554	11·71738	70	4·49772	1·43333
26	32·5043	10·83350	71	4·36622	1·38546
27	30·1407	10·04543	72	4·24650	1·34156
28	28·0260	9·34030	73	4·13150	1·29905
29	26·1262	8·70678	74	4·02134	1·25799
30	24·4142	8·13582	75	3·91554	1·21821
31	22·8643	7·61888	76	3·81391	1·17964
32	21·4574	7·14956	77	3·71629	1·14222
33	20·1766	6·72226	78	3·62241	1·10586
34	19·0077	6·33221	79	3·53214	1·07051
35	17·9369	5·97482	80	3·44533	1·03612
36	16·9542	5·64676	81	3·36163	1·00255
37	16·0506	5·34502	82	3·28113	·96986
38	15·2169	5·06654	83	3·20353	·93792
39	14·4465	4·80911	84	3·12871	·90669
40	13·7332	4·57066	85	3·05658	·87615
41	13·0719	4·34950	86	2·98703	·84623
42	12·4569	4·14371	87	2·91989	·81690
43	11·8843	3·95199	88	2·85504	·78811
44	11·3505	3·77316	89	2·79248	·75984
45	+ 10·8517	− 3·60591	90	+ 2·73200	− ·73202

	$s = \text{p}\,\frac{r\omega_2}{180}$	$S = \text{p}\,\frac{r\omega_2'}{180}$		$s = \text{p}\,\frac{r\omega_2}{180}$	$S = \text{p}\,\frac{r\omega_2'}{180}$
$r = 91$	+ 2·67366	− ·70470	$r = 136$	+ 1·28989	+ ·37141
92	2·61713	·67770	137	1·27562	·39420
93	2·56253	·65112	138	1·26183	·41679
94	2·50974	·62490	139	1·24846	·43929
95	2·45868	·59900	140	1·23564	·46140
96	2·40928	·57339	141	1·22310	·48357
97	2·36141	·54802	142	1·21105	·50543
98	2·31541	·52310	143	1·19934	·52716
99	2·27023	·49804	144	1·18798	·54877
100	2·22679	·47337	145	1·17710	·56994
101	2·18467	·44886	146	1·16657	·59090
102	2·14303	·42402	147	1·15639	·61163
103	2·10347	·39980	148	1·14662	·63195
104	2·06523	·37580	149	1·13720	·65196
105	2·02809	·35187	150	1·12805	·67180
106	1·99207	·32803	151	1·11932	·69112
107	1·95715	·30429	152	1·11101	·70987
108	1·92327	·28063	153	1·10290	·72851
109	1·89043	·25705	154	1·09514	·74669
110	1·85856	·23352	155	1·08773	·76436
111	1·82745	·20990	156	1·08073	·78133
112	1·79759	·18656	157	1·07394	·79807
113	1·76849	·16317	158	1·06757	·81405
114	1·74016	·13974	159	1·06147	·82956
115	1·71272	·11637	160	1·05558	·84476
116	1·68605	·09298	161	1·05011	·85909
117	1·66069	·07010	162	1·04484	·87306
118	1·63554	·04674	163	1·03999	·88610
119	1·61108	·02333	164	1·03528	·89891
120	1·58746	− ·00000	165	1·03099	·91073
121	1·56439	+ ·02336	166	1·02690	·92210
122	1·54194	·04681	167	1·02323	·93242
123	1·52275	·06744	168	1·01969	·94244
124	1·49926	·09343	169	1·01651	·95155
125	1·47876	·11682	170	1·01360	·95994
126	1·45887	·14019	171	1·01103	·96738
127	1·43961	·16348	172	1·00861	·97447
128	1·42091	·18677	173	1·00653	·98058
129	1·40275	·21002	174	1·00473	·98590
130	1·38509	·23331	175	1·00327	·99023
131	1·36797	·25651	176	1·00229	·99340
132	1·35141	·27959	177	1·00120	·99620
133	1·33527	·30273	178	1·00050	·99840
134	1·31961	·32580	179	1·00010	·99960
135	+ 1·30458	+ ·34856	180	+ 1·00000	+1·00000

11. Now, put $z = \frac{y}{m}$ in the equation

$$z^3 + az^2 - 4b = 0\,;$$

then $$y^3 + amy^2 - 4bm^3 = 0,$$

or, if $$m^3 = \frac{1}{b} = T,$$

then $$y^3 + amy^2 - 4 = 0\,;$$

and, comparing this with

$$\mathrm{p}^3 u + 3\mathrm{p}\,(iu\sqrt{3})\,\mathrm{p}^2 u - 4 = 0,$$

when $$g_2 = 0, \quad g_3 = 4\,;$$

then $$\mathrm{p}u = y,$$

if $$\mathrm{p}\,(iu\sqrt{3}) = \tfrac{1}{3}am = \frac{S}{3T^{\frac{1}{3}}} = \left(\frac{J}{J-1}\right)^{\frac{1}{3}};$$

and then $$z = \frac{\mathrm{p}u}{T^{\frac{1}{3}}}, \quad x = \frac{T^{\frac{1}{3}}}{\mathrm{p}u}.$$

Then, if two roots of the equation are imaginary, the value of $\frac{S}{3T^{\frac{1}{3}}}$ lies between $-\infty$ and 1; and, to solve the cubic, look out the value of $\mathrm{p}\,(iu\sqrt{3})$ corresponding to $\frac{S}{3T^{\frac{1}{3}}}$, and then the corresponding values of $\mathrm{p}u$ on the same horizontal line; and then the value of x is $\frac{T^{\frac{1}{3}}}{\mathrm{p}u}$; the other two values of x being $\frac{T^{\frac{1}{3}}}{\mathrm{p}\,(u \pm \frac{2}{3}\omega_2')}$.

If the three roots are real, the value of $\frac{S}{3T^{\frac{1}{3}}}$ lies between ∞ and 1; so that $iu\sqrt{3}$ is real; and therefore, putting

$$iu\sqrt{3} = v,$$

and looking out the value of v corresponding to

$$\mathrm{p}v = \frac{S}{3T^{\frac{1}{3}}},$$

the value of x will be $\frac{T^{\frac{1}{3}}}{\mathrm{p}\frac{1}{3}u}$; the other two roots being $\frac{T^{\frac{1}{3}}}{\mathrm{p}\,(\frac{1}{3}u \pm \frac{2}{3}\omega_2)}$.

This method of solution of the cubic has the advantage of requiring only the tabulated values of a doubly periodic function of finite periods.

B. *Solution of the Quartic Equation.*

12. Next, suppose the general quartic equation

$$U_x = (a, b, c, d, e)(x, 1)^4 = 0$$

is presented for solution.

Denoting the Hessian, changing however its sign to what is usually employed, by

$$H_x = (b^2-ac)\,x^4-2\,(ad-bc)\,x^3+(3c^2-ae-2bd)\,x^2$$
$$-2\,(be-cd)\,x+d^2-ce\,;$$

then, if G_x denote the sextic covariant,

$$G_x^2 = 4H_x^3 - g_2 H_x U_x^2 - g_3 U_x^3,$$

where
$$g_2 = ae-4bd+3c^2,$$
$$g_3 = ace+2bcd-ad^2-eb^2-c^3,$$
$$= \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix};$$

the *quadrinvariant* and the *cubinvariant* respectively.

Then, if we put
$$s = \frac{H_x}{U_x},$$
$$\frac{ds}{dx} = \frac{H_x' U_x - H_x U_x'}{U_x^2}$$
$$= 2\,\frac{G_x}{U_x^2}$$

(Cayley, *Elliptic Functions*, page 347),

and
$$4s^3-g_2 s-g_3 = \frac{G_x^2}{U_x^3};$$

so that, if we put the general elliptic integral

$$\int \frac{dx}{\sqrt{(U_x)}} = u,$$

then
$$du = \frac{dx}{\sqrt{(U_x)}} = \tfrac{1}{2}\,\frac{ds}{\sqrt{(4s^3-g_2 s-g_3)}};$$

or
$$2u = \int \frac{ds}{\sqrt{(4s^3-g_2 s-g_3)}};$$

so that, in Weierstrass's notation, we may put

$$s = \frac{H_x}{U_x} = \mathrm{p}\ (2u;\ g_2, g_3);$$

and

$$-\frac{ds}{du} = 2\frac{G_x}{U_x^{\frac{3}{2}}} = 2\mathrm{p}'\ (2u;\ g_2, g_3).$$

We may, therefore, use the notation

$$u = \int \frac{dx}{\sqrt{(U)}} = \tfrac{1}{2}\mathrm{p}^{-1}\left(\frac{H}{U};\ g_2, g_3\right),$$

expressing the general elliptic integral as a function of the *covariants* U and H.

Mr. Robert Russell, of Trinity College, Dublin, has also shown how to reduce the general elliptic integral to one of Legendre's or Jacobi's canonical form, as a function of the quotients of the quadratic factors of the sextic covariant G, the squares of these quadratic factors being

$$H_x - e_1 U_x,\quad H_x - e_2 U_x,\quad H_x - e_3 U_x.$$

13. Suppose, now, that $x = \infty$ when $u = a$, then

$$\mathrm{p}\, 2a = \frac{b^2 - ac}{a}.$$

On the assumption that it is possible to express x as a linear function of $\mathrm{p}u$, then the roots of the quartic will correspond to the infinite values of $\mathrm{p}\,2u$; so that $x = x_0, x_1, x_2, x_3$, the roots of the quartic, when $u = 0, \omega_1, \omega_2, \omega_3$, in the notation of Weierstrass, previously explained.

Thus, when

$$u = \int_{x_0}^{x} \frac{dx}{\sqrt{(U_x)}},$$

we have

$$\mathrm{p}2u = \frac{H_x}{U_x};$$

x_0 denoting the root of the quartic $U_x = 0$, corresponding to $u = 0$.

To express $\mathrm{p}u$ as a function of x, we can employ Klein's formula (54) (*Hyperelliptische Sigmafunctionen*, *Math. Ann.*, XXVII., p. 454),

$$\mathrm{p}u = \frac{\sqrt{U^x}\sqrt{U_{x_0}} + \frac{1}{12}\left(x_0\frac{\delta}{\delta x} + y_0\frac{\delta}{\delta y}\right)^2 U_{(x,y)}}{2\,(x - x_0)^2},$$

where y and y_0 are replaced by unity after differentiation; and then, when x_0 is a root of $U_x = 0$,

$$\mathrm{p}u = \frac{(a, b, c)(x_0, 1)^2 x^2 + 2\,(b, c, d)(x_0, 1)^2 x + (c, d, e)(x_0, 1)^2}{2\,(x - x_0)^2};$$

and, by (55),

$$\wp' u = -\frac{(a, b, c, d)(x_0, 1)^3 x + (b, c, d, e)(x_0, 1)^3}{(x-x_0)^3} U_x.$$

Then, supposing $x = \infty$ when $u = \alpha$,

$$\wp\alpha = \tfrac{1}{2}(a, b, c)(x_0, 1)^2,$$

$$\wp'\alpha = -\sqrt{a}\,(a, b, c, d)(x_0, 1)^3;$$

so that
$$\wp u - \wp\alpha = \frac{(a, b, c, d)(x_0, 1)^3}{x-x_0}$$

$$= -\frac{\wp'\alpha}{\sqrt{a}\,(x-x_0)};$$

or
$$x - x_0 = -\frac{\wp'\alpha}{\sqrt{a}\,(\wp u - \wp\alpha)}.$$

Or, otherwise, putting

$$x - x_0 = \frac{A}{\wp u - \wp\alpha},$$

where A is some constant, to be determined hereafter; and then, with the notation previously explained, we can put

$$x - x_1 = \frac{A}{\wp u - \wp\alpha}\,\frac{\wp u - e_1}{\wp\alpha - e_1},$$

$$x - x_2 = \frac{A}{\wp u - \wp\alpha}\,\frac{\wp u - e_2}{\wp\alpha - e_2},$$

$$x - x_3 = \frac{A}{\wp u - \wp\alpha}\,\frac{\wp u - e_3}{\wp\alpha - e_3}.$$

Then
$$U^2 = a\,(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$= a\,\frac{A^4}{(\wp u - \wp\alpha)^4}\,\frac{\wp'^2 u}{\wp'^2 \alpha}.$$

Also
$$\frac{dx}{du} = -\frac{A\wp' u}{(\wp u - \wp\alpha)^2};$$

so that
$$du = \frac{dx}{\sqrt{(U)}} = \frac{-\dfrac{A\wp' u}{(\wp u - \wp\alpha)^2}\,du}{\sqrt{a}\,\dfrac{A^2}{(\wp u - \wp\alpha)^2}\,\dfrac{\wp' u}{\wp'\alpha}},$$

and therefore
$$A = -\frac{\wp'\alpha}{\sqrt{a}}.$$

Then
$$x_0-x_1 = \frac{A}{\wp\alpha-e_1} = -\frac{1}{\sqrt{a}}\frac{\wp'\alpha}{\wp\alpha-e_1},$$
$$x_0-x_2 = \frac{A}{\wp\alpha-e_2} = -\frac{1}{\sqrt{a}}\frac{\wp'\alpha}{\wp\alpha-e_2},$$
$$x_0-x_3 = \frac{A}{\wp\alpha-e_3} = -\frac{1}{\sqrt{a}}\frac{\wp'\alpha}{\wp\alpha-e_3};$$

so that, since
$$x_0+x_1+x_2+x_3 = -4\frac{b}{a},$$
$$3x_0-x_1-x_2-x_3 = 4\left(x_0+\frac{b}{a}\right)$$
$$= -\frac{1}{\sqrt{a}}\left(\frac{\wp'\alpha}{\wp\alpha-e_1}+\frac{\wp'\alpha}{\wp\alpha-e_2}+\frac{\wp'\alpha}{\wp\alpha-e_3}\right);$$

or
$$x_0+\frac{b}{a} = -\frac{1}{4\sqrt{a}}\left(\frac{\wp'\alpha}{\wp\alpha-e_1}+\frac{\wp'\alpha}{\wp\alpha-e_2}+\frac{\wp'\alpha}{\wp\alpha-e_3}\right);$$

and, therefore,
$$x_1+\frac{b}{a} = -\frac{1}{4\sqrt{a}}\left(\frac{-3\wp'\alpha}{\wp\alpha-e_1}+\frac{\wp'\alpha}{\wp\alpha-e_2}+\frac{\wp'\alpha}{\wp\alpha-e_3}\right),$$
$$x_2+\frac{b}{a} = -\frac{1}{4\sqrt{a}}\left(\frac{\wp'\alpha}{\wp\alpha-e_1}+\frac{-3\wp'\alpha}{\wp\alpha-e_2}+\frac{\wp'\alpha}{\wp\alpha-e_3}\right),$$
$$x_3+\frac{b}{a} = -\frac{1}{4\sqrt{a}}\left(\frac{\wp'\alpha}{\wp\alpha-e_1}+\frac{\wp'\alpha}{\wp\alpha-e_2}+\frac{-3\wp'\alpha}{\wp\alpha-e_3}\right).$$

Then
$$U = \frac{1}{a}\frac{\wp'^2\alpha\,\wp'^2u}{(\wp u-\wp\alpha)^4} = \frac{1}{a}\{\wp(u-\alpha)-\wp(u+\alpha)\}^2,$$

as in M. Halphen's paper "Sur l'inversion des Integrales Elliptiques," *Journal de l'Ecole Polytechnique*, 1884.

14. These preceding investigations indicate the advantage of the substitution of § 61 of Burnside and Panton's *Theory of Equations*, where the second term of the quartic
$$ax^4+4bx^3+6cx^2+4dx+e = 0$$
is removed by means of the substitution
$$z = ax+b,$$
or
$$x = \frac{z}{a}-\frac{b}{a};$$

so that the quartic becomes

$$V = z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0,$$

where $$H = ac - b^2,$$

$$G = a^2d - 3abc + 2b^3.$$

Then the quadrinvariant and the cubinvariant of V are

$$G_2 = a^2 g_2,$$

$$G_3 = a^3 g_3;$$

also $$dv = \frac{dz}{\sqrt{(V)}} = \frac{dx}{\sqrt{(aU)}} = \frac{du}{\sqrt{a}},$$

so that $$u = v\sqrt{a},$$

agreeing with the formula

$$\wp(u;\ g_2, g_3) = \frac{1}{a}\wp(v;\ G_2, G_3).$$

15. It will simplify matters, without any restriction on generality, to suppose hereafter a is replaced by unity whenever necessary, so that u and v are the same; also $G_2 = g_2$, $G_3 = g_3$; and then, α denoting the value of u which makes x, and therefore z, infinite,

$$\wp 2\alpha = H, \text{ and } \wp' 2\alpha = -G;$$

and, denoting the roots of the quartic in z by z_0, z_1, z_2, z_3; then, as before,

$$z - z_0 = \frac{-\wp'\alpha}{\wp u - \wp\alpha},$$

$$z - z_1 = \frac{-\wp'\alpha}{\wp u - \wp\alpha}\,\frac{\wp u - e_1}{\wp\alpha - e_1},$$

$$z - z_2 = \frac{-\wp'\alpha}{\wp u - \wp\alpha}\,\frac{\wp u - e_2}{\wp\alpha - e_2},$$

$$z - z_3 = \frac{-\wp'\alpha}{\wp u - \wp\alpha}\,\frac{\wp u - e_3}{\wp\alpha - e_3};$$

also $$z_0 = -\tfrac{1}{4}\left(\frac{\wp'\alpha}{\wp\alpha - e_1} + \frac{\wp'\alpha}{\wp\alpha - e_2} + \frac{\wp'\alpha}{\wp\alpha - e_3}\right),$$

$$z_1 = -\tfrac{1}{4}\left(\frac{-3\wp'\alpha}{\wp\alpha - e_1} + \frac{\wp'\alpha}{\wp\alpha - e_2} + \frac{\wp'\alpha}{\wp\alpha - e_3}\right),$$

$$z_2 = -\tfrac{1}{4}\left(\frac{p'a}{pa-e_1} + \frac{-3p'a}{pa-e_2} + \frac{p'a}{pa-e_3}\right),$$

$$z_3 = -\tfrac{1}{4}\left(\frac{p'a}{pa-e_1} + \frac{p'a}{pa-e_2} + \frac{-3p'a}{pa-e_3}\right).$$

16. By means of the notation explained by Schwarz in *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen*, we can transform the above expressions for the roots z_0, z_1, z_2, z_3, into

$$z_0 = 2\frac{\sigma' a}{\sigma a} - \frac{\sigma' 2a}{\sigma 2a} = -\tfrac{1}{2}\frac{p'' a}{p' a},$$

$$z_1 = 2\frac{\sigma_1' a}{\sigma_1 a} - \frac{\sigma' 2a}{\sigma 2a} = -\tfrac{1}{2}\frac{p'' a}{p' a} + \frac{p' a}{pa-e_1} = -\tfrac{1}{2}\frac{p''(a+\omega_1)}{p'(a+\omega_1)},$$

$$z_2 = 2\frac{\sigma_2' a}{\sigma_2 a} - \frac{\sigma' 2a}{\sigma 2a} = -\tfrac{1}{2}\frac{p'' a}{p' a} + \frac{p' a}{pa-e_2} = -\tfrac{1}{2}\frac{p''(a+\omega_2)}{p'(a+\omega_2)},$$

$$z_3 = 2\frac{\sigma_3' a}{\sigma_3 a} - \frac{\sigma' 2a}{\sigma 2a} = -\tfrac{1}{2}\frac{p'' a}{p' a} + \frac{p' a}{pa-e_3} = -\tfrac{1}{2}\frac{p''(a+\omega_3)}{p'(a+\omega_3)};$$

or, in another form,

$$z_0 = \frac{\sigma_1 2a}{\sigma 2a} + \frac{\sigma_2 2a}{\sigma 2a} + \frac{\sigma_3 2a}{\sigma 2a},$$

$$z_1 = \frac{\sigma_1 2a}{\sigma 2a} - \frac{\sigma_2 2a}{\sigma 2a} - \frac{\sigma_3 2a}{\sigma 2a},$$

$$z_2 = -\frac{\sigma_1 2a}{\sigma 2a} + \frac{\sigma_2 2a}{\sigma 2a} - \frac{\sigma_3 2a}{\sigma 2a},$$

$$z_3 = -\frac{\sigma_1 2a}{\sigma 2a} - \frac{\sigma_2 2a}{\sigma 2a} + \frac{\sigma_3 2a}{\sigma 2a};$$

equivalent to

$$z_0 = \sqrt{(p2a-e_1)} + \sqrt{(p2a-e_2)} + \sqrt{(p2a-e_3)},$$

$$z_1 = \sqrt{(p2a-e_1)} - \sqrt{(p2a-e_2)} - \sqrt{(p2a-e_3)},$$

$$z_2 = -\sqrt{(p2a-e_1)} + \sqrt{(p2a-e_2)} - \sqrt{(p2a-e_3)},$$

$$z_3 = -\sqrt{(p2a-e_1)} - \sqrt{(p2a-e_2)} + \sqrt{(p2a-e_3)},$$

agreeing with the expressions on page 117 of Burnside and Panton's *Theory of Equations*.

Then $$(x_0-x_1)(x_2-x_3) = (z_0-z_1)(z_2-z_3)$$
$$= 4(p2a-e_3-p2a+e_2) = 4(e_2-e_3);$$

and, similarly,

$$(x_0-x_2)(x_3-x_1) = (z_0-z_2)(z_3-z_1) = 4\,(e_3-e_1),$$
$$(x_0-x_3)(x_1-x_2) = (z_0-z_3)(z_1-z_2) = 4\,(e_1-e_2)\,;$$

e_1, e_2, e_3 denoting the roots of the *reducing cubic*

$$4s^3-g_2s-g_3=0,$$

and replacing a by unity.

17. The simplest expression, however, of the roots of a quartic is obtained by increasing the roots of the equation in z by $\dfrac{\sigma' 2a}{\sigma 2a}$, equivalent to putting

$$\frac{\sigma' 2a}{\sigma 2a} = -b$$

in the quartic equation

$$y^4+4by^3+6cy^2+4dy+e=0\,;$$

when the roots of the quartic are

$$y_0 = 2\frac{\sigma' a}{\sigma a},\quad y_1 = 2\frac{\sigma_1' a}{\sigma_1 a},\quad y_2 = 2\frac{\sigma_2' a}{\sigma_2 a},\quad y_3 = 2\frac{\sigma_3' a}{\sigma_3 a}.$$

Then $$y_1-y_0 = \frac{\wp' a}{\wp a-e_1},\quad y_2-y_0 = \frac{\wp' a}{\wp a-e_2},\quad y_3-y_0 = \frac{\wp' a}{\wp a-e_3}\,;$$

so that $$(y_1-y_0)(y_2-y_3) = \frac{\wp'^2 a\,(e_2-e_3)}{(\wp a-e_1)(\wp a-e_2)(\wp a-e_3)} = 4\,(e_2-e_3),$$

and, similarly, $$(y_2-y_0)(y_3-y_1) = 4\,(e_3-e_1),$$
$$(y_3-y_0)(y_1-y_2) = 4\,(e_1-e_2).$$

Then, generally, $$y = \frac{\sigma'(u+a)}{\sigma(u+a)} - \frac{\sigma'(u-a)}{\sigma(u-a)},$$

if $$u = \int \frac{dy}{\sqrt{(U_y)}}.$$

18. We may now compare these substitutions with those given by Halphen (*Journal de l'Ecole Polytechnique*, 1884), where

$$z = \tfrac{1}{2}\,\frac{\wp' u-\wp' v}{\wp u-\wp v}$$
$$= \frac{\sigma'(u+v)}{\sigma(u+v)} - \frac{\sigma' u}{\sigma u} - \frac{\sigma' v}{\sigma v}\,;$$

or, in our notation, with

$$z = \tfrac{1}{2}\frac{\wp'(u-a)-\wp'2a}{\wp(u-a)-\wp 2a}$$

$$= \tfrac{1}{2}\frac{\wp'(u+a)+\wp'2a}{\wp(u+a)-\wp 2a} = \tfrac{1}{2}\frac{\wp'(u-a)-\wp'(u+a)}{\wp(u-a)-\wp(u+a)}$$

$$= \frac{\sigma'(u+a)}{\sigma(u+a)}-\frac{\sigma'(u-a)}{\sigma(u-a)}-\frac{\sigma' 2a}{\sigma 2a},$$

or $$y = \frac{\sigma'(u+a)}{\sigma(u+a)}-\frac{\sigma'(u-a)}{\sigma(u-a)},$$

with the notation of the last article (§ 16).

Then $$V = \frac{\wp'^2 a\, \wp'^2 u}{(\wp u-\wp a)^4}$$

$$= \{\wp(u-a)-\wp(u+a)\}^2,$$

and $$U = \frac{1}{a}\{\wp(u-a)-\wp(u+a)\}^2,$$

or $$\sqrt{U} = \frac{1}{\sqrt{a}}\{\wp(u-a)-\wp(u+a)\},$$

agreeing with Halphen's expression.

19. It remains to investigate the conditions for the reality or otherwise of the roots of the quartic.

(A) When the discriminant

$$\Delta = g_2^3-27g_3^2$$

is negative, two of the roots e_1, e_2, e_3 of the reducing cubic

$$4s^3-g_2^3-g_3=0$$

are imaginary; and then two of the roots of the quartic are imaginary and two are real.

(B) When the discriminant Δ is positive, e_1, e_2, e_3 are all real, and the roots of the quartic may be all real, or all imaginary.

The roots will be all real when $\wp c$ and $\wp' c$ are real; that is, when $\wp 2c$ or H lies between ∞ and e_1; otherwise, all the roots will be imaginary.

The solution of the *quintic* by means of Weierstrass's functions has been considered by Kiepert in *Crelle*, Vol. 87, page 120.

20. Let us apply the preceding theories to the motion of a prolate solid of revolution, moving through infinite liquid, under no forces.

Then, as explained in the *Quar. Jour. of Math.* (No. 62, 1879), if $\cos\theta$ is denoted by x,

$$\dot{x}^2 = a_0(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$
$$= a_0x^4 - 4a_1x^3 + 6a_2x^2 - 4a_3x + a_4,$$

where $$a_0 = \frac{F^2}{c_4}\left(\frac{1}{c_3}-\frac{1}{c_1}\right) = M, \quad \text{also } a_1 = 0;$$

and if $$\wp 2c = \frac{a_2}{a_0}, \quad \wp' 2c = \frac{a_3}{a_0};$$

then $$x_0 = -\tfrac{1}{2}\frac{\wp'' c}{\wp' c}, \quad x_1 = -\tfrac{1}{2}\frac{\wp''(c+\omega_1)}{\wp'(c+\omega_1)}, \quad x_2 = \ldots, \quad x_3 = \ldots;$$

and $$x = \frac{\sigma'(u+c)}{\sigma(u+c)} - \frac{\sigma'(u-c)}{\sigma(u-c)} - \frac{\sigma' 2c}{\sigma 2c}$$
$$= \tfrac{1}{2}\frac{\wp'(u-c)-\wp' 2c}{\wp(u-c)-\wp 2c},$$

and $$x - x_0 = \frac{\sigma'(u+c)}{\sigma(u+c)} - \frac{\sigma'(u-c)}{\sigma(u-c)} - 2\frac{\sigma' c}{\sigma c}$$
$$= \frac{-\wp' c}{\wp u - \wp c},$$

$$x - x_1 = \frac{-\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_1}{\wp c - e_1},$$

$$x - x_2 = \frac{-\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_2}{\wp c - e_2},$$

$$x - x_3 = \frac{-\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_3}{\wp c - e_3};$$

so that $$(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$
$$= \frac{\wp'^4 c}{(\wp u - \wp c)^4}\,\frac{\wp'^2 u}{\wp'^2 c}$$
$$= \frac{\wp'^2 c\, \wp'^2 u}{(\wp u - \wp c)^4}$$
$$= \{\wp(u-c) - \wp(u+c)\}^2;$$

also $$\frac{dx}{du} = \frac{\wp' c\, \wp' u}{(\wp u - \wp c)^2} = \wp(u-c) - \wp(u+c),$$

and therefore $$du = \sqrt{a_0}\,dt,$$

or $$u = \sqrt{a_0}\,t + \text{constant}.$$

The constant must be taken to be ω_3, for $\wp u$ to range between e_2 and e_3, and therefore x to range between x_2 and x_3.

21. Then $$\frac{d\psi}{dt} = \frac{G+c_6 n}{2c_4}\frac{1}{1+\cos\theta} + \frac{G-c_6 n}{2c_4}\frac{1}{1-\cos\theta},$$

$$\frac{d\psi}{du} = \tfrac{1}{2}\frac{G+c_6 n}{2c_4\sqrt{M}}\frac{1}{1+\cos\theta} + \frac{G-c_6 n}{2c_4\sqrt{M}}\frac{1}{1-\cos\theta},$$

$$= \tfrac{1}{2}i\,\frac{\sqrt{(1+x_0\,.\,1+x_1\,.\,1+x_2\,.\,1+x_3)}}{1+\cos\theta} + \tfrac{1}{2}i\,\frac{\sqrt{(1-x_0\,.\,1-x_1\,.\,1-x_2\,.\,1-x_3)}}{1-\cos\theta}.$$

Now, suppose $u = a$ when $\cos\theta = -1$,

$u = b$,, $\cos\theta = +1$;

then $$1+\cos\theta = \frac{\wp' c}{\wp a - \wp c} - \frac{\wp' c}{\wp u - \wp c} = \frac{\wp' c\,(\wp u - \wp a)}{(\wp a - \wp c)(\wp u - \wp c)},$$

$$1-\cos\theta = \frac{\wp' c}{\wp u - \wp c} - \frac{\wp' c}{\wp b - \wp c} = \frac{\wp' c\,(\wp b - \wp u)}{(\wp b - \wp c)(\wp u - \wp c)},$$

$$1+x_0\,.\,1+x_1\,.\,1+x_2\,.\,1+x_3 = \frac{\wp'^2 a\,\wp'^2 c}{(\wp a - \wp c)^4},$$

$$1-x_0\,.\,1-x_1\,.\,1-x_2\,.\,1-x_3 = \frac{\wp'^2 b\,\wp'^2 c}{(\wp b - \wp c)^4};$$

also $\wp' a$ is negative imaginary, and $\wp' b$ positive imaginary,

so that $$\frac{d\psi}{du} = \tfrac{1}{2}i\,\frac{\dfrac{\wp' a\,\wp' c}{(\wp a - \wp c)^2}}{\dfrac{\wp' c\,(\wp u - \wp a)}{(\wp a - \wp c)(\wp u - \wp c)}} - \tfrac{1}{2}i\,\frac{\dfrac{\wp' b\,\wp' c}{(\wp b - \wp c)^2}}{\dfrac{\wp' c\,(\wp b - \wp u)}{(\wp b - \wp c)(\wp u - \wp c)}}$$

$$= \tfrac{1}{2}i\,\frac{\wp' a\,(\wp u - \wp c)}{(\wp a - \wp c)(\wp u - \wp a)} + \tfrac{1}{2}i\,\frac{\wp' b\,(\wp u - \wp c)}{(\wp b - \wp c)(\wp u - \wp b)},$$

$$2\frac{d\psi}{diu} = \frac{\wp' a}{\wp a - \wp c} + \frac{\wp' a}{\wp u - \wp a} - \frac{\wp' b}{\wp b - \wp c} - \frac{\wp' b}{\wp u - \wp b}$$

$$= \frac{\sigma'}{\sigma}(a+c) + \frac{\sigma'}{\sigma}(a-c) - 2\frac{\sigma'}{\sigma}a - \frac{\sigma'}{\sigma}(u+a) + \frac{\sigma'}{\sigma}(u-a) + 2\frac{\sigma'}{\sigma}a,$$

$$+\frac{\sigma'}{\sigma}(b+c) - \frac{\sigma'}{\sigma}(b-c) + 2\frac{\sigma'}{\sigma}b - \frac{\sigma'}{\sigma}(u+b) + \frac{\sigma'}{\sigma}(u-b) - 2\frac{\sigma'}{\sigma}b;$$

so that $$\psi = \tfrac{1}{2}i \log \frac{\sigma(u-a)\,\sigma(u-b)}{\sigma(u+a)\,\sigma(u+b)} + \tfrac{1}{2}iPu,$$

where $$P = \frac{\sigma'}{\sigma}(a+c) + \frac{\sigma'}{\sigma}(a-c)$$

$$+\frac{\sigma'}{\sigma}(b+c) + \frac{\sigma'}{\sigma}(b-c),$$

or $$e^{-2i\psi} = e^{Pu}\frac{\sigma(u-a)\,\sigma(u-b)}{\sigma(u+a)\,\sigma(u+b)}.$$

22. Then

$$x_0 = -\tfrac{1}{2}\frac{\wp''c}{\wp'c} = 2\frac{\sigma_0'}{\sigma_0}c - \frac{\sigma'}{\sigma}2c = \frac{\sigma_1}{\sigma}2c + \frac{\sigma_2}{\sigma}2c + \frac{\sigma_3}{\sigma}2c,$$

$$x_1 = -\tfrac{1}{2}\frac{\wp''(c+\omega_1)}{\wp'(c+\omega_1)} = 2\frac{\sigma_1'}{\sigma_1}c - \frac{\sigma'}{\sigma}2c = \frac{\sigma_1}{\sigma}2c - \frac{\sigma_2}{\sigma}2c - \frac{\sigma_3}{\sigma}2c,$$

$$x_2 = -\tfrac{1}{2}\frac{\wp''(c+\omega_2)}{\wp'(c+\omega_2)} = 2\frac{\sigma_2'}{\sigma_2}c - \frac{\sigma'}{\sigma}2c = -\frac{\sigma_1}{\sigma}2c + \frac{\sigma_2}{\sigma}2c - \frac{\sigma_3}{\sigma}2c,$$

$$x_3 = -\tfrac{1}{2}\frac{\wp''(c+\omega_3)}{\wp'(c+\omega_3)} = 2\frac{\sigma_3'}{\sigma_3}c - \frac{\sigma'}{\sigma}2c = -\frac{\sigma_1}{\sigma}2c - \frac{\sigma_2}{\sigma}2c + \frac{\sigma_3}{\sigma}2c\,;$$

and, generally,

$$x = \tfrac{1}{2}\frac{\wp'(u-c) - \wp'2c}{\wp(u-c) - \wp\,2c} = \frac{\sigma'}{\sigma}(u+c) - \frac{\sigma'}{\sigma}(u-c) - \frac{\sigma'}{\sigma}2c,$$

$$\frac{dx}{dt} = \sqrt{M}\frac{\wp'c\,\wp'u}{(\wp u - \wp c)^2} = \sqrt{M}\left\{\wp(u-c) - \wp(u+c)\right\}.$$

23. In order to agree with the notation of the *Quarterly Journal*, we must suppose

$$x_0 = \delta, \quad x_1 = \alpha, \quad x_2 = \beta, \quad x_3 = \gamma\,;$$

and then $$(\alpha-\gamma)(\beta-\delta) = 4(e_1-e_3)\,;$$

also, in order for $\wp u$ to oscillate in value between e_2 and e_3, we must have $$u = \sqrt{M}\,t + \omega_3.$$

Comparing Weierstrass's notation with Jacobi's, we have

$$\mathrm{p}u - e_1 = \left(\frac{\sigma_1 u}{\sigma u}\right)^2 = (e_1 - e_3)\,\frac{\mathrm{cn}^2}{\mathrm{dn}^2}\,\sqrt{(e_1-e_3)}\,u$$

or

$$(e_1 - e_3)\,\mathrm{cd}^2\sqrt{(e_1-e_3)}\,u;$$

$$\mathrm{p}u - e_2 = \left(\frac{\sigma_2 u}{\sigma u}\right)^2 = (e_1 - e_3)\,\mathrm{ds}^2\sqrt{(e_1-e_3)}\,u,$$

$$\mathrm{p}u - e_3 = \left(\frac{\sigma_3 u}{\sigma u}\right)^2 = (e_1 - e_3)\,\mathrm{ns}^2\sqrt{(e_1-e_3)}\,u$$

(Schwarz, page 30);

and $$\sqrt{(e_1-e_3)}\,u = \tfrac{1}{2}\sqrt{\{M(\alpha-\gamma)(\beta-\delta)\}}\,t + 2iK',$$

agreeing with the notation of the *Quarterly Journal*.

The determination of α, β, γ, δ attempted in that article has thus been effected, in terms of Weierstrass functions of c, the invariants being g_2 and g_3, the invariants of the quartic $\dot{x}^2$ in terms of x.

24. Now $$1+\delta = \frac{\mathrm{p}'c}{\mathrm{p}a - \mathrm{p}c}, \quad 1-\delta = \frac{-\mathrm{p}'c}{\mathrm{p}b - \mathrm{p}c};$$

so that $$\frac{1-\delta}{1+\delta} = -\frac{\mathrm{p}a-\mathrm{p}c}{\mathrm{p}b-\mathrm{p}c};$$

and, similarly, $$\frac{1-\alpha}{1+\alpha} = -\frac{\mathrm{p}a-\mathrm{p}c}{\mathrm{p}b-\mathrm{p}c}\,\frac{\mathrm{p}b-e_1}{\mathrm{p}a-e_1},$$

$$\frac{1-\beta}{1+\beta} = -\frac{\mathrm{p}a-\mathrm{p}c}{\mathrm{p}b-\mathrm{p}c}\,\frac{\mathrm{p}b-e_2}{\mathrm{p}a-e_2},$$

$$\frac{1-\gamma}{1+\gamma} = -\frac{\mathrm{p}a-\mathrm{p}c}{\mathrm{p}b-\mathrm{p}c}\,\frac{\mathrm{p}b-e_3}{\mathrm{p}a-e_3};$$

so that $$\frac{\dfrac{1-\alpha}{1+\alpha}}{\dfrac{\sigma_1^2 b}{\sigma_1^2 a}} = \frac{\dfrac{1-\beta}{1+\beta}}{\dfrac{\sigma_2^2 b}{\sigma_2^2 a}} = \frac{\dfrac{1-\gamma}{1+\gamma}}{\dfrac{\sigma_3^2 b}{\sigma_3^2 a}} = \frac{\dfrac{1-\delta}{1+\delta}}{\dfrac{\sigma^2 b}{\sigma^2 a}},$$

and if this is put $= \dfrac{1}{x}$, and

$$A = \frac{\sigma_1^2 b}{\sigma_1^2 a}, \quad B = \frac{\sigma_2^2 b}{\sigma_2^2 a}, \quad C = \frac{\sigma_3^2 b}{\sigma_3^2 a}, \quad D = \frac{\sigma^2 b}{\sigma^2 a},$$

the biquadratic

$$4x^4 - 2x^3(A+B+C+D)$$
$$+2x(BCD+CDA+DAB+ABC) - 4ABCD = 0,$$

obtained by putting $\alpha+\beta+\gamma+\delta=0$,

has a root $$x_0 = -\frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma^2 b}{\sigma^2 a} = -\frac{\sigma(a+c)\,\sigma(a-c)}{\sigma(b+c)\,\sigma(b-c)},$$

the other roots being

$$-\frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma_1^2 b}{\sigma_1^2 a},\quad -\frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma_2^2 b}{\sigma_2^2 a},\quad -\frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma_3^2 b}{\sigma_3^2 a}.$$

Denoting these four roots by x_0, x_1, x_2, x_3; then

$$x_1-x_0 = \frac{(\wp a-\wp c)}{(\wp b-\wp c)}\frac{\sigma^2 b}{\sigma^2 a}\frac{\wp a-\wp b}{\wp a-e_1},$$

$$x_2-x_3 = \frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma^2 b}{\sigma^2 a}\frac{(\wp a-\wp b)(e_2-e_3)}{(\wp a-e_2)(\wp a-e_3)};$$

so that

$$(x_1-x_0)(x_2-x_3) = \left(\frac{\wp a-\wp c}{\wp b-\wp c}\right)^2\frac{\sigma^4 b}{\sigma^4 a}\frac{(\wp a-\wp b)^2}{\wp'^2 a}4\,(e_2-e_3),$$

so that the S and T of the reducing cubic of the last quartic becomes

$$S = m^4 g_2,\quad T = m^4 g_3,$$

where $$m = \frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma^2 b}{\sigma^2 a}\frac{\wp a-\wp b}{\wp' a};$$

and this complication is sufficient to explain the difficulty experienced previously in the attempt to solve the biquadratic (6) and its reducing cubic.

25. Compared with the previous expressions in the *Quarterly Journal*, Vol. XVI.,

$$\mathrm{sn}^2 ia' = \frac{\alpha-\gamma}{\alpha-\delta}\frac{1+\delta}{1+\gamma} = \frac{\dfrac{1+\alpha}{1+\gamma}-1}{\dfrac{1+\alpha}{1+\delta}-1} = \frac{\dfrac{\wp a-e_1}{\wp c-e_1}\dfrac{\wp c-e_3}{\wp a-e_3}-1}{\dfrac{\wp a-e_1}{\wp c-e_1}-1} = \frac{e_1-e_3}{\wp a-e_3},$$

$$\mathrm{cn}^2 ia' = \frac{\gamma-\delta}{\alpha-\delta}\frac{1+\alpha}{1+\gamma} = \frac{1-\dfrac{1+\delta}{1+\gamma}}{1-\dfrac{1+\delta}{1+\alpha}} = \frac{1-\dfrac{\wp c-e_3}{\wp a-e_3}}{1-\dfrac{\wp c-e_1}{\wp a-e_1}} = \frac{\wp a-e_1}{\wp a-e_3},$$

$$\mathrm{dn}^2 ia' = \frac{\gamma-\delta}{\beta-\delta}\frac{1+\beta}{1+\gamma} = \frac{1-\dfrac{1+\delta}{1+\gamma}}{1-\dfrac{1+\delta}{1+\beta}} = \frac{1-\dfrac{\wp c-e_3}{\wp a-e_3}}{1-\dfrac{\wp c-e_2}{\wp a-e_2}} = \frac{\wp a-e_2}{\wp a-e_3},$$

and similarly $$\operatorname{sn}^2(ib'+K) = \frac{e_1-e_3}{\wp b-e_3},$$

$$\operatorname{cn}^2(ib'+K) = \frac{\wp a-e_1}{\wp a-e_3},$$

$$\operatorname{dn}^2(ib'+K) = \frac{\wp a-e_2}{\wp a-e_3};$$

indicating that $a = r\omega_3,\quad b = \omega_1+s\omega_3,$

where r and s are proper fractions, with Schwarz's notation (*Formeln*, p. 74). Also, to the complementary modulus,

$$\operatorname{sn}^2 a' = \frac{1-\dfrac{1+\gamma}{1+\alpha}}{1-\dfrac{1+\gamma}{1+\delta}} = \frac{1-\dfrac{\wp a-e_3}{\wp c-e_3}\,\dfrac{\wp c-e_1}{\wp a-e_1}}{1-\dfrac{\wp a-e_3}{\wp c-e_3}} = -\frac{e_1-e_3}{\wp a-e_1}$$

$$\operatorname{cn}^2 a' = \frac{\wp a-e_3}{\wp a-e_1},$$

$$\operatorname{dn}^2 a' = \frac{\wp a-e_2}{\wp a-e_1},$$

$$\operatorname{sn}^2 b' = \frac{1-\dfrac{1-\beta}{1-\delta}}{1-\dfrac{1-\beta}{1-\alpha}} = \frac{1-\dfrac{\wp b-e_2}{\wp c-e_2}}{1-\dfrac{\wp b-e_2}{\wp c-e_2}\,\dfrac{\wp c-e_1}{\wp b-e_1}} = -\frac{\wp b-e_1}{e_1-e_2},$$

$$\operatorname{cn}^2 b' = \frac{\wp b-e_2}{e_1-e_2},$$

$$\operatorname{dn}^2 b' = \frac{\wp b-e_3}{e_1-e_3}.$$

26. Now put $x = \cos\theta = \dfrac{1-y}{1+y}$, so that $y = \tan^2\tfrac{1}{2}\theta$;

then $$4\dot{y}^2 = -(A_0y^4-4A_1y^3+6A_2y^2-4A_3y+A_4);$$

or, if $x = \dfrac{z-1}{z+1}$, so that $z = \cot^2\tfrac{1}{2}\theta$,

then $$4\dot{z}^2 = -(A_4z^4-4A_3z^3+6A_2z^2-4A_1z+A_0),$$

where $$A_0 = \frac{(G+c_6n)^2}{c_4^2},\quad A_4 = \frac{(G-c_6n)^2}{c_4^2}.$$

Then the quadrinvariant G_2 and the cubinvariant G_3 of these reciprocal quartics in y and z are the same, and

$$G_2 = 2^4 g_2, \quad G_3 = -2^6 g_3;$$

so that, if $y = \infty$ when $t = a$, $z = \infty$ when $t = b$, we may put, changing from G_2 and G_3 to g_2 and g_3,

$$\sqrt{A_0}\,y = \frac{\sigma'}{\sigma}\tfrac{1}{2}(t+a) - \frac{\sigma'}{\sigma}\tfrac{1}{2}(t-a),$$

$$\sqrt{A_4}\,z = \frac{\sigma'}{\sigma}\tfrac{1}{2}(t+b) - \frac{\sigma'}{\sigma}\tfrac{1}{2}(t-b),$$

and $$\frac{d\psi}{dt} = \tfrac{1}{4}\sqrt{A_0}\sec^2\tfrac{1}{2}\theta + \tfrac{1}{4}\sqrt{A_4}\operatorname{cosec}^2\tfrac{1}{2}\theta$$

$$= \tfrac{1}{4}(A_0+A_4) + \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t+a) - \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t-a)$$

$$+\tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t+b) - \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t-b),$$

or $$\frac{d\psi}{dt} = \tfrac{1}{2}\frac{G}{c_4} + \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t+a) - \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t-a)$$

$$+\tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t+b) - \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t-b),$$

so that $$\psi = \tfrac{1}{2}\frac{G}{c_4}t + \tfrac{1}{2}\log\frac{\sigma\tfrac{1}{2}(t+a)}{\sigma\tfrac{1}{2}(t-a)} + \tfrac{1}{2}\log\frac{\sigma\tfrac{1}{2}(t+b)}{\sigma\tfrac{1}{2}(t-b)}.$$

Also $$\sqrt{A_0}\,y_0 = 2\frac{\sigma'}{\sigma}\tfrac{1}{2}a, \quad \sqrt{A_4}\,z_0 = 2\frac{\sigma'}{\sigma}\tfrac{1}{2}b;$$

so that $$\sqrt{A_0}(y-y_0) = \frac{-\wp' a}{\wp u - \wp a},$$

with G_2 and G_3.

Arranged in descending order of magnitude, we have

$$\infty > x_1 > 1 > x_2 > x > x_3 > -1 > x_0 > -\infty;$$

also, when $x = \infty$, $y = -1$, $z = -1$, $u = c$;

$x = x_1$, $y = y_1$, $z = z_1$, $u = \omega_1$;

$x = 1$, $y = 0$, $z = \infty$, $u = b$;

$x = x_2$, $y = y_2$, $z = z_2$, $u = \omega_2$;

and now x, y, z have the real values of the problem; also

$$x = x_3, \quad y = y_3, \quad z = z_3, \quad u = \omega_3;$$
$$x = -1, \quad y = \infty, \quad z = 0, \quad u = a;$$
$$x = x_0, \quad y = y_0, \quad z = z_0, \quad u = 0;$$
$$x = -\infty, \quad y = -1, \quad z = -1, \quad u = c.$$

Then

$$\dot{x}^2 = M(x-x_0)(x-x_1)(x-x_2)(x-x_3),$$
$$4\dot{y}^2 = -A_0(y-y_0)(y-y_1)(y-y_2)(y-y_3),$$
$$4\dot{z}^2 = -A_4(z-z_0)(z-z_1)(z-z_2)(z-z_3),$$

and now $\wp a$ lies between e_1 and e_2 and $\wp' a$ is positive imaginary,

$\wp b$ „ e_3 and $-\infty$ and $\wp' b$ is negative „

$\wp u$ „ e_2 and e_3,

and

$$u = t + \omega_3.$$

APPENDIX.

Let us apply the Table to the solution of two representative Cubic Equations.

(i.) To solve $x^3+x^2-1 = 0$, a modular equation of the 23rd order, the real root of which is $\sqrt[12]{(16kk')}$, when $K'/K = \sqrt{23}$.

Here

$$a = 1, \quad b = \tfrac{1}{4}, \quad m = \sqrt[12]{4};$$
$$\wp(iu\sqrt{3}) = \tfrac{1}{3}am = {\cdot}529;$$

so that, to the nearest integral value of r,

$$iu\sqrt{3} = \tfrac{143}{180}\omega_2', \quad u = \tfrac{143}{180}\omega_2,$$
$$\wp u = {\cdot}07882, \quad \text{and} \quad z = {\cdot}7551.$$

(ii.) To solve $x^3-5x^2+6x-1 = 0$, the roots of which are

$$4\sin^2\frac{\pi}{14}, \quad 4\sin^2\frac{3\pi}{14}, \quad 4\sin^2\frac{5\pi}{14}.$$

Put $x = \dfrac{z}{2z+1}$, then

$$7z^3+7z^2-1 = 0;$$

here

$$a = 1, \quad m = \sqrt[3]{28};$$

and $$\mathrm{p}\,(iu\sqrt{3}) = \tfrac{1}{3}\sqrt[3]{28} = 1{\cdot}012,$$

so that, to the nearest integral value of r,

$$iu\sqrt{3} = \tfrac{171}{180}\omega_2\,(\pm 2\omega_2), \quad u = \tfrac{57}{180}\omega_2'\,(\pm \tfrac{2}{3}\omega_2');$$

$$\mathrm{p}u = -2{\cdot}226, \quad +{\cdot}996, \quad -1{\cdot}803;$$

and $$x = 1{\cdot}573, \quad {\cdot}198, \quad 3{\cdot}16.$$

On the Cremonian Congruences which are contained in a Linear Complex. By Dr. T. Archer Hirst, F.R.S.

[*Read May 13th*, 1886.]

1. In his well-known memoir,* published in the *Monats Bericht* of the Academy of Berlin (17th January, 1878), Kummer drew attention to the existence of two different, and equally general, congruences of the third order and third class. One of these is contained in a linear complex; the other, which for distinction might be termed the skew cubic congruence, is such that the three rays thereof, proceeding from an arbitrary point in space, are not, in general, coplanar. The properties of the latter congruence were fully developed by Kummer; whilst those of the former were only very briefly alluded to by him.

2. A year ago, in a paper communicated to the London Mathematical Society, I had occasion to study a special case of the abovementioned skew cubic congruence.† It was of the Cremonian

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* *Über diejenigen Flächen, welche mit ihren reciprok polaren Flächen von derselben Ordnung sind und die gleichen Singularitäten besitzen.*

† *On Congruences of the Third Order and Class*, "Proceedings of the London Mathematical Society," Vol. xvi., pp. 232—38, 1885.

I may here mention that, in 1882, Dr. Roccella published, at Piazza Armerina, in Sicily, an interesting thesis entitled, *Sugli enti geometrici dello spazio di retti generati dalle intersezione dè complessi corrispondenti in due o più fasci proiettivi di complessi lineari*, in which, amongst other things, he speaks of a congruence of the third order and class, definable as *the locus of a right line constantly incident with three corresponding rays of three given projective pencils, arbitrarily situated in space.* This congruence, as I have recently shown, in a communication to the *Circolo Matematico di Palermo* ("Rendiconti," t. i., *seduta del* 21 *febrajo* 1886), is itself a special case of the one studied by me, and referred to in the text.

I am also informed by Prof. Sturm, of Münster, that he has been led, still more recently, and quite independently, to a somewhat similar, purely descriptive method of generating the congruence described in my paper of 1885. In place of one of the three projective pencils employed by Roccella, he simply substitutes a quadric regulus, one of whose generators coincides with its corresponding ray in one of the two remaining projective pencils.