PROPERTIES OF LOGARITHMICO-EXPONENTIAL FUNCTIONS

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I.

Introduction and Summary of Results.

1. The contents of this paper are designed as a supplement to those of my recently published pamphlet, "Orders of Infinity, &c." (*Cambridge Tracts in Mathematics*, No. 12, 1910). When writing this pamphlet I was compelled by considerations of space to condense the proof of one theorem at any rate almost to the point of obscurity, and to omit the proofs of others altogether. Apart from the new results contained in this paper, the interest of the theorems that I refer to seems to me sufficient to justify a return to the subject.

2. I shall use throughout the paper the system of notation explained in the first section of my tract,* which attaches a special sense to the symbols

$$\succ,\quad \prec,\quad \asymp,\quad \stackrel{\asymp}{=},\quad \sim,\quad \succcurlyeq,\quad \preccurlyeq,$$

$$O,\quad K,\quad \delta,\quad \Delta,\quad x_0,\quad \epsilon.$$

To the conventions there explained I propose to add two others.

(i) I shall frequently employ formulæ containing constants restricted to be *rational*. Such constants I shall denote by small italic letters s, t, ... When the constant is not restricted to be rational, I shall use large italic letters A, B, ... Thus Ax^s is an algebraic function, but x^A is in general transcendental.

(ii) I shall frequently (when no confusion is likely to arise from the practice) repeat a letter or letters without implying that the numbers denoted by them are the same as when the letters were previously used. For example, I shall write

$$\int^x At^s\,dt = Ax^s,$$

* "Orders of Infinity," pp. 2–7. I shall refer to this tract henceforth simply as "O.I."

meaning thereby, "if f is a constant multiple of a rational power of x, then its integral is a function of the same kind." I might add, "unless $s=-1$"; it would, of course, be evident that this clause referred to the s on the left-hand side of the equation.

3. Let $A(x)$ denote generally a real one-valued algebraical function of x—it will, of course, be a branch of an algebraical function of a complex variable, which we suppose to be real for $x > x_0$ (*i.e.*, for all real values of x from some definite value onwards). Thus $A(x)$ might be

$$x,\quad \sqrt{x},\quad -\sqrt{x},\quad Ax^s,\quad \sqrt{(x+1)}-\sqrt[3]{(x+1)},$$

or be defined as an implicit function by an equation which, like

$$y^5-y-x=0,$$

has one root which has the property stated.

Then in my tract I defined a *logarithmico-exponential function* (shortly, an *L-function*) substantially as follows:—

An L-function is a real one-valued function defined by a finite combination of the functional symbols

$$A(\ldots),\quad \log(\ldots),\quad e^{(\ldots)}$$

*operating on the variable x and on real constants.**

It is, of course, to be understood that, if necessary, we confine our attention to values of x greater than some definite value x_0.

4. I also classified L-functions according to *orders*, by a method due in principle to Liouville.† Thus

$$e^x,\quad \log x,\quad \frac{e^x+\sqrt{(\log x)}}{\sqrt{(x+1)}-(\log x)^2}$$

are of order 1;

$$e^{e^x},\quad \log\log x,\quad x^x=e^{x\log x},\quad x^{\sqrt{2}}=e^{\sqrt{2}\log x},$$

of order 2; and so on.‡ It should be observed that it is not obvious that

* "O.I.," p. 17. I there confined myself to the case in which the algebraical functions are explicit; but I pointed out that no additional difficulties arose from considering the more general case. I also pointed out (p. 18) that "the result of working out the value of the function, by substituting the value of x in the formula defining it, is to be real at all stages of the work."

† *Journal de Mathématiques* (1), t. 2, pp. 71 *et seq.*

‡ "O.I.," p. 18.

(*e.g.*) $\log\log x$ is really of order 2. It is obviously of order *not greater than* 2. That it cannot be expressed as an L-function of lower order demands a proof. Such a proof was, in fact, given by Liouville*; another proof follows from results which will be established in this paper.

The following additional definitions will be found useful.

We shall say that f_n, an L-function of order n, is *integral* if it is of the form†

$$\Sigma \rho_{n-1} e\sigma_{n-1} (l\tau^{(1)}_{n-1})^{\kappa_1} (l\tau^{(2)}_{n-1})^{\kappa_2} \dots (l\tau^{(h)}_{n-1})^{\kappa_h},$$

where the functions with suffix $n-1$ are L-functions of order $n-1$ and $\kappa_1, \kappa_2, \dots, \kappa_h$ are positive integers. We call

$$\kappa_1+\kappa_2+\dots+\kappa_h$$

the *logarithmic degree*, or, simply, the *degree*, of the typical term of f_n; if λ is the greatest value of $\kappa_1+\kappa_2+\dots+\kappa_h$, we say that f_n is *of logarithmic degree* λ. If the number of terms of degree λ in f_n is μ, we say that f_n is *of logarithmic type* (λ, μ).‡ We shall in general denote integral L-functions by the letter M, with or without suffixes, indices, &c.

If an integral L-function is of degree 0, *i.e.*, of the form

$$\Sigma \rho_{n-1} e\sigma_{n-1},$$

we shall say that it is *exponential*.

If an integral exponential L-function contains ϖ terms, we shall say that it is *of type* ϖ; if $\varpi = 1$, we shall say that it is *simply exponential*. Thus $(lx)^2 e^{e^x lx}$ is a simply exponential L-function of order 2, while

$$(lx)^2 (l_2 x)^2 e^{e^x lx}$$

is an integral function of order 2, of type (2, 1). We shall in general denote integral exponential L-functions by the letter N.

If f_n is the quotient of two integral functions, *i.e.*, of the form M_1/M_2, we shall say that it is *rational*.

If M_1 and M_2 are exponential, *i.e.*, if f_n is of the form N_1/N_2, we shall describe f_n as a *rational exponential* L-function.

* *L.c.*, pp. 99 *et seq.*

† I write lx, l_2x, ..., ex, e_2x, ... for $\log x$, $\log\log x$, ..., e^x, e^{e^x}, ..., as in my tract.

‡ A function of type (λ, μ) may be immediately reducible to a lower type. Thus

$$l_2 x + l_2(1+x) = l\{lx\, l(1+x)\}.$$

The left-hand side is of type (1, 2), the right-hand side of type (1, 1). In some kinds of argument it might be essential so to frame our definitions as to be free from such ambiguities; but they in no way affect the arguments of this paper. See further, § 6.

5. We shall constantly be making use of the following facts :—

(i) The derivative of a simply exponential function is a simply exponential function, with the same exponential factor.

(ii) The derivative of an integral exponential function of type ϖ is an integral exponential function of type ϖ. If one of the terms of the original function is a constant, the derivative is of type $\varpi-1$.

(iii) The derivative of an integral function of logarithmic type (λ, μ) is in general a function of the same type. If the exponential factor $\rho_{n-1}e\sigma_{n-1}$ of one of the terms of degree λ is a constant, then the derivative is of type $(\lambda, \mu-1)$; if $\mu = 1$, the derivative is of degree $\lambda-1$.

(iv) In general, the derivative of an L-function of order n is an L-function of order n. In exceptional cases the derivative may be of order $n-1$.

These facts are all immediate consequences of the formal rules of the differential calculus. The only one which possibly requires a word of additional explanation is (iii). The derivative of

$$(l\tau_{n-1}^{(1)})^{\kappa_1}(l\tau_{n-1}^{(2)})^{\kappa_2}\dots(l\tau_{n-1}^{(h)})^{\kappa_h},$$

where $\kappa_1+\kappa_2+\dots+\kappa_h = \lambda$, plainly consists of h terms each of degree $\lambda-1$ only. When we differentiate

$$\rho_{n-1}e\sigma_{n-1}(l\tau_{n-1}^{(1)})^{\kappa_1}\dots(l\tau_{n-1}^{(h)})^{\kappa_h},$$

we obtain one term of degree λ (by differentiating the exponential factor) and h of degree $\lambda-1$.

6. Before proceeding further, I wish to make a few remarks about the relation between my results and those of Liouville. The subject-matter appears very much the same, but the methods used and the results obtained are entirely dissimilar.

I regard an L-function essentially as the embodiment of an "order of infinity," as expressing a certain rate of increase or decrease or of approach to a limit; and for this reason I consider only functions of a real variable which are real and one-valued and (as I shall show) ultimately monotonic, excluding altogether oscillating functions such as $\sin x$. These ideas do not appear in Liouville's work at all. He was interested solely in problems of functional form: $\sin x$ was for him exactly on the same footing as $\log x$ or e^x. From his point of view, each of my orders of

L-functions would present itself simply as a sub-class of his corresponding order of "elementary transcendents."* But his classes do not, as classes, possess any of the properties with which I am concerned in this paper; and so, even when we seem to be engaged by very similar questions, our results are in reality widely divergent.

Thus Liouville proves that $\log\log x$ and $x^{\sqrt{2}}$ cannot be expressed as elementary transcendents of order one. I prove that the same functions cannot be expressed as L-functions of order 1; and in so doing I do indeed prove a part of what Liouville proved. But the substance of what I prove is something quite different from any question raised by Liouville at all, viz., that *the rates of increase specified by* $\log\log x$ and $x^{\sqrt{2}}$ cannot belong to any L-function of the first order.

It is then only to be expected that the processes of argument used by Liouville should be entirely unlike any of mine—how different a single consideration is enough to show. Liouville considers, as I do, functions of order n built up by means of the simple functions

$$e\sigma_{n-1}, \qquad l\tau_{n-1}$$

(see § 4 above). In all his arguments it is absolutely vital to suppose these simple functions genuinely independent and reduced to the smallest possible number—that is to say, to suppose that no algebraical relation connects them with one another and transcendents of lower order. This assumption plays no part in any of my arguments: it will, in point of fact, usually be satisfied, but it is in no way essential.

7. It is easy to see that any L-function (or any elementary transcendent in Liouville's sense) is a solution of an algebraic differential equation, which may without loss of generality be supposed to be of the form

$$P(x, y, y', \ldots, y^{(k)}) = 0,$$

where P is a polynomial. And in the case of an L-function (more generally, a Liouville's transcendent) of order n, this equation cannot be of order less than n. The lines of a proof have been indicated by Königsberger,† who has also established some simple results as to the forms of transcendents that can satisfy equations of specified order.

* For a short account of his classification, see my tract, "The Integration of Functions of a Single Variable" (*Cambridge Tracts*, No. 2, 1905).

† "Bemerkungen zur Liouville's Classificirung der Transcendenten" (*Math. Annalen*, Bd. 28, S. 483).

The classification of transcendents adopted by Liouville, however, does not run parallel with any obvious classification of algebraic differential equations. Thus e^{e^x} satisfies an equation of the second order, viz.,

$$yy'' = (y+y')y'.$$

So do the functions of the first order

$$y = e^x + \log x, \qquad y = e^x + e^{\frac{1}{2}x^2},$$

the equations being

$$x^2(y''-y')+x+1 = 0, \qquad (x-1)y''-x^2y'+(x^2-x+1)y = 0.$$

Generally, an integral exponential L-function, of order 1 and type ϖ, satisfies a linear and homogeneous equation of order ϖ, whose coefficients are algebraic functions of x; and other particular remarks of the same nature may be made. And it is possible to obtain a certain number of more or less general results concerning the possible modes of increase of the solutions of algebraic differential equations of specified types: I shall make a few further observations on this point at the end of this paper. But it is clearly impracticable to base the fundamental properties of L-functions upon the nature of the differential equations satisfied by them. As soon as we adopt the standpoint of the differential equation, the properties of L-functions are lost in those of larger and vaguer aggregates of functions.

It would be unjustifiable to conclude from this (as Königsberger appears to have done) that Liouville's classification is in some sense illegitimate or trivial or uninteresting. It must not be forgotten that the particular is often more interesting than the general: and, in my opinion, the main interest of Liouville's classification lies in its application to two special problems—indefinite integration in finite terms on the one hand, orders of infinity on the other.

8. It will probably be convenient if I give a rapid summary of the results which I propose to prove. Some of the proofs, I am afraid, are long and rather tedious: that this should be so is, I think, inevitable from the nature of the subject. The results, too, are of the class that seem more obvious than they are.

In Section II (§§ 9–11) I prove that every L-function is continuous and monotonic from a certain value of x onwards. This theorem and its corollaries form the basis of all the subsequent work.*

* A proof of this theorem (but of no other proved in this paper) is contained in my tract ("O.I.," pp. 18–20). The proof given here is not different in principle, but has been remodelled in such a way that I hope it will be found simpler and clearer.

In Section III (§§ 12–16) I discuss the limits of rapidity of the increase of an L-function of given order. I state in my tract,* but without proof, that an L-function of order n cannot increase more rapidly than $e_n(x^{\Delta})$ or (if it tends to infinity at all) more slowly than $(l_n x)^{\delta}$. I now prove more precise results of which these are corollaries.

In Section IV (§§ 17–19) I apply the results of the preceding section to determine generally the order of the integral of a given L-function.

In Section V (§§ 20–22) I consider systems of standard forms for the increase of L-functions of given order. I give a full investigation in the case of $n = 1$, which shows incidentally that such modes of increase as are given by l_2x, e_2x, x^{ν}, $x^{\sqrt{2}}$, ... are impossible for functions of order 1. I then state the corresponding results for $n = 2$ and for higher orders; but I have not written out a detailed proof of these results. It would be long and tedious, and the nature of the arguments employed will be clear from the simpler discussions which precede.

In Section VI (§§ 23–26) I discuss shortly the construction of functions whose rate of increase corresponds to a gap in the logarithmico-exponential scales of infinity, and in Section VII (§§ 27–28) various topics of a miscellaneous character.

II.

Proof of the Fundamental Theorem.

9. THEOREM 1.—*Any L-function is ultimately continuous, of constant sign, and monotonic; and tends, as $x \to \infty$, to ∞, or to zero or some other definite limit. Further, if f and ϕ are L-functions, one or other of the relations*

$$f \succ \phi, \quad f \asymp \phi, \quad f \prec \phi,$$

holds between them.

Two preliminary remarks will be useful.

(1) If f and ϕ are L-functions, f/ϕ is an L-function, whose order is not greater than the greater of the orders of f and ϕ. Thus the second part of the theorem is a mere corollary of the first part; for it follows from the first part that f/ϕ must tend to infinity, or to zero or some other limit.

(2) The derivative of an L-function is an L-function of not higher order [§ 5 (iv)]. From this it follows that if all L-functions are ulti-

* P. 20.

mately continuous and of constant sign they are all also ultimately monotonic; for their derivatives are also ultimately of constant sign.

10. The results of the theorem are certainly true of functions of zero order, that is to say, of purely algebraical L-functions. Any such function can, in fact, be expressed in the form

$$\Sigma x^{a_r} P_r(x^{-b_r}),$$

where a_r and b_r are rational, and P_r is a power series convergent for sufficiently large values of x. It is therefore sufficient to prove that, if the results of Theorem 1 are true of functions of order $n-1$, then they are true also of functions of order n.

(1) *The results of the theorem are true of any simply exponential function of order n.*

It is, in fact, obvious that

$$f_n = \rho_{n-1} e\sigma_{n-1}$$

is ultimately continuous and of constant sign; and the same is true of its derivative

$$f_n' = (\rho_{n-1}' + \rho_{n-1}\sigma_{n-1}') e\sigma_{n-1}.$$

(2) *The results are true of any integral exponential function of order n.*

This has just been proved when the function is of type 1 (§ 4). Let us then assume it true for functions of type $\varpi-1$: and let

$$f_n = \Sigma \rho_{n-1} e\sigma_{n-1}$$

be of type ϖ.

If $\rho_{n-1} e\sigma_{n-1}$ is any one of the terms of f_n, the function

$$\bar{f}_n = f_n/(\rho_{n-1} e\sigma_{n-1})$$

is of type ϖ, with one term a constant (unity). And so $\bar{f}_n'$ is of type $\varpi-1$ [§ 5 (ii)]. Hence $\bar{f}_n'$ is ultimately continuous and of constant sign: and so the same is true of $\bar{f}_n$, and therefore of f_n. Finally, f_n' is of type ϖ, and so (after what has just been proved) ultimately of constant sign; and so f_n is ultimately monotonic.

(3) *The results are true of any integral function of order n.*

Suppose that

$$f_n = \Sigma \rho_{n-1} e\sigma_{n-1} (l\tau_{n-1}^{(1)})^{\kappa_1} (l\tau_{n-1}^{(2)})^{\kappa_2} \dots (l\tau_{n-1}^{(h)})^{\kappa_h}$$

is of logarithmic type (λ, μ). The results have been proved true when $\lambda = 0$. Hence it is enough to prove

(i) that, if true for functions of logarithmic degree $\lambda-1$, they are true for functions of degree λ and type $(\lambda, 1)$;

(ii) that, if true for functions of type $(\lambda, \mu-1)$, they are true for functions of type (λ, μ).

Suppose that the typical term written above in the expression of f_n is one of the terms of degree λ, and let $\bar{f}_n = f_n/(\rho_{n-1} e\sigma_{n-1})$ as before. Then $\bar{f}_n'$ is of type $(\lambda, \mu-1)$, unless $\mu = 1$, when it is of degree $\lambda-1$ [§ 5 (iii)]. Hence, whichever of the inductions (i), (ii) we are engaged in proving, $\bar{f}_n'$ is ultimately continuous and of constant sign; and we deduce as before that f_n is ultimately continuous, of constant sign, and monotonic.

(4) We are now in a position to complete the proof of the theorem. Any L-function f_n is of the form

$$\begin{aligned} f_n &= A\{e\phi_{n-1}^{(1)}, e\phi_{n-1}^{(2)}, \ldots, e\phi_{n-1}^{(r)}, l\psi_{n-1}^{(1)}, \ldots, l\psi_{n-1}^{(s)}, \chi_{n-1}^{(1)}, \ldots, \chi_{n-1}^{(t)}\} \\ &= A(z_1, z_2, \ldots, z_q), \end{aligned}$$

say, where $$q = r+s+t.$$

There is therefore an identical relation

$$F(x, y) \equiv M_0 y^p + M_1 y^{p-1} + \ldots + M_p = 0,$$

where $y = f_n$, and the coefficients $M_1, M_2, \ldots, M_p$ are integral L-functions of order n.

The derivatives of these coefficients are also integral. It therefore follows from what has already been proved that

$$\frac{\partial F}{\partial x} = \sum_{(i)} \frac{dM_i}{dx} y^{p-i}, \qquad \frac{\partial F}{\partial y} = \sum_{(i)} (p-i) M_i y^{p-i-1},$$

considered as functions of the two variables x, y, are continuous for all sufficiently large values of x and for all values of y.

Let ξ, η be a pair of values of x and y satisfying the equation $F = 0$. Then, if only ξ is large enough, $\partial F/\partial y$ cannot vanish for $x = \xi$, $y = \eta$. For, if F and $\partial F/\partial y$ both vanish for $x = \xi$, $y = \eta$, then the eliminant of y between $F = 0$ and $\partial F/\partial y = 0$ vanishes for $x = \xi$. But this eliminant is plainly an integral L-function of order n, and so cannot vanish for values of x surpassing all limit.

Now there is a well known theorem which asserts that* if F is a

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* Goursat, *Cours d'Analyse*, t. i, p. 40; Young, *Proc. London Math. Soc.*, Ser. 2, Vol. 7, pp. 397 *et seq.*

function of x and y which vanishes for $x = \xi$, $y = \eta$, and has derivatives $\partial F/\partial x$, $\partial F/\partial y$ continuous in a region including (ξ, η), and if $\partial F/\partial y$ does not vanish for $x = \xi$, $y = \eta$, then there is a unique continuous and differentiable function y of x, which is equal to η when x is equal to ξ, and satisfies the equation $F(x, y) = 0$ identically. Hence f_n is an ultimately continuous function of x.

Moreover f_n is ultimately of constant sign. For $f_n = 0$ involves $M_p = 0$, and we have already seen that it is impossible that this equation should be satisfied for values of x surpassing all limit.

Finally, f_n' is a function of order n. Hence, applying the same conclusions to f_n', we see that f_n is ultimately monotonic. The proof of the theorem is thus completed.*

11. COROLLARY.—*If f, ϕ, and F are L-functions, then*

$$F(f)/F(\phi), \qquad F(f)-F(\phi)$$

tend to infinity or to some definite limit.

This requires no further proof. It is of some interest in connection with Pincherle's extensions of some of Du Bois-Reymond's work.†

III.

The Limits of the Increase of an L-Function.

12. I also stated, but without proof, the following theorems‡ :—

THEOREM 2.—*An L-function of order n cannot satisfy*

$$f_n \succ e_n(x^{\Delta}).$$

THEOREM 3.—*An L-function of order n cannot satisfy*

$$1 \prec f_n \prec (l_n x)^{\delta},$$

or

$$(l_n x)^{-\delta} \prec f_n \prec 1.$$

Theorem 2 is not only obvious to the eye of common sense, but very easy to prove. Assume the result true for functions of order $n-1$: it is, of course, true for functions of zero order.

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* I should mention that the idea of attempting to construct a formal proof of this theorem was suggested to me some years ago by Mr. V. Ramaswami Aiyar.

† "O.I.," pp. 13 *et seq.*

‡ "O.I.," p. 20.

Any function of order n is an algebraical function of certain arguments $e\phi_{n-1}, \ldots, l\psi_{n-1}, \ldots, \chi_{n-1}, \ldots$, the increase of any one of which is *ex hypothesi* less than that of

$$e(e_{n-1}x^{\Delta}) = e_n(x^{\Delta})$$

for some value of Δ. Hence the increase of the function is less than that of

$$(e_n x^{\Delta})^{\Delta_1}$$

for some values of Δ and Δ_1; and so less than that of $e_n(x^{\Delta_2})$ for some value of Δ_2. Thus the theorem is established.

Theorem 3 appears to be much harder to prove. It is indeed natural to suppose that it ought to be deducible as a corollary of Theorem 2. But for such a deduction we appear to need some such theorem as the following:*—

If f_n is an L-function of order n, tending to infinity with x, and $\bar{f}$ is the function inverse to f_n, then there is an L-function ϕ_n, of degree n at most, such that

$$\bar{f} \sim \phi_n.$$

If this is true, Theorem 3 may be at once deduced from Theorem 2. For the inverse of $(l_n x)^{\delta}$ is $e_n(x^{1/\delta})$; and so, if

$$1 \prec f_n \prec (l_n x)^{\delta},$$

it follows that $\bar{f} \succcurlyeq e_n(x^{\Delta})$ for sufficiently large values of x,† and so $\phi_n \succ e_n(x^{\Delta})$, which is impossible. I have, however, not been able to prove the result assumed in this deduction, and I am not inclined to commit myself to a definite opinion as regards the probability of its being true.‡

13. It therefore appears to be necessary to proceed to the proof of Theorem 3 by a different road. It is convenient to recall one or two theorems from my tract which will be needed in the proof.

THEOREM 4.—*The relations $f \succ \phi$, $f \asymp \phi$, $f \prec \phi$ involve the corresponding relations*

$$\int_a^x f(t)\,dt \;\substack{\succ \\ \asymp \\ \prec}\; \int_a^x \phi(t)\,dt,$$

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* A less precise result might be sufficient. The possibility of the truth of the theorem was suggested to me by Mr. Littlewood.

† "O.I.," p. 16.

‡ I shall return to this point in Section VII.

if either of the integrals $\int^{\infty} f dx$, $\int^{\infty} \phi dx$ *is divergent. If both integrals are convergent, then*

$$\int_x^{\infty} f(t)\,dt \; \begin{matrix} \succ \\ \asymp \\ \prec \end{matrix} \; \int_x^{\infty} \phi(t)\,dt.$$

Here f and ϕ are supposed to be positive, continuous and monotonic—there is, however, no difficulty in interpreting the results so as to apply them to negative functions.* The theorem may be stated briefly thus: *relations of the type* $f \succ \phi$, ..., *may be integrated.*

THEOREM 5.—*The relation* $f \succ \phi$ *involves* $f' \succ \phi'$, *unless* $f \asymp 1$; $f \asymp \phi$ *involves* $f' \asymp \phi'$, *unless* $f \asymp \phi \asymp 1$; $f \prec \phi$ *involves* $f' \prec \phi'$, *unless* $\phi \asymp 1$; *provided always that it is known that one of the relations* $f' \succ \phi'$, $f' \asymp \phi'$, $f' \prec \phi'$ MUST *hold between* f' *and* ϕ'.

The exceptional cases mentioned in the enunciation are in reality of a trivial character, and are due to the fact that, if $f \asymp 1$, then f, when regarded as the integral of f', is dominated by a constant of integration. Thus, if $f = 1 + e^{-x}$, $\phi = 1/x$, then $f \succ \phi$, but $f' \prec \phi'$.

Theorem 5 may be stated thus: *relations of the type* $f \succ \phi$, ... *may* (except in certain special cases) *be differentiated, provided we are assured a priori that some relation of this type must hold between the derivatives.*

When f and ϕ are L-functions, so are their derivatives: thus we obtain:

THEOREM 6.—*Relations of the type* $f \succ \phi$, ..., *holding between* L-*functions, may be integrated and differentiated, subject to certain restrictions relating to the cases in which* $f \asymp 1$ *or* $\phi \asymp 1$.

14. I shall now establish Theorem 3 as a corollary of a more precise theorem.

THEOREM 7.—*If* f_n *is an* L-*function of order* n, *and*

$$(l_{n-1}x)^{-\delta} \prec f \prec (l_{n-1}x)^{\delta},$$

then

$$f_n \asymp (l_n x)^s,$$

where s *is a rational number. In particular, if* f_n *is rational,* s *must be an integer; and if* f_n *is integral, a positive integer; and if* f_n *is an integral or rational exponential function, then* s *must be zero.*

* "O.I.," pp. 36 *et seq.*; this reference applies also to Theorems 5 and 6.

I shall base my proof upon two lemmas:

LEMMA A.—*If* $$(l_m x)^{-\delta} \prec f \prec (l_m x)^{\delta},$$
and it is not true that $f \asymp 1$, *then*
$$(l_m x)^{-\delta} \prec x\, lx \ldots l_m x \,.\, f' \prec (l_m x)^{\delta}.$$

The truth of this follows immediately upon differentiation.

LEMMA B.—*If* $$(l_m x)^{-\delta} \prec f/\phi \prec (l_m x)^{\delta},$$
while, for some positive γ,
$$\phi \prec (l_m x)^{-\gamma} \quad or \quad \phi \succ (l_m x)^{\gamma},$$
then $$(l_m x)^{-\delta} \prec f'/\phi' \prec (l_m x)^{\delta}.$$

For

(1) $$\phi (l_m x)^{-\delta} \prec f \prec \phi (l_m x)^{\delta}.$$

It is evident from the conditions that no one of the relations
$$\phi (l_m x)^{-\delta} \asymp 1, \quad f \asymp 1, \quad \phi (l_m x)^{\delta} \asymp 1$$
is possible, if δ is small enough.

Hence (Theorem 5) we may differentiate, and so we obtain

(2) $$\frac{d}{dx}\{\phi (l_m x)^{-\delta}\} \prec f' \prec \frac{d}{dx}\{\phi (l_m x)^{\delta}\}.$$

Now $$\frac{d}{dx}\{\phi (l_m x)^{-\delta}\} = \phi (l_m x)^{-\delta}\left(\frac{\phi'}{\phi} - \frac{\delta}{x\, lx \ldots l_m x}\right)$$

But $$l\phi \succcurlyeq \gamma l_{m+1} x,$$

and so $$\frac{\phi'}{\phi} \succcurlyeq \frac{\gamma}{x\, lx \ldots l_m x}.$$

When γ is fixed we can find K and x_0, so that
$$\left|\frac{\phi'}{\phi}\right| > \frac{K}{x\, lx \ldots l_m x} \; (x \geqslant x_0)$$

Thus, if $\delta < \frac{1}{2}K$, the ratio of the functions
$$\frac{\phi'}{\phi} - \frac{\delta}{x\, lx \ldots l_m x}, \quad \frac{\phi'}{\phi}$$

lies, for $x > x_0$, between $\frac{1}{2}$ and $\frac{3}{2}$. As they are L-functions, we must have

$$\frac{\phi'}{\phi} - \frac{\delta}{x\,lx\,\ldots\,l_m x} \succeq \frac{\phi'}{\phi}.$$

Thus

$$\frac{d}{dx}\{\phi\,(l_m x)^{-\delta}\} \succeq \phi'\,(l_m x)^{-\delta};$$

and a similar result holds when $-\delta$ is replaced by δ. The truth of the lemma now follows immediately from (2).

15. We shall now assume Theorem 7 to be true for functions of order $n-1$, and prove it true for functions of order n. As it is plainly true for $n=0$ (when $l_{n-1}x$ is e^x and $l_n x$ is x), it will follow that it is true generally.

(1) *If f_n is a simply exponential function, and*

$$(l_{n-1}x)^{-\delta} \prec f_n \prec (l_{n-1}x)^{\delta},$$

then $f_n \succeq 1$.

Suppose, *e.g.*, that $f_n \succ 1$. Then

$$1 \prec lf_n \prec l_n x,$$

and so

$$\frac{1}{x\,lx\,\ldots\,l_{n-2}x\,(l_{n-1}x)^{1+\delta}} \prec \frac{f_n'}{f_n} \prec \frac{1}{x\,lx\,\ldots\,l_{n-1}x};$$

the second relation following from the differentiation of $lf_n \prec l_n x$, the first being a consequence of the fact that

$$\int^{\infty} \frac{dx}{x\,lx\,\ldots\,l_{n-2}x\,(l_{n-1}x)^{1+\delta}}$$

is convergent.

Hence, if $f_n = \rho_{n-1} e\sigma_{n-1}$, we obtain

$$(l_{n-1}x)^{-\delta} \prec (x\,lx\,\ldots\,l_{n-1}x)\left(\frac{\rho_{n-1}'}{\rho_{n-1}} + \sigma_{n-1}'\right) \prec 1,$$

which is impossible, as the function in the middle is of order $n-1$.

(2) *The same result holds for any integral exponential function.*

Suppose that

$$f_n = \Sigma \rho_{n-1} e\sigma_{n-1}$$

is of type ϖ, and that the result has been proved for functions of type $\varpi-1$.

Two cases are conceivable. Either *all* the terms of f_n satisfy

$$\rho_{n-1} e\sigma_{n-1} \succeq 1,$$

or there is at least one which does not. We consider the latter case first.

(i) Let
$$\rho_{n-1} e\sigma_{n-1} = 1/\phi$$
be one of the terms which does not. Then, by what has already been proved,
$$\phi \prec (l_{n-1}x)^{-\gamma} \quad or \quad \phi \succ (l_{n-1}x)^{\gamma}$$
for some positive γ.

Let $f_n = \bar{f}_n/\phi$, so that
$$(l_{n-1}x)^{-\delta} \prec \bar{f}_n/\phi \prec (l_{n-1}x)^{\delta}.$$
The conditions of Lemma B are satisfied, and so
$$(l_{n-1}x)^{-\delta} \prec \bar{f}_n'/\phi' \prec (l_{n-1}x)^{\delta}.$$
But $\bar{f}_n'/\phi'$ is clearly of type $\varpi - 1$. Hence these last relations are only possible if
$$\bar{f}_n' \asymp \phi',$$
i.e., if $\bar{f}_n \asymp \phi$ or $f_n \asymp 1$.

(ii) If every $\rho_{n-1} e\sigma_{n-1}$ satisfies $\rho_{n-1} e\sigma_{n-1} \asymp 1$, we define ϕ as above by means of any one of them.

In this case it is plain that $f_n \preccurlyeq 1$.

Now
$$(l_{n-1}x)^{-\delta} \prec \bar{f}_n \prec (l_{n-1}x)^{\delta},$$
and so, by Lemma A,
$$(l_{n-1}x)^{-\delta} \prec x\, lx \ldots l_{n-1}x \,.\, \bar{f}_n' \prec (l_{n-1}x)^{\delta},$$
unless $\bar{f}_n \asymp 1$ or $f_n \asymp 1$. But $x\, lx \ldots l_{n-1}x \,.\, \bar{f}_n'$ is of type $\varpi - 1$. Hence we must have
$$x\, lx \ldots l_{n-1}x \,.\, \bar{f}_n' \asymp 1,$$
which involves
$$f_n \asymp \bar{f}_n \asymp l_n x,$$
and this is impossible, since $f_n \preccurlyeq 1$. Thus we are driven back on the conclusion that $f_n \asymp 1$.

(3) *If f_n is any integral function, and*
$$(l_{n-1}x)^{-\delta} \prec f_n \prec (l_{n-1}x)^{\delta},$$
then
$$f_n \asymp (l_n x)^q,$$
where q is a positive integer.

We shall prove this by establishing, as in § 10 (3), inductions from functions of degree $\lambda-1$ to functions of degree λ and type $(\lambda, 1)$, and from functions of type $(\lambda, \mu-1)$ to functions of type (λ, μ).

Suppose, then, that f_n is of type (λ, μ), and consider the factors $\rho_{n-1} e\sigma_{n-1}$ of the μ terms in f_n of degree λ. We must distinguish two cases exactly as under (2) above.

(i) Suppose that among these μ factors there is one of which it is not true that

$$\rho_{n-1} e\sigma_{n-1} = 1/\phi \succeq\!\!\preceq 1,$$

and let $f_n = \bar{f}_n/\phi$. Arguing precisely as above, we find

$$(l_{n-1}x)^{-\delta} \prec \bar{f}_n'/\phi' \prec (l_{n-1}x)^{\delta}.$$

The function in the middle is of type $(\lambda, \mu-1)$, or if $\mu = 1$, of degree $\lambda-1$. Hence we must have

$$\bar{f}_n' \succeq\!\!\preceq (l_n x)^q \phi',$$

where q is a positive integer. Now

$$\phi \prec (l_{n-1}x)^{-\gamma} \quad or \quad \phi \succ (l_{n-1}x)^{\gamma},$$

so that

$$l\phi \succcurlyeq \gamma l_n x,$$

$$\frac{\phi'}{\phi} \succcurlyeq \frac{1}{x\, lx \dots l_{n-1}x} \succ \frac{1}{x\, lx \dots l_n x}.$$

Thus

$$\frac{\phi'}{\phi} \succ \frac{d}{dx} \log \{(l_n x)^q\},$$

or

$$(l_n x)^q \phi' \sim \frac{d}{dx} \{(l_n x)^q \phi\}.$$

Hence

$$\bar{f}_n' \succeq\!\!\preceq \frac{d}{dx} \{(l_n x)^q \phi\},$$

$$\bar{f}_n \succeq\!\!\preceq (l_n x)^q \phi,$$

$$f_n = \bar{f}_n/\phi \succeq\!\!\preceq (l_n x)^q.$$

Thus in this case the required induction is established.

(ii) Suppose that all the μ functions $\rho_{n-1} e\sigma_{n-1}$ satisfy $\rho_{n-1} e\sigma_{n-1} \succeq\!\!\preceq 1$. Then

$$(l_{n-1}x)^{-\delta} \prec \bar{f}_n \prec (l_{n-1}x)^{\delta},$$

and so, by Lemma A,

$$(l_{n-1}x)^{-\delta} \prec x\, lx \dots l_{n-1}x \,.\, \bar{f}_n' \prec (l_{n-1}x)^{\delta},$$

unless $\bar{f}_n \asymp 1$ or $f_n \asymp 1$. Apart from this case we must have

$$x\, lx \ldots l_{n-1}x \,.\, \bar{f}_n' \asymp (l_n x)^q;$$

and, dividing by the logarithmic factors on the left and integrating, we obtain

$$f_n \asymp \bar{f}_n \asymp (l_n x)^{q+1}.$$

Thus again our induction is established, and the proof of (3) is completed.

(4) *If* f_n *is any rational function, and*

$$(l_{n-1}x)^{-\delta} \prec f_n \prec (l_{n-1}x)^{\delta},$$

then $$f_n \asymp (l_n x)^q,$$

where q *is a positive* (*or negative*) *integer.*

Suppose that $$f_n = M_1/M_2,$$

where M_1 and M_2 are integral. When M_2 is simply exponential, the proposition reduces to one already proved. We shall prove first that it is true when M_2 is any integral exponential function. In these cases q may be restricted to be positive.

Let $\rho_{n-1} e\sigma_{n-1}$ be one of the ϖ terms of M_2; let

$$\rho_{n-1} e\sigma_{n-1} = 1/\phi, \quad M_1 = \bar{M}_1/\phi, \quad M_2 = \bar{M}_2/\phi;$$

and let us assume that the proposition has been proved when M_2 is of type $\varpi - 1$. Then

(1) $$(l_{n-1}x)^{-\delta} \prec \bar{M}_1/\bar{M}_2 \prec (l_{n-1}x)^{\delta}.$$

If $$(l_{n-1}x)^{-\delta} \prec \bar{M}_2 \prec (l_{n-1}x)^{\delta},$$

we must, by what has already been proved, have

$$\bar{M}_2 \asymp (l_n x)^{q_2}$$

(q_2 a positive integer). In this case M_1 satisfies similar relations, and so

$$\bar{M}_1 \asymp (l_n x)^{q_1}, \quad f_n = \bar{M}_1/\bar{M}_2 \asymp (l_n x)^{q_1 - q_2},$$

which is what we want to prove. Thus this case may be dismissed, and we may assume that

$$\bar{M}_2 \prec (l_{n-1}x)^{-\gamma} \quad \text{or} \quad \bar{M}_2 \succ (l_{n-1}x)^{\gamma}.$$

Then it follows from (1), by Lemma B, that

(2) $$(l_{n-1}x)^{-\delta} \prec \bar{M}_1'/\bar{M}_2 \prec (l_{n-1}x)^{\delta}.$$

As $\bar{M}_2'$ is of type $\varpi - 1$, we must have

$$\bar{M}_1'/\bar{M}_2' \asymp (l_n x)^q,$$

$$\bar{M}_1' \asymp (l_n x)^q \bar{M}_2',$$

where q is a positive integer. It then follows, just as at the end of (3) (i), that

$$\bar{M}_1 \asymp (l_n x)^q \bar{M}_2 \quad or \quad f_n \asymp (l_n x)^q.$$

Thus the induction from $\varpi - 1$ to ϖ is established; and (4) is proved when M_2 is any integral exponential function, q being so far necessarily positive or zero.

We now suppose M_2 to be of logarithmic type (λ, μ), and we have to establish our customary inductions from degree $\lambda - 1$ to degree λ and type $(\lambda, 1)$, and from type $(\lambda, \mu - 1)$ to type (λ, μ).

Suppose, then, that the relations (1) are established. The assumption that $M_2 \asymp (l_n x)^{q_2}$ leads, precisely as before, to the conclusion that

$$f_n \asymp (l_n x)^{q_1 - q_2}.$$

Rejecting this hypothesis, and pursuing the same train of argument, we arrive at the relations (2). As $\bar{M}_2'$ is of type $(\lambda, \mu - 1)$, or, if $\mu = 1$, of degree $\lambda - 1$, we must have

$$\bar{M}_1'/\bar{M}_2' \asymp (l_n x)^q.$$

If $q = 0$, we have

$$\bar{M}_1' \asymp \bar{M}_2', \quad \bar{M}_1 \asymp \bar{M}_2, \quad f_n \asymp 1.$$

Suppose then $q > 0$, so that

$$\bar{M}_1' \asymp (l_n x)^q \bar{M}_2'.$$

If $$\bar{M}_2' \asymp (l_n x)^{q_2},$$

we have $$\bar{M}_1' \asymp (l_n x)^{q+q_2},$$

$$\bar{M}_1 \asymp x (l_n x)^{q+q_2}, \qquad \bar{M}_2 \asymp x (l_n x)^{q_2},*$$

$$f \asymp (l_n x)^q.$$

* It is easy to see that $\int (l_n x)^q \, dx \sim x (l_n x)^q$: see Section IV.

If it is not true that $\bar{M}_2' \asymp (l_n x)^{q_2}$,

we must have $\bar{M}_2' \prec (l_{n-1}x)^{-\gamma}$ *or* $\bar{M}_2' \succ (l_{n-1}x)^{\gamma}$,

and then we prove that $f_n \asymp (l_n x)^q$ precisely as at the end of (3) (i). The case in which q is negative may be treated similarly, and so the proof of (4) is completed.

Before proceeding further let us point out that it is easily proved (by reasoning of the same character) that, *if f_n is a rational exponential function, then q must be zero.*

16. We are now in a position to complete the proof of Theorem 7.

If f_n is any L-function of order n, there is [§ 10, (4)] an equation

$$M_0 f_n^p + M_1 f_n^{p-1} + \ldots + M_p = 0,$$

wherein the coefficients are integral L-functions of order n. If we denote by $X_0, X_1, \ldots, X_p$ the various terms of this equation, any pair X_i, X_j must satisfy one of the relations

$$X_i \succ X_j, \quad X_i \asymp X_j, \quad X_i \prec X_j$$

(Theorem 1). Moreover, as their sum is zero, there must be *at least one* pair such that

$$X_i \asymp X_j.$$

Thus $$f_n^{j-i} \asymp M_j/M_i.$$

Hence, if $$(l_{n-1}x)^{-\delta} \prec f_n \prec (l_{n-1}x)^{\delta},$$

we have also $$(l_{n-1}x)^{-\delta} \prec M_j/M_i \prec (l_{n-1}x)^{\delta},$$

and so $$M_j/M_i \asymp (l_n x)^q,$$

where q is an integer. Therefore

$$f_n \asymp (l_n x)^s,$$

where s is rational, being in fact an integral multiple of $1/(j-i)$.

It is possible to obtain, for the different cases which we have considered, more precise information as to the indices of the powers $(l_n x)^q$, $(l_n x)^s$ which figure in the theorem. To do so would however, lead us into a more elaborate analysis of the possible forms of L-functions of order n; and the theorem embodies all that we shall require in the sequel.

IV.

The Integration of L-Functions.

17. We are now in a position to establish a general system of rules which define the orders of greatness or smallness of the integrals

$$\int_a^x f_n(t)\,dt, \qquad \int_x^\infty f_n(t)\,dt.$$

We choose the first form of the integral or the second, of course, according as $\int^\infty f_n(x)\,dx$ is divergent or convergent. In either case we denote the integral by $F(x)$.

We observe first, that if f is an L-function, then either (i) $f \succ x^\Delta$ or (ii) $f \prec x^{-\Delta}$ or (iii) $f = x^a f_1$, where

$$x^{-\delta} \prec f_1 \prec x^\delta.*$$

THEOREM 8a.—*If $f \succ x^\Delta$ or $f \prec x^{-\Delta}$, then*

$$F \sim f^2/f'.$$

If $f = x^a f_1$, then $$F \sim \frac{x^{a+1}}{a+1} f_1,$$

unless $a = -1$, when further investigation is necessary.

(1) If $f \succ x^\Delta$ the integral is obviously divergent, and

$$\int_a^x f\,dt = \int_a^x f' \frac{f}{f'}\,dt = \frac{\{f(x)\}^2}{f'(x)} - \frac{\{f(a)\}^2}{f'(a)} - \int_a^x f \frac{d}{dt}\left(\frac{f}{f'}\right) dt.\dagger$$

Now, since $f \succ x^\Delta$, we have

$$\log f \succ \log x, \qquad f'/f \succ 1/x,$$

$$x \succ f/f', \qquad 1 \succ \frac{d}{dx}\left(\frac{f}{f'}\right)$$

$$\int_a^x f\,dt \succ \int_a^x f \frac{d}{dt}\left(\frac{f}{f'}\right) dt,$$

and so $$\int_a^x f\,dt \sim \frac{\{f(x)\}^2}{f'(x)}.$$

* "O.I.," p. 21.

† We suppose a chosen so that the functions are continuous for $x \geqslant a$.

The case in which $f \prec x^{-\Delta}$, when the integral is obviously convergent, may be settled in the same way.

(2) If $f = x^a f_1$, where $a > -1$, the integral is divergent. We have then

$$\int_a^x t^a f_1\, dt = \frac{x^{a+1}}{a+1} f_1(x) - \frac{a^{a+1}}{a+1} f_1(a) - \frac{1}{a+1} \int_a^x t^{a+1} f_1'\, dt.$$

But $x^{-\delta} \prec f_1 \prec x^{\delta}$, and so

$$\log f_1 \prec \log x, \qquad f_1'/f_1 \prec 1/x,$$

$$x^{a+1} f_1' \prec x^a f_1,$$

$$\int_a^x t^a f_1\, dt \sim \frac{x^{a+1}}{a+1} f_1(x).$$

The case in which $a < -1$, when the integral is convergent, may be treated similarly. But when $a = -1$ (when the integral may be convergent or divergent) further analysis is required.

18. Suppose now that $\qquad f = f_1/x \quad (x^{-\delta} \prec f_1 \prec x^{\delta}).$

Then either (i) $f_1 \succ (lx)^{\Delta}$ or (ii) $f_1 \prec (lx)^{-\Delta}$ or (iii) $f_1 = (lx)^{a_1} f_2$, where

$$(lx)^{-\delta} \prec f_2 \prec (lx)^{\delta}.$$

THEOREM 8*b*.—*If* $\qquad f = f_1/x \quad (x^{-\delta} \prec f_1 \prec x^{\delta}),$

then, if $f_1 \succ (lx)^{\Delta}$ *or* $f_1 \prec (lx)^{-\Delta}$, *we have*

$$F \sim f_1^2/x f_1'.$$

If $f_1 = (lx)^{a_1} f_2$, *where* $(lx)^{-\delta} \prec f_2 \prec (lx)^{\delta}$, *we have*

$$F \sim \frac{(lx)^{a_1+1}}{a_1+1} f_2,$$

unless $a_1 = -1$, *when further investigation is needed.*

We have, in fact (taking the integral to be divergent),

$$\int_a^x \frac{f_1}{t}\, dt = \int_{\log a}^{\log x} f_1(e^{\tau})\, d\tau = \int_{\log a}^{\log x} F_1(\tau)\, d\tau,$$

say. If, *e.g.*, $f_1 \succ (lx)^{\Delta}$, we have clearly $F_1 \succ \tau^{\Delta}$, and so

$$\int_a^x \frac{f_1}{t}\, dt \sim \frac{\{F_1(\log x)\}^2}{F_1'(\log x)} = \frac{f_1^2}{x f_1'}.$$

Similarly in the other cases.

It is evident that this process may be continued: thus we obtain

THEOREM 8c.—*If* $f = f_2/x\,lx$, $(lx)^{-\delta} \prec f_2 \prec (lx)^{\delta}$,

then if $$f_2 \succ (l_2x)^{\Delta} \quad or \quad f_2 \prec (l_2x)^{-\Delta},$$

we have $$F \sim \frac{f_2^2}{x\,lx\,f_2'}.$$

If $f_2 = (l_2x)^{a_2} f_3$, *where* $(l_2x)^{-\delta} \prec f_3 \prec (l_2x)^{\delta}$, *we have*

$$F \sim \frac{(l_2x)^{a_2+1}}{a_2+1} f_3;$$

unless $a_2 = -1$, *when further investigation is necessary. And so on generally.*

19. THEOREM 9.—*The various forms of Theorem* 8 $(a, b, c, \ldots)$ *apply to* ALL *L-functions.*

Consider first L-functions of order 1. The only case not settled by 8a is that in which
$$f = f_1/x,$$
where $$x^{-\delta} \prec f_1 \prec x^{\delta}.$$
By Theorem 7 we must then have
$$f_1 \sim A\,(lx)^{a_1}.$$
The only case not settled by Theorem 8b is that in which
$$f_1 \sim \frac{A}{lx}, \quad f \sim \frac{A}{x\,lx}, \quad F \sim A\,l_2x.$$
Similarly L-functions of order 2 are dealt with by 8a or 8b or 8c, unless
$$f_2 \sim \frac{A}{l_2x}, \quad f_1 \sim \frac{A}{lx\,l_2x}, \quad f \sim \frac{A}{x\,lx\,l_2x}, \quad F \sim A\,l_3x;$$
and so on generally.

V.

Standard Forms for the Increase of L-functions.

20. The preceding theorems enable us to establish systems of standard forms for the increase of L-functions of a given order.

THEOREM 10.—*Any L-function* f_1 *of the first order, ultimately positive,* *may be expressed in one or other of the forms*

$$e^{Ax^s(1+\epsilon)}, \qquad Ax^s(\log x)^t(1+\epsilon).*$$

If f_1 is rational, t must be an integer; if f_1 is integral, a positive integer; if f_1 is an exponential function (integral or rational), t must be zero.

(1) *The conclusions of Theorem* 10 *are valid when f_1 is a simply exponential function $\rho_0 e\sigma_0$.*

If $f_1 \asymp 1$, there is nothing to prove. If $f_1 \succ 1$ or $f_1 \prec 1$, then

$$\log f_1 \succ 1:$$

the only difference between the two cases lies in the fact that in the first case $\log f_1$ is ultimately positive and in the second ultimately negative. Let us suppose, to fix our ideas, that $f_1 \succ 1$. We have then

$$\log f_1 = \int^x \left(\frac{\rho_0'}{\rho_0}+\sigma_0'\right) dt = A\int^x t^s(1+\epsilon)\,dt,$$

since the subject of integration is algebraical. Since $\log f_1 \succ 1$, $s \geqslant -1$. It is evident that, unless $s=-1$, we obtain†

$$\log f_1 = Ax^s(1+\epsilon), \qquad f_1 = e^{Ax^s(1+\epsilon)}.$$

Here, of course, A and s are positive. If $s=-1$, we have

$$\frac{\rho_0'}{\rho_0}+\sigma_0' = \frac{A}{x}(1+\epsilon).$$

But ϵ is algebraical. Hence we must have

$$\epsilon = O(x^{-a}) \quad (a>0),$$

$$\frac{f_1'}{f_1} = \frac{A}{x}+O(x^{-1-a})$$

$$\log f_1 = A\log x + B+\epsilon,$$

$$f_1 = A_1 x^A(1+\epsilon).$$

It remains to be seen that A is rational and may be replaced by s.

Now σ_0', being the derivative of an algebraical function, cannot, when expanded in descending powers of x, contain any term in $1/x$. And ρ_0'/ρ_0, being the logarithmic derivative of an algebraical function, can contain such a term only with a rational coefficient.‡ Thus A must be rational.

* As explained in § 2, s, t denote *rational* numbers.

† Not, of course, with the *same* A and s; see the remarks in § 2. Of course, too, "unless $s=-1$" refers to the *first* s.

‡ The logarithm of an algebraical function is necessarily of the form $a\log x+\ldots$, where a is rational, and the remaining terms involve negative powers of x.

The case in which $f \prec 1$ may be treated in the same way.

(2) *The conclusions of the theorem are valid for any integral exponential function*

$$f_1 = \Sigma \rho_0 e\sigma_0.$$

Assume this proved when f_1 is of type $\varpi-1$. If we divide f_1 by $\rho_0 e\sigma_0$, differentiate, and restore $\rho_0 e\sigma_0$ to the other side, we obtain

(1)
$$f_1' + s_0 f_1 = \phi_1,$$

where ϕ_1 is an integral exponential function of type $\varpi-1$ and

$$s_0 = -\frac{d}{dx}\log(\rho_0 e\sigma_0) = -\frac{\rho_0'}{\rho_0} - \sigma_0'$$

is algebraical.

In the equation (1) there must be at least one pair of terms which can be connected by the symbol $\asymp$. We have thus to distinguish *three* possibilities.

(i) We may have
$$s_0 f_1 \asymp \phi_1.$$

As ϕ_1/s_0 is of type $\varpi-1$, our induction from $\varpi-1$ to ϖ is in this case immediately established.

(ii) We may have
$$f_1' \asymp \phi_1.$$

As ϕ_1 is of type $\varpi-1$, it follows that f_1' must have one of the forms

$$e^{Ax^s(1+\epsilon)}, \qquad Ax^s(1+\epsilon);$$

and it then follows immediately, from Theorem 8*a*, that f_1 itself has one of these forms, except in the special case when in the second form $s = -1$. In this case

$$f_1' \sim A/x, \qquad f_1 \sim A\log x.$$

But this is impossible, obviously if $f_1 \prec 1$, and, by Theorem 7, if $f_1 \succ 1$.

(iii) We may have
$$f_1' \asymp s_0 f_1,$$

in which case
$$f_1'/f_1 = Ax^s(1+\epsilon).$$

Unless $s = -1$, we at once see that f_1 is of the form $e^{Ax^s(1+\epsilon)}$. If $s = -1$, we have

$$\frac{f_1'}{f_1} = \frac{A}{x}(1+\epsilon),$$

and the same argument as was used under (1) above may be used again to show that A is rational.

Again,
$$\epsilon = (xf_1' - Af_1)/Af_1$$

is a rational exponential function, and must therefore,* by Theorem 7, be of the form

$$O(x^{-a}) \quad (a > 0).$$

* See the last remark in § 15.

Hence
$$\frac{f_1'}{f_1} = \frac{s}{x} + O(x^{-1-\alpha}),^*$$
$$\log f_1 = s \log x + A + \epsilon,$$
$$f_1 = Ax^s(1+\epsilon).$$

Thus (2) is completely established.

(3) *The conclusions of the theorem are valid for any integral function*
$$f_1 = \Sigma \rho_0 e\sigma_0 (l\tau_0^{(1)})^{\kappa_1} (l\tau_0^{(2)})^{\kappa_2} \dots (l\tau_0^{(h)})^{\kappa_h}.$$

We shall prove this by our usual process of induction from degree $\lambda-1$ to type $(\lambda, 1)$, and from type $(\lambda, \mu-1)$ to type (λ, μ).

Dividing by $\rho_0 e\sigma_0$, differentiating, and restoring $\rho_0 e\sigma_0$ to the other side, we obtain

(1) $$f_1' + s_0 f_1 = \phi_1,$$

where s_0 is the same function as before, and ϕ_1 is of type $(\lambda, \mu-1)$ or, if $\mu = 1$, of degree $\lambda-1$. We have again three possibilities.

(i) If $s_0 f_1 \succeq \phi_1$, our conclusion follows immediately.

(ii) If $f_1' \asymp \phi_1$, we have $f_1' = e^{Ax^s(1+\epsilon)}$

or
$$f_1' = Ax^s (lx)^t (1+\epsilon).$$

It then follows, from Theorem 8*a* and Theorem 8*b*, that f_1 itself has one of these forms. The case of exception to Theorem 8*b*, viz., $s = -1$, $t = -1$, cannot occur, since t is *ex hypothesi* zero or a positive integer.

(iii) If $f_1' \asymp s_0 f_1$, we have
$$f_1'/f_1 = Ax^s(1+\epsilon),$$
and our conclusion again follows at once unless $s = -1$.

Now
$$\epsilon = (xf_1' - Af_1)/Af_1$$
is rational, and so, either $\epsilon \prec x^{-\gamma}$,

for some positive γ, or $\epsilon \asymp (lx)^{-q}$,

where q is a positive integer. If $\epsilon \prec x^{-\gamma}$ or $q > 1$, we have
$$\frac{f_1'}{f_1} = \frac{A}{x} + O\left(\frac{1}{x^{1+\gamma}}\right) \quad or \quad \frac{A}{x} + O\left\{\frac{1}{x(lx)^q}\right\},$$
$$\log f_1 = A \log x + B + \epsilon,$$

(2) $$f_1 = A_1 x^A (1+\epsilon).$$

* We have replaced A, proved to be rational, by s.

If $q = 1$, we have $$\frac{f_1'}{f_1} = \frac{A}{x} + \frac{B}{x\,lx}(1+\epsilon),$$

where now $$\epsilon = O\{(lx)^{-q}\} \quad (q \geqslant 1).$$

Hence we obtain $$lf_1 = Alx + Bllx + C + \epsilon,$$

(3) $$f_1 = A_2 x^A (lx)^B (1+\epsilon).$$

It may be shown as before that A is rational: that B is rational follows at once from Theorem 7. Thus in any case the proof of our induction is completed.

(4) *The conclusions of the theorem are valid for any rational function of order n.*

Let $$f_1 = M_1/M_2.$$

The truth of (4) has already been established when M_2 is simply exponential. We require to establish inductions

(i) from the case in which M_2 is exponential and of type $\varpi - 1$ to the case in which it is exponential and of type ϖ,

(ii) from the case in which M_2 is of logarithmic degree $\lambda - 1$ to the case in which it is of degree λ and type $(\lambda, 1)$,

(iii) from the case in which is of type $(\lambda, \mu - 1)$ to the case in which it is of type (λ, μ).

In any case let $$\rho_0 e\sigma_0 = 1/\phi,$$

a factor of one of the terms of M_2, chosen as usual, and let

$$M_1 = \bar{M}_1/\phi, \qquad M_2 = \bar{M}_2/\phi, \qquad f_1 = \bar{M}_1/\bar{M}_2.$$

Each of $\bar{M}_1$, $\bar{M}_2$ is of one of the forms

$$e^{Ax^s(1+\epsilon)}, \qquad Ax^s(lx)^t(1+\epsilon),$$

where s is rational and t is integral.

There is only one case in which it is not evident that f_1 itself is of one of these forms. This case occurs when $\bar{M}_1$ and $\bar{M}_2$ are both of the first form, with the same values of A and s. Then

$$\log \bar{M}_1 \sim \log \bar{M}_2 \sim Ax^s,$$

$$\bar{M}_1'/\bar{M}_1 \sim \bar{M}_2'/\bar{M}_2,$$

and so $$f_1 \sim \bar{M}_1'/\bar{M}_2'.$$

Now $\bar{M}_2'$ is of type $(\lambda, \mu - 1)$, or of degree $\lambda - 1$, or of type $\varpi - 1$, according to the induction which we are proving. Thus in any case the induction is established.

(5) We can now complete the proof of Theorem 10. For f_1 satisfies an equation

$$M_0 f_1^p + M_1 f_1^{p-1} + \ldots + M_p = 0$$

whose coefficients are integral functions of order 1. Of the terms $X_0, X_1, \ldots, X_p$ of this equation, one pair at least must satisfy a relation $X_i \asymp X_j$. Thus

$$f_1^{j-i} \asymp M_j / M_i ;$$

and the theorem follows now as a corollary of (4).

21. Theorem 10 enables us to recognize at once that certain types of function cannot be L-functions of order 1. Thus

$$x^a = e^{a \log x},$$

where a is irrational, is of order 2. This was proved quite differently by Liouville.* The same is true of

$$\log \log x, \dagger \quad e^{e^x}, \quad x^x.$$

Again :

THEOREM 11.—*No function of order* 1 *can satisfy*

$$x^{\Delta} \prec f \prec e^{x^{\delta}}$$

or

$$(\log x)^{\Delta} \prec f \prec x^{\delta}.$$

Thus

$$e^{(\log x)^a} \quad (a \neq 1)$$

is of order 2.

22. Let us pass to functions of order 2. It is easy to recognize

$$e^{e^{ax^s}}, \qquad e^{ax^s (lx)^t}, \qquad ax^s (lx)^t (l_2 x)^u$$

as fundamentally distinct types of increase for such functions. By arguments similar to those of § 20 we can establish the theorem—

THEOREM 12.—*An L-function of order* 2, *positive and tending to infinity with x, can be expressed in one or other of the following forms :—*

$$\text{(i)} \quad e^{e^{ax^s (1+\epsilon)}},$$

$$\text{(ii)} \quad e^{ax^s (lx)^t (1+\epsilon)},$$

$$\text{(iii)} \quad x^s e^{\pm a (lx)^t (1+\epsilon)},$$

$$\text{(iv)} \quad ax^s (lx) (l_2 x)^u (1+\epsilon),$$

* *Journal de Math.*, T. 2, pp. 94–98.

† *Ibid.*, pp. 99–102.

where a is positive, and s, t, u rational, and, moreover,

(i) $s > 0$;

(ii) $s > 0$ *or* $s = 0$, $t > 1$;

(iii) $s > 0$, $0 < t < 1$ *or* $s = 0$, $0 < t < 1$ (*when the positive sign must be taken*), *or* $s = 0$, $t = 1$ (*when again the positive sign must be taken, and a is irrational**);

(iv) $s > 0$, *or* $s = 0$, $t > 0$, *or* $s = 0$, $t = 0$, $u > 0$.

By means of this theorem we recognize, for example, that

$$l_3 x, \quad e_3 x, \quad x^{x^x}, \quad (lx)^a \quad (a \text{ irrational}),$$
$$e^{e^{(\log x)^a}} \; (a \neq 1), \quad e^{(\log\log x)^a} \; (a \neq 1),$$
$$\dots \quad \dots \quad \dots \quad \dots$$

are in reality of order *three.*

Again, *no L-function of order 2 can satisfy*

$$e^{(lx)^\Delta} \prec f \prec e^{x^\delta}.$$

It is, of course, possible to proceed further in this direction. Thus for functions of order 3 the standard forms are

$$e^{e^{e^{ax^s}(1+\epsilon)}},$$
$$e^{e^{ax^s (lx)^t (1+\epsilon)}},$$
$$e^{x^s e^{\pm a (lx)^t (1+\epsilon)}},$$
$$e^{ax^s (lx)^t (l_2 x)^u (1+\epsilon)},$$
$$x^s e^{\pm a (lx)^t (l_2 x)^u (1+\epsilon)},$$
$$x^s (lx)^t e^{\pm a (l_2 x)^u (1+\epsilon)},$$
$$x^s (lx)^t (l_2 x)^u (l_3 x)^v (1+\epsilon);$$

and so on generally. No function of order 3 can satisfy

$$e^{(lx)^\Delta} \prec f \prec e^{e^{(lx)^\delta}}.$$

No function of order 4 can satisfy

$$e^{e^{(llx)^\Delta}} \prec f \prec e^{e^{(lx)^\delta}};$$

no function of order 5 can satisfy

$$e^{e^{(llx)^\Delta}} \prec f \prec e^{e^{e^{(llx)^\delta}}};$$

and so on generally. We shall make use of this remark in the sequel.†

* Were a rational, $e^{a \log x}$ could be replaced by x^a.

† On the subject of the functions $e_r (l_s x)^a$, see "O.I.," pp. 21 *et seq.*

VI.

On Gaps in the Logarithmico-Exponential Scales.

23. I have in my tract considered the question of the construction of functions whose rate of increase cannot be measured by functions of the logarithmico-exponential scales; *i.e.*, of functions f such that the relation

$$f \asymp \phi$$

is impossible for any L-function ϕ.

Such functions are furnished, for example, by any function f such that

$$f \succ e_{\Delta}(x) \quad \textit{or} \quad 1 \prec f \prec l_{\Delta}(x).$$

A simple example of the first type is given in my tract.* An example of the second type can be constructed by means of the functional equation

$$\phi(e^x) = e\phi(x).$$

It is easy to define, by means of a geometrical construction, a solution of this equation which tends steadily and continuously to infinity with x. The fact that

$$l_k(e^x) \succ l_k x,$$

for all values of k, at once suggests that the increase of ϕ must be slower than that of any logarithm; and it is easy to prove that this is in fact so.

A still more interesting question is that of defining functions whose increase cannot be measured by any L-function, although it falls within the limits of the logarithmico-exponential scales. I showed in my tract† how we could, by a geometrical construction, define a continuous and monotonic solution of the equation

$$\phi\phi(x) = e^x.$$

I was compelled, by considerations of space, to content myself with the briefest indications of a proof, which I will now complete.

24. The solution of the equation was defined as follows. Let

$$x_0 = 0, \quad x_1 = \tfrac{1}{2}, \quad x_2 = 1, \quad x_3 = \sqrt{e}, \quad x_4 = e, \quad \ldots,$$

* "O.I.," p. 33.
† "O.I.," p. 34.

and generally
$$x_{2n} = e_n(0) = e_{n-1}(1),$$
$$x_{2n+1} = e_{n-1}(\sqrt{e}).$$

Then $\phi(x)$ is defined by the equations

$$\phi(x) = e_{-2}(l_{-2}x)^{\sqrt{e}} = x+\tfrac{1}{2} \quad (x_0 < x < x_1),^*$$
$$\phi(x) = e_{-1}(l_{-2}x)^{1/\sqrt{e}} = e^{x-\frac{1}{2}} \quad (x_1 < x < x_2),$$
$$\phi(x) = e_{-1}(l_{-1}x)^{\sqrt{e}} = x\sqrt{e} \quad (x_2 < x < x_3),$$

and generally
$$\phi(x) = \lambda_{n-2}(x) = e_{n-2}(l_{n-2}x)^{\sqrt{e}} \quad (x_{2n} < x < x_{2n+1}),$$
$$\phi(x) = \mu_{n-2}(x) = e_{n-1}(l_{n-2}x)^{1/\sqrt{e}} \quad (x_{2n+1} < x < x_{2n+2}).$$

Then†
$$\lambda_0 \prec \lambda_1 \prec \lambda_2 \prec \dots \prec \mu_2 \prec \mu_1 \prec \mu_0.$$

I shall now prove that (as was asserted in my tract)‡

$$\lambda_p \prec \phi \prec \mu_q$$

for all values of p and q.

25. In order to prove that $\phi \succ \lambda_p$ for all values of p, it is plainly sufficient to prove that
$$\phi \geqslant \lambda_{n-2} \quad (x \geqslant x_{2n}).$$

Now, if $x > x_{2n}$, ϕ is equal either to one of the functions $\lambda_{m-2}\ (m \geqslant n)$ or to one of the functions $\mu_{m-2}\ (m \geqslant n)$.

Now
$$\lambda_{n-1} \geqslant \lambda_{n-2},$$
if
$$e(l_{n-1}x)^{\sqrt{e}} \geqslant (l_{n-2}x)^{\sqrt{e}};$$
or, putting $y = l_{n-2}x$, if
$$(ly)^{\sqrt{e}} \geqslant \sqrt{e}\, ly,$$
$$ly \geqslant (\sqrt{e})^{1/(\sqrt{e}-1)}.$$

Now, for $x = x_{2n} = e_{n-1}(1)$, we have $y = e$. And
$$e > (\sqrt{e})^{1/(\sqrt{e}-1)},$$

* We may agree to interpret $l_{-k}x$ as e_kx and $e_{-k}x$ as l_kx.
† "O.I.," pp. 21 *et seq.*
‡ "O.I.," p. 35.

if
$$\sqrt{e}-1>\tfrac{1}{2},$$
which is obviously true. Hence $\lambda_{n-1} \geqslant \lambda_{n-2}$ for $x \geqslant x_{2n}$; similarly $\lambda_n \geqslant \lambda_{n-1}$ for $x \geqslant x_{2n+2}$, and so on. It follows that
$$\phi(x) \geqslant \lambda_{n-2}(x) \quad (x \geqslant x_{2n}),$$
certainly throughout the intervals in which ϕ coincides with one of the functions λ.

Again
$$\mu_{n-2} \geqslant \lambda_{n-2},$$
if
$$e\left(y^{1/\sqrt{e}}\right) \geqslant y^{\sqrt{e}},$$
or, putting $y = z^{\sqrt{e}}$, if
$$e^z \geqslant z^e.$$
Now the equation $e^z = z^e$ has a root when $z = e$: and $e^z > z^e$ if $z > e$. Thus $\mu_{n-2} \geqslant \lambda_{n-2}$, if $y \geqslant e^{\sqrt{e}}$, or
$$x \geqslant e_{n-1}(\sqrt{e}) = x_{2n+1}.$$
Similarly $\mu_{n-1} \geqslant \lambda_{n-1}$, if $x > x_{2n+3}$, and, *a fortiori*, $\mu_{n-1} \geqslant \lambda_{n-2}$, and so on. Thus
$$\phi(x) \geqslant \lambda_{n-2}(x) \quad (x \geqslant x_{2n}),$$
also in the intervals in which ϕ is equal to one of the functions μ. And so
$$\phi(x) \geqslant \lambda_{n-2}(x) \quad (x \geqslant x_{2n}).$$
Similarly we can prove that $\phi(x) \leqslant \mu_{n-2}(x) \quad (x \geqslant x_{2n+1})$, and from this it at once follows that
$$\lambda_p \prec \phi \prec \mu_q,$$
for all values of p and q.

Thus the function ϕ enables us to divide all L-functions into two classes, such that if L_1 and L_2 denote any members of the respective classes, then
$$L_1 \prec \phi \prec L_2.$$
The increase of ϕ would, in Borel's notation,* be denoted by
$$\sqrt{\omega}.$$
We could similarly define the mode of increase $\omega^{n+\frac{1}{2}}$, where n is any

* *Leçons sur la théorie de la croissance*, pp. 14 *et seq.*

integer, positive or negative: and indeed ω^a for any rational a. *All these modes of increase are distinct from those of all L-functions.*

26. There is another method of constructing functions whose increase, so to say, hits off a gap in the logarithmico-exponential scales. This method is simpler but less interesting than that just discussed, and I shall content myself with some very summary indications.

Consider the function $$f = e^{x+1/x}.$$

Since $$e^{x+1/x} \sim e^x,$$

the increase of the function is the same as that of the simpler function e^x.

A series of closer approximations to the function f would be furnished by
$$e^x\left(1+\frac{1}{x}\right), \quad e^x\left(1+\frac{1}{x}+\frac{1}{2x^2}\right), \quad \ldots .$$

But we cannot in this way find a function ϕ such that
$$f = \phi+\epsilon.$$

However many terms of the series for $e^{1/x}$ we retain, the difference $f-\phi$ remains of order greater than x^{Δ}. To put it roughly, we can only express f with the degree of approximation implied by the equation $f = \phi+\epsilon$, by taking ϕ equal to f itself. Similar considerations apply, evidently, to any function
$$e^x P(1/x),$$

where P is a convergent power series.

If now $$F = e^f = e^{e^{x+1/x}},$$

we see that, in order to express F by a simpler function ϕ, with the degree of approximation indicated by $$F = \phi(1+\epsilon)$$

—*i.e.*, with sufficient approximation to fix the increase of F—we must take
$$\phi = F.$$

Similar considerations apply, of course, to the more general function
$$e^{e^x P(1/x)}.$$

Thus, by taking $P(1/x)$ to be a power series *whose sum is not an ele-*

mentary function—such a series as

$$1+\frac{1}{1^2}\,\frac{1}{x}+\frac{1}{1^2.2^2}\,\frac{1}{x^2}+\frac{1}{1^2.2^2.3^2}\,\frac{1}{x^3}+\ldots$$

—we are led to functions whose increase is not equal to that of any L-function.*

These considerations can be extended to cases in which $P(1/x)$ is a summable asymptotic series, and the factor e^x is replaced by a more general factor—we are thus led to such functions as

$$e^{\Gamma(x+1)}=e^{x^{x+\frac{1}{2}}e^{-x}\sqrt{(2\pi)}P(1/x)}.$$

VII.

Miscellaneous Remarks and Conclusion.

27. *Inverse Functions.*—The present is a convenient moment at which to make a few additional remarks as to the possible modes of increase of the inverse of an L-function (a question already raised in § 12).

Suppose that y is an L-function of order 1, ultimately positive and tending to infinity with x. Then, by Theorem 10,

$$y=e^{Ax^s(1+\epsilon)}$$

or $$y=Ax^s(lx)^t(1+\epsilon).$$

In the first case $$x=A_1(ly)^{t_1}(1+\epsilon),$$

and in the second $$x=A_1y^{s_1}(ly)^{t_1}(1+\epsilon),$$

unless $s=0$, when $$x=e^{A_1y^{t_1}(1+\epsilon)}.$$

Thus *the standard forms for the increase of the inverses of L-functions of order* 1 *are the same as for the L-functions themselves.*

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* Several theorems have been proved with reference to the possible forms of power series whose sum is an elementary transcendent in Liouville's sense, or, more generally, a solution of an algebraic differential equation: see, for example, Hermite, *Cours* (4th edition), p. 195, where a theorem originally due to Eisenstein is proved. This theorem has been extended widely: see Heine, *Crelle*, Bd. 45 and Bd. 48; Hurwitz, *Annales de l'École Normale*, t. 6 (series 3); Fouet, *Leçons sur la théorie des fonctions analytiques* (2nd edition), t. 2, p. 123, where further references are given.

Hermite (*l.c.*) also refers to an extremely interesting theorem of Tschebyschef, of which no proof appears to have been published (see Wassilief, *P. L. Tschebyschef und seine wissenschaftlichen Leistungen*, Leipzig, 1900, p. 42 and p. 54).

Similarly, if y is of order 2, it has one of the forms

$$(1)\ e^{e^{Ax^s(1+\epsilon)}}, \qquad (2)\ e^{Ax^s(lx)^t(1+\epsilon)}, \qquad (2a)\ e^{A(lx)^t(1+\epsilon)}\ (t>1),$$

$$(3)\ x^s e^{A(lx)^t(1+\epsilon)}\ (t<1), \qquad (3a)\ e^{A(lx)^t(1+\epsilon)}\ (t<1),$$

$$(4)\ Ax^s(lx)^t(l_2x)^u(1+\epsilon), \qquad (4a)\ A(lx)^t(l_2x)^u(1+\epsilon), \qquad (4b)\ A(l_2x)^u(1+\epsilon).$$

It is easily verified that the forms

	1,	2,	$2a$,	3,	$3a$,	4,	$4a$,	$4b$
invert into	$4b$,	$4a$,	$3a$,	3,	$2a$,	4,	2,	1.*

Thus the standard forms are also the same in the case of order 2, and this conclusion may be extended to any order.

It is easy to see that many of the theorems proved in this paper are true not only of L-functions but also of their inverses. This is certainly so, for example, with Theorems 2, 3, 7, 10, 11, and 12. Again, if $\bar{f}$ and $\bar{\phi}$ are inverses of L-functions, one of the relations $\bar{f} \succ \bar{\phi}$, $\bar{f} \asymp \bar{\phi}$, $\bar{f} \prec \bar{\phi}$ holds between them; and if f is an L-function, and $\bar{\phi}$ the inverse of an L-function, one of the relations $f \succ \bar{\phi}$, ... holds between them.

But whether or not it is true that, given an L-function ϕ and its inverse $\bar{\phi}$, there must be an L-function ψ, such that

$$\bar{\phi} \sim \psi,$$

I cannot say; and, as I said in § 12, I am very doubtful whether this is so.

Consider, for example, the equation

$$y = x\,lx: \tag{1}$$

an equation whose solution was proved by Liouville† not to be expressible explicitly in finite terms. Then

$$x \sim y/ly. \tag{2}$$

If, now, we consider the equation

$$y = lx\,l_2x, \tag{3}$$

* The special case of $2a$ or $3a$, in which $t = 1$, A being *irrational*, inverts into itself.

† *Journal de Mathématiques*, t. iii, pp. 526–531.

we have

(4) $$x = e^{\left\{ \frac{y}{ly}(1+\epsilon) \right\}}.$$

More precisely* $$x = e^{\left\{ \frac{y}{ly} + \frac{y\,l_2 y}{(ly)^2} + \frac{y\,(l_2 y)^2}{(ly)^3} - \frac{y\,l_2 y}{(ly)^3}(1+\epsilon) \right\}}.$$

But in order to express the solution of (3) in the form

$$x = \psi(y)(1+\epsilon),$$

we should have to express the solution of (1) in the form

$$x = \chi(y) + \epsilon,$$

χ being an L-function: and it seems to me very improbable that this is possible.

In any case it should be observed that all the information that we have acquired concerning the modes of increase of the inverses of L-functions has been obtained by the use of Theorem 7 and its consequences, and so cannot be used in order to simplify the proof of that theorem.

28. *Algebraic Differential Equations.*—Borel† has indicated the lines of extremely interesting researches concerning the possible modes of increase of functions defined by differential equations. Thus he has proved that the equation

(1) $$f(x, y, y') = 0,$$

where x and y are real, and f is a polynomial, cannot have an increasing solution of order as great as $$e^{e^x}.$$

Similarly the corresponding equation of the second order cannot have a solution whose increase is as great as that of $e_3 x$; and there can be no doubt of the truth of the corresponding general theorem.

The discussion even of the two simplest cases is rather elaborate when so general hypotheses as M. Borel's are adopted. At present I propose only to point out how, in the case of the equation of the first order, we can obtain much more precise information about the possible modes of

* "O.I.," p. 46.

† "Mémoire sur les séries divergentes," *Annales de l'École Normale*, t. 16, pp. 25 *et seq.*: see also Boutroux, *Leçons sur les fonctions définies par les équations différentielles du premier ordre.*

increase of the solutions, if we assume this increase to be, so to say, sufficiently regular.

Let us assume that the equation (1) has a solution y, such that any function of the form

$$x^m y^n (y')^p,$$

where m, n, p are integers, is ultimately monotonic. We shall express this by saying that y is *of strictly regular increase*. It is possible to prove that, in the case of the particular equation

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}, \tag{2}$$

where P and Q are polynomials, any solution which is ultimately continuous is of strictly regular increase; but it would carry us too far to enter upon the details of a proof at present.

It then follows that any two of the terms of (1) must satisfy a relation of one of the types $\phi \succ \psi$, $\phi \asymp \psi$, $\phi \prec \psi$, and there must be at least one pair which satisfy a relation of the second type. We must therefore have a relation of the type

$$y^\nu y' \sim A x^\mu. \tag{3}$$

If neither μ nor ν is equal to -1, we obtain a relation

$$y \sim A x^\alpha. \tag{4}$$

If ν (but not μ) is equal to -1, we obtain

$$y = e^{Ax^\alpha (1+\epsilon)}. \tag{5}$$

If μ (but not ν) is equal to -1, we obtain

$$y \sim A (lx)^\alpha. \tag{6}$$

If μ and ν are each equal to -1, we obtain

$$ly \sim A\, lx,$$

$$y = x^{A+\epsilon}. \tag{7}$$

This equation is less precise than the relation (4). To make it more precise, we must put $y = x^A z$ and form the equation satisfied by z.

Let us (to cut the matter short) confine ourselves to the equation (2). We find that y *must be expressible in one of the forms*

$$e^{Ax^\alpha (1+\epsilon)}, \qquad A x^\beta (lx)^\gamma (1+\epsilon). \tag{8}$$

It is not difficult to see that α and γ must be *rational.* This is not true of β—as is obvious from the fact that $y = x^\beta$ satisfies

$$y' = \beta y/x,$$

whether β be rational or irrational.*

Apart from this circumstance, the forms (8) are the standard forms of Theorem 10. This parallelism at once suggests a number of further questions which demand a more careful analysis of the equation (1) and its extensions : to some of these questions I hope to return on another occasion.

* More precisely, the possible forms are

$$y = e^{Ax^p(1+\epsilon)},$$

$$y = A\,(x^q lx)^{1/p}(1+\epsilon), \qquad A\left(\frac{x^q}{lx}\right)^{1/p}(1+\epsilon),$$

$$y = Ax^\beta(1+\epsilon).$$

Here p and q are integers.