

XX.—*On Fresnel's Formulæ for the Intensity of Reflected and Refracted Light.*
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INTRODUCTION.

IT is well known, that when light is incident on a refracting surface, a portion of it is reflected, whilst both the transmitted and the reflected light undergo polarization. The obvious mode of accounting for this, is to attribute to the particles on whose motion light is supposed to depend, the property of transmitting one class of vibrations more freely than another, limited, however, by the direction and mode of action of the adjacent particles. M. FRESNEL, in order to determine the intensity of light reflected and refracted under different circumstances, assumed that the density of the particles of ether is greater in refracting media than *in vacuo*. By means of this assumption, and other subsidiary ones, he deduced formulæ for the intensity of the reflected and refracted light, by means of which the amount of polarization, as well as the change which the plane of polarization undergoes, can be readily deduced. The obvious interpretation of the formulæ coincided precisely with discoveries which had been long known, and the more difficult deductions from them have been tested by numerous experiments of Sir DAVID BREWSTER and others. It appears that, although for highly refractive media, they may be only approximations, yet, in most cases, they are so close as to deserve the most careful attention of those who endeavour to establish a correct mechanical theory.

M. CAUCHY, in different memoirs, has laboured to deduce M. FRESNEL'S formulæ from the equations of motion, and, in one instance, from assumed conditions of a nature not widely different from M. FRESNEL'S own. The fact that these expressions had been deduced from the assumption of a greater density within refracting media than without, appeared to throw a doubt over the truth either of the molecular hypothesis, which seemed to require the reverse, or of the formulæ themselves.

Whilst M. CAUCHY is tossed about with various and conflicting conclusions, Mr M'CULLAGH is led, by totally different considerations, to one of the most important of them, viz. that the vibrations which constitute light polarized in the plane of incidence are vibrations effected in that plane, a result which is direct-

ly opposed to that of M. FRESNEL. How these philosophers have succeeded in the more complex case of crystalline reflexion, it concerns us not to inquire, until the principles which guide their hypotheses shall be shewn to be sound and mechanical.

It appears, however, that M. CAUCHY has actually inferred, from mechanical principles, that the vibrations of polarized light are the opposite to those assumed by M. FRESNEL. Of the amount of evidence which M. CAUCHY adduces I am altogether ignorant; but it ought to be overpowering indeed to shake our faith in an hypothesis which has so successfully overcome all difficulties, and brought the apparently complex phenomena of double refraction to the level of common optics.

However this be, the matter is not yet set at rest, for a paper has just been printed for the next part of the CAMBRIDGE TRANSACTIONS, in which M. FRESNEL's hypothesis as to the direction of vibration is assumed to hold, and his formulæ corresponding to light polarized in the plane of incidence are established, whilst an approximate demonstration is offered for those corresponding to the perpendicular plane.

My primary object in drawing up the present memoir has been to remove from the molecular theory some difficulties in which Mr GREEN's researches seem to involve it. As a preliminary step, I will therefore point out the most important of these, and endeavour to shew that the arguments which naturally arise out of them are such as can be answered without compromising any of the principles on which the molecular hypothesis is based. Having done this, I shall apply the equations of motion deduced from molecular forces, to shew that the formulæ result in the most satisfactory manner from the state which such forces induce.

To effect my purpose of explaining the difficulties which Mr GREEN's memoir opposes to the molecular theory, it will be requisite that I point out in few words the nature and results of that theory.

Almost all mathematicians have admitted the idea of discrete molecules to be philosophical; but very few have attached any weight to the results to which this hypothesis leads. LAPLACE, in his *Mécanique Céleste*, supposes the atoms of matter to be permeated by the molecules of caloric; but he assigns forces to the molecules, which are conceived to diminish with great rapidity as the distance from the molecule is increased, and actually to vanish at all appreciable distances. By a similar hypothesis, in the same great work, he solves the problem of Capillary Attraction.

POISSON also, in his Memoir on the Equilibrium and Motion of Fluids, as well as in his Capillary Attraction and Theory of Heat, conceives the particles to be separated by finite intervals, and makes use of a force which results from this circumstance; but neither he nor LAPLACE appears to have investigated the complex arrangement of actions and their counteracting opposites, to which this force

is due. This last investigation was reserved for M. CAUCHY, who managed it with great skill, in his *Exercices de Mathématiques*, vol. iii. p. 188, and vol. iv. p. 129. In the fifth volume, M. CAUCHY applied his results to the theory of light; but his success was not complete at first, owing to the circumstance that he had recourse to the method of expansion so universally adopted in physical investigations. In subsequent publications, however, M. CAUCHY has solved the difficult problem of obtaining a relation between the velocity of transmission and the length of the wave. This very important result, which removed from the undulatory theory almost the only obstacle to its being entitled to the designation of a true *physical theory*, appeared in 1830. Since that time M. CAUCHY has published various memoirs on the reflexion of light, and on other points of the theory, in one of which he has determined the law of force by which the particles act on one another to be that of the inverse fourth power of the distance.

In a memoir of my own (*Transactions of the Cambridge Philosophical Society*, vol. vi. p. 153), another law is arrived at, viz. that of the inverse square of the distance. This conclusion, agreeing as it does with the great law of gravitation, and necessary, moreover, as it appears to be, from the very condition of attraction, I have retained in all my subsequent investigations. One important corollary from it will be found in page 180 of the same memoir, viz. that the vibrations are altogether transversal to the direction of a *wave*. This conclusion Professor LLOYD has also obtained from CAUCHY'S law of the inverse fourth power. His paper was read to the Royal Irish Academy. It would be too wide a field to enter on the discoveries of Sir WILLIAM HAMILTON. Copious information on the subject, together with a translation of M. CAUCHY'S most important memoir, will be found in the pages of the *Philosophical Magazine*.

Nor is the arrangement and action of force thus assumed less in consistence with statical than with dynamical truths. The great problem of cohesion, as connected with expansion, &c., appeared to defy a law of force such as that of the inverse square, until M. MOSSOTTI, by a most skilful application of analysis, removed the most glaring difficulties. The same subject has been commenced by myself in the *Transactions of the Cambridge Philosophical Society*, vol. vii., in which I have deduced results which demonstrate the possibility, or at least afford argument for the probability, of the universality of the law of universal gravitation. This, therefore, is the present state of the molecular theory: it coincides with the great law of attraction, and is the extreme limit to that law; it accounts for the complicated phenomena of light, which defy more simple investigation, at the same time that it requires the introduction of no modification into those processes which are adequate to effect their purposes without its aid; it demonstrates the necessity of a circumstance which had previously been only suspected to exist, the perfect transversality of vibration; and, lastly, it promises an insight into the perplexing phenomena of absorption. Having thus pointed out the na-

ture and results of the molecular hypothesis, I return to the examination of the memoir, which appears in some points to argue against it.

Mr GREEN states, that two waves will result from giving a motion to a fluid, such as that commonly supposed to be the medium the vibrations of which constitute light, the one transversal and the other normal. On a careful examination of his memoir, I cannot discover this normal vibration; the nearest approach to it appears to result from the circumstance, that two waves, the incident and the reflected, may be transmitted at the same time, and therefore cross each other. If, then, in this case, the angle of incidence be an angle of 45° , one vibration may be at right angles to the other; but this circumstance does not in the slightest degree militate against any conclusions which have been arrived at by the molecular hypothesis. The coexistence of vibrations travelling in different directions, is distinctly recognised in that theory. It may be well to state clearly, that the point, and I think a most important one, which has been proved from the molecular hypothesis, is this; that *one* wave cannot consist partly of normal, partly of transversal vibrations. Of course, the definition of the wave restricts it to a state of motion transmitted in *one* direction with *one* velocity.

There can be little doubt, however, that the normal vibration to which Mr GREEN refers, is supposed to be contained in that function which he introduces in the body of his memoir, as the result of the change of motion from an incident and reflected to a refracted one. This vibration is, however, merely a vibratory motion, *not* transmitted in the same direction as the incident; and in the sequel of the present memoir, it will appear that it is really and *bonâ fide* a transverse vibration. Thus a statement, which at the first sight appears to argue powerfully against the molecular theory, does, when attentively examined, afford strong presumptive evidence in its favour.

I have deemed it right to be explicit on this subject, as the admission of Mr GREEN's statement, if it left hypotheses such as LAPLACE's as to the constitution of media uninjured, would absolutely crush the more probable hypothesis of the Newtonian law of gravitation applied to the ultimate atoms.

There is another point in Mr GREEN's paper which, although not so important as the one just noticed, will require an answer of a very different nature, and ought consequently to be attended to. It is this: in order to obtain the law which FRESNEL has deduced for the intensity of light polarized in the plane of incidence, it is found requisite to assume that the velocity of transmission varies inversely as the square root of the density.

This overthrows, apparently, all the previous conclusions of the molecular hypothesis; for all its advocates, as far as I recollect, have come to the conclusion that the density of the caloric within refracting media is less than it is *in vacuo*. But it is desirable that great caution should be exercised in judging of this and like apparent oppositions. We have no very precise notion of the pro-

per signification of density within a medium, nor, if we had, is it quite obvious that the aggregate attraction of combined molecules should of necessity vary as the density. It must be recollected that, in estimating the effect of forces resulting from molecules, the whole result consists of the sum of a large number of terms, not diminishing in magnitude with the same rapidity in all cases. It does not then follow, that the attraction varies as the density, nor even according to any simple function of it.

But I do not stop here. Allowing the assumptions of FRESNEL to be correct,—and from the coincidence of Mr GREEN's conclusions with his, most persons will be inclined to think them substantially so,—all the discrepancy between the molecular hypothesis as viewed by M. CAUCHY, and that deviation from it adopted by M. POISSON and Mr GREEN, amounts to this, that one party (suppose the former) have misinterpreted the formulæ relative to the density of the particles. I shall shew presently that the formulæ themselves are not at all affected by the apparent contradiction of conclusion, since the results of M. FRESNEL may be deduced as easily, and I think with as little assumption, by the molecular hypothesis as by the other.

ANALYTICAL INVESTIGATION.

My object in the investigation which follows, is to deduce M. FRESNEL's formulæ for the intensity of rays reflected at the common surface of two media, air and glass, the incident rays being polarized. It will not be requisite in this place to enter into a discussion of the results obtained by grouping particles. Suffice it to say, that, by strict mathematical investigation, it can be shewn that the assumption of NEWTON's law of force for the particles of the media surrounding the material particles, gives rise to an expression of the following form, for the aggregate attraction or repulsion of those particles which surround one particle of matter

$$f = m \cdot e^{-\alpha a} \frac{1 + \alpha a}{a^2}$$

a being the distance between two material particles. This expression is insensible at sensible distances, and consequently we may limit our summation in the subsequent process to such distances. We will adopt the following notation.

The media being both perfectly symmetrical, and bounded by a plane surface; let that plane be called the plane of $y z$, the axis of z being parallel to the line at which the front of the wave cuts the plane, and that of x' the direction of transmission. When the incident vibrations are polarized in the plane of incidence, all the motion will be in a direction parallel to the axis of z .

Take x, y, z as the co-ordinates of any particle in a state of rest; $x, y, z + \gamma$

those of the same particle at the time t ,

$$x + \delta x, y + \delta y, z + \delta z, \text{ and } z + \delta z + \delta \gamma$$

the corresponding quantities for another particle in the upper medium ;

$$x + \delta x, y + \delta y, z + \delta z, z + \delta z + \delta \gamma,$$

the co-ordinates of another particle in the lower medium at the same time.

When discussing the lower medium separately, we will adopt x, y, z, γ , etc. in all cases for which we use x, y, z, γ , etc. in the upper.

r is the distance between the two particles in a state of rest.

$r + \rho$ their distance in a state of motion.

Let $r \phi r$ be the force on the particle under consideration, arising from another particle at the distance r . If, however, it be thought requisite, we may consider $r \phi r$ as the aggregate attraction of a group of particles about a material particle. The law of force, whenever a law is wanted, will be assumed to be that of NEWTON.

Let $r', \delta x$ etc. denote the distance, etc. of particles in one medium from those in the other.

The notation $\gamma_{x, + \delta x, y, + \delta y}$, denotes the value which γ , assumes when $x, + \delta x, y, + \delta y$, are written for x , and y .

Slight deviations from these arrangements will occasionally be made, which will be pointed out when they occur.

SECTION I.

ON LIGHT, CONSISTING OF VIBRATIONS PERPENDICULAR TO THE PLANE OF INCIDENCE.

We shall adopt the following process : first, deduce the equations of motion on the supposition that the force is insensible except at very small distances from its origin, and then take the law of force, varying inversely as the square of the distance. The object of the first process is the discovery of the *form* which the results assume, to serve as a guide to the more complex calculations of the second.

The two media will be supposed to be arranged in a perfectly symmetrical manner, so that all terms which involve the odd powers of the distances of the particles, will vanish when the sums of such terms are taken, extending throughout the whole mass.

1. The expression for the force on the particle P, resolved parallel to the axis of z , is $\Sigma \phi (r + \rho) (\delta z + \delta \gamma)$.

$$\begin{aligned} &= \Sigma (\phi r + \phi' r \cdot \rho) (\delta z + \delta \gamma) \\ &= \Sigma (\phi r + \frac{\phi' r}{r} \delta z \delta \gamma) (\delta z + \delta \gamma) \\ &= \Sigma \{ \phi r \cdot \delta z + \frac{\phi' r}{r} \delta z^2 \delta \gamma + \phi r \cdot \delta \gamma \} \end{aligned}$$

$$\text{Now} \quad \delta \gamma = \frac{d\gamma}{dx} \delta x + \frac{d^2\gamma}{dx^2} \frac{\delta x^2}{2} + \frac{d^3\gamma}{dx^3} \frac{\delta x^3}{6} + \dots$$

δx being supposed small for those limits to which the force produces a sensible effect. We conclude that

$$\begin{aligned} \frac{d^2\gamma}{dt^2} &= \Sigma \left\{ \phi r \delta z + \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \cdot \left(\frac{d\gamma}{dx} \delta x + \frac{1}{2} \frac{d^2\gamma}{dx^2} \delta x^2 \right) \right\} + \text{etc.} \\ &= \Sigma \left\{ \phi r + \frac{\phi' r}{r} \delta z^2 \right\} \left\{ \frac{d\gamma}{dx} \delta x + \frac{d^2\gamma}{dx^2} \frac{\delta x^2}{2} + \frac{d^3\gamma}{dx^3} \frac{\delta x^3}{6} \right\} \dots \dots (1) \end{aligned}$$

$$\frac{d^2\gamma_i}{dt^2} = \Sigma \left\{ \phi r_i + \frac{\phi' r_i}{r_i} \delta z_i^2 \right\} \left\{ \frac{d\gamma_i}{dx_i} \delta x_i + \frac{d^2\gamma_i}{dx_i^2} \frac{\delta x_i^2}{2} + \frac{d^3\gamma_i}{dx_i^3} \frac{\delta x_i^3}{6} \right\} \dots (2)$$

the symbol Σ being supposed to extend to the sensible limits of the force; but the particle in each case not near the confines of the medium.

It is clear that the term involving $\frac{d\gamma}{dx} \delta x$ vanishes, and the equations are thereby simplified.

By the same process, we may obtain the equation of motion of a particle in the upper medium, near the confines, to be the following:

$$\begin{aligned} \frac{d^2\gamma}{dt^2} &= \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \left(\frac{d\gamma}{dx} \delta x + \frac{d^2\gamma}{dx^2} \frac{\delta x^2}{2} + \frac{d^3\gamma}{dx^3} \frac{\delta x^3}{6} \right) \\ &+ \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta z'^2 \right) \left(\frac{d\gamma}{dx'} \delta x' + \frac{d^2\gamma}{dx'^2} \frac{\delta x'^2}{2} + \dots \right) \dots \dots (3) \end{aligned}$$

the symbol now extending indefinitely on one side, but being bounded by the common surface on the other.

$$\begin{aligned} \text{Also,} \quad \frac{d^2\gamma_i}{dt^2} &= \Sigma \left(\phi r_i + \frac{\phi' r_i}{r_i} \delta z_i^2 \right) \left(\frac{d\gamma_i}{dx_i} \delta x_i + \frac{d^2\gamma_i}{dx_i^2} \frac{\delta x_i^2}{2} + \frac{d^3\gamma_i}{dx_i^3} \frac{\delta x_i^3}{6} \right) \\ &+ \Sigma \left(\phi r'_i + \frac{\phi' r'_i}{r'_i} \delta z_i'^2 \right) \left(\frac{d\gamma_i}{dx'_i} \delta x'_i + \frac{d^2\gamma_i}{dx_i'^2} \frac{\delta x_i'^2}{2} + \dots \right) \dots \dots (4) \end{aligned}$$

If the particles last considered coincide in the bounding surface, the results will be simplified, as we shall see presently.

Let $\Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \frac{\delta x^2}{2}$ in equation (1) be designated by n^2 , it is clear that

$$\Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \frac{\delta y^2}{2} \text{ is also equal to } n^2.$$

Denote $\Sigma \left(\phi r_i + \frac{\phi' r_i}{r_i} \delta z_i^2 \right) \frac{\delta x_i^2}{2}$ the expression in equation (2) by n_i^2

$$\Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \delta x \text{ in (3) by } P$$

$$\Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta z'^2 \right) \delta x' = \Sigma \left(\phi r_i + \frac{\phi' r_i}{r_i} \delta z_i^2 \right) \delta x_i \text{ in equation (3) by } P,$$

then the four equations in order become

$$\begin{aligned}\frac{d^2 \gamma}{dt^2} &= n^2 \left(\frac{d^2 \gamma}{dx^2} + \frac{d^2 \gamma}{dy^2} \right) \\ \frac{d^2 \gamma_i}{dt^2} &= n_i^2 \left(\frac{d^2 \gamma}{dx^2} + \frac{d^2 \gamma_i}{dy_i^2} \right) \\ \frac{d^2 \gamma}{dt^2} &= \frac{n^2}{2} \left(\frac{d^2 \gamma}{dx^2} + \frac{d^2 \gamma}{dy^2} \right) + \frac{n_i^2}{2} \left(\frac{d^2 \gamma_i}{dx_i^2} + \frac{d^2 \gamma_i}{dy_i^2} \right) + P \frac{d\gamma}{dx} + P_i \frac{d\gamma_i}{dx_i} \\ \frac{d^2 \gamma_i}{dt^2} &= \frac{n^2}{2} \left(\frac{d^2 \gamma}{dx^2} + \frac{d^2 \gamma}{dy^2} \right) + \frac{n_i^2}{2} \left(\frac{d^2 \gamma_i}{dx_i^2} + \frac{d^2 \gamma_i}{dy_i^2} \right) + P \frac{d\gamma}{dx} + P_i \frac{d\gamma_i}{dx_i}\end{aligned}$$

that last two equations requiring that x have the value o written for it.

2. Since a particle at the confines of the medium must be so acted on that it is in equilibrium when $\gamma=0$, it is easy to perceive that

$$\Sigma \phi r \delta x = - \Sigma \phi r_i \delta x_i \text{ taken as in equation (3).}$$

This must of course arise from the variation of density near the common surface. On examining the expression, it will appear that, when expressed in language, it is equivalent to the equalization of the sum of a series of terms of different values, but of given dimensions. Now $\frac{\phi' r}{r} \delta z^2 \delta x$ is of the same dimensions as the above term; hence we should expect that

$$\Sigma \frac{\phi' r}{r} \delta z^2 \delta x = - \Sigma \frac{\phi' r_i}{r_i} \delta z_i^2 \delta x_i$$

and

$$P = - P_i$$

3. The solution of equation (1) is

$$\gamma = f(ax + by + ct) + F(-ax + by + ct)$$

the function f corresponding to the incident, and F to the reflected wave; that of (2) is

$$\begin{aligned}\gamma_i &= f_i(a_i x_i + b y + ct) \\ &= f_i(a, x + b y + ct)\end{aligned}$$

by writing x as the general symbol. Now we suppose the wave motion to continue unbroken, so that the equations (3) and (4) give the same results respectively as (1) and (2).

If, then, we substitute the results already obtained, we shall satisfy the two equations (3) and (4).

$$\begin{aligned}c^2 \{f''(by + ct) + F''(by + ct)\} = \\ \frac{n^2}{2} (a^2 + b^2) \{f''(by + ct) + F''(by + ct)\} + \frac{n_i^2}{2} (a_i^2 + b^2) \{f_i''(by + ct)\} \\ + P \{af'(by + ct) - aF'(by + ct) - a_i f_i'(by + ct)\}\end{aligned}$$

And the right hand side of equation (4) is the same as this. Let us now write f for $f(by + ct)$ and so on, then taking notice that by (1) and (2)

$$\begin{aligned} c^2 &= n^2 (a^2 + b^2) \\ &= n_i^2 (a_i^2 + b^2) \end{aligned}$$

we reduce the two equations to these

$$\begin{aligned} f'' + F'' &= f_i'' \\ &= \frac{1}{2} (f'' + F'' + f_i'') + \frac{P}{c^2} \{a f' - a F' - a_i f_i'\} \end{aligned}$$

or

$$f'' + F'' - f_i'' = \frac{2P}{c^2} (a f' - a F' - a_i f_i')$$

$$f_i'' - F'' - f'' = \frac{2P}{c^2} (a f' - a F' - a_i f_i')$$

whence

$$\begin{aligned} f'' + F'' - f_i'' &= 0 \\ a f' - a F' - a_i f_i' &= 0 \end{aligned}$$

which equations give

$$\begin{aligned} F'' &= \frac{a - a_i}{a + a_i} \cdot f'' \\ f_i'' &= \frac{2a}{a + a_i} \cdot f'' \end{aligned}$$

by differentiating the second and eliminating successively f_i'' and F''

4. Now if ϕ be the angle of incidence,

ϕ , that of refraction,

$r = x \cos \phi + y \sin \phi$ is the space described in a given time without the medium,

$r_i = x_i \cos \phi_i + y_i \sin \phi_i$ within ;

and if $\lambda, \frac{\lambda}{\mu} \left(= \frac{\lambda}{\sin \phi} \sin \phi_i \right)$ be the lengths of the waves respectively,

$$a = l \cdot \frac{\cos \phi}{\lambda} \quad b = l \cdot \frac{\sin \phi}{\lambda}$$

$$a_i = l \cdot \frac{\cos \phi_i}{\lambda} \cdot \frac{\sin \phi}{\sin \phi_i} = \frac{l \cdot \sin \phi \cos \phi_i}{\lambda \sin \phi_i}$$

$$b = l \cdot \frac{\sin \phi_i}{\lambda \sin \phi_i} \sin \phi = \frac{l \sin \phi}{\lambda}$$

$$F'' = \frac{\sin \phi \cos \phi_i - \sin \phi_i \cos \phi}{\sin \phi \cos \phi_i + \sin \phi_i \cos \phi} \cdot f''$$

$$= \frac{\sin (\phi - \phi_i)}{\sin (\phi + \phi_i)} f''$$

$$f_i'' = \frac{2 \cos \phi \sin \phi_i}{\sin (\phi + \phi_i)} \cdot f''$$

The notation F , &c. is the same as that used by Mr GREEN, and the present page is added merely to make the subject complete.

5. These are the results deduced, in a manner apparently widely different, by M. FRESNEL. That the results should coincide is not by any means a matter

of surprise, even supposing in both cases the argument fallacious; for in all the ways of establishing them the same grand assumption extends throughout the whole, viz. that the particles at the common surface of the media have motions resulting from, and conversely affecting, the motion without the latter medium, and that these motions are regulated by the usual laws of the result of forces. Perhaps I shall be better understood if I illustrate my meaning by giving the following demonstration of the results in question.

A particle at the surface is acted on by three sets of forces, in the directions respectively of the directions of incidence, reflection, and refraction: not that the particle is urged in these directions, but is acted by a force which gives it a motion as much depending on the direction as though it were. We have then three forces acting on the particle, and any one may be considered as the resultant of the other two. If this be allowed, we know by the laws of mechanics, that each force is in the proportion of the sine of the angle contained by the other two.

Let then I R and T denote the incident reflected and transmitted vibration;

$$\begin{aligned} \text{then} \quad \frac{R}{I} &= \frac{\sin \overline{\phi - \phi'}}{\sin \overline{\phi + \phi'}} \\ \frac{T}{I} &= \frac{\sin 2 \phi}{\sin \overline{\phi + \phi'}} \\ &= \frac{2 \sin \phi \cos \phi}{\sin \overline{\phi + \phi'}} \\ &= \frac{2 \mu \sin \phi, \cos \phi}{\sin \overline{\phi + \phi'}} \end{aligned}$$

μ being the refractive index.

But when the motion actually takes place within the medium, the length of the wave has to be diminished in the ratio $\frac{1}{\mu} : 1$; if, then, we conceive the new wave to remain similar to the old one, as we doubtless ought, we must diminish the vibration in the same ratio: hence the value of the vibration within the medium is

$$\begin{aligned} \frac{1}{\mu} T &= \frac{1}{\mu} I \cdot \frac{2 \sin \phi \cos \phi}{\sin \overline{\phi + \phi'}} \\ &= I \cdot \frac{2 \sin \phi, \cos \phi}{\sin \overline{\phi + \phi'}} \end{aligned}$$

the same result as before.

This consideration, then, leads us to M. FRESNEL'S formulæ.

6. Next let us adopt the molecular hypothesis, without having recourse to the approximations mentioned in the introduction: then

$$\begin{aligned} \frac{d^2 \gamma}{dt^2} &= \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \delta \gamma \\ \frac{d^2 \gamma_i}{dt^2} &= \Sigma \left(\phi r_i + \frac{\phi' r_i}{r_i} \delta z_i^2 \right) \delta \gamma_i \end{aligned}$$

and if we assume

$$\gamma = a \cos (ex + fy + ct) + b \cos (-ex + fy + ct + g)$$

$$\gamma_1 = a_1 \cos (e_1 x_1 + f_1 y_1 + ct + h)$$

then
$$\begin{aligned} \delta \gamma &= a \cos (ex + e \delta x + fy + ct + f \delta y) - a \cos \overline{ex + fy + ct} \\ &\quad + b \cos (-ex - e \delta x + fy + ct + g + f \delta y) - b \cos (-ex + fy + ct + g) \\ &= -a \cos (ex + fy + ct) (1 - \cos e \delta x + f \delta y) \\ &\quad - a \sin (ex + fy + ct) \sin (e \delta x + f \delta y) + \text{etc.} \end{aligned}$$

Let
$$\begin{array}{ll} ex + fy & \text{be abbreviated by } \rho \\ ex - fy & \text{..... } R \\ e_1 x_1 + f_1 y_1 & \text{..... } \rho_1 \end{array}$$

and
$$\begin{aligned} \delta \gamma &= -a \cos \overline{\rho + ct} (1 - \cos \delta \rho) - a \sin \overline{\rho + ct} \sin \delta \rho \\ &\quad - b \cos (-R + ct + g) (1 - \cos \delta R) + b \sin (-R + ct + g) \sin \delta R \\ &= -I \cdot 2 \sin^2 \frac{\delta \rho}{2} - R \cdot 2 \sin^2 \frac{\delta R}{2} \\ &\quad + \frac{1}{e} \frac{dI}{dx} \sin \delta \rho + \frac{1}{e} \frac{dR}{dx} \sin \delta R \end{aligned}$$

denoting γ by $I + R$.

Now the wave is similarly situated with respect to the line along which R is measured, and that along which ρ is measured; hence

$$\Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \sin^2 \frac{\delta \rho}{2} = \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \sin^2 \frac{\delta R}{2}$$

which gives each of them $= \frac{c^2}{2}$.

For
$$\begin{aligned} \frac{d^2 \gamma}{dt^2} &= -\Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) 2 \sin^2 \frac{\delta \rho}{2} \cdot \gamma \\ \&c. &= \&c. \\ \therefore c^2 &= \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) 2 \sin^2 \frac{e \delta x + f \delta y}{2} \\ &= \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) 2 \sin^2 \frac{e_1 \delta x_1 + f_1 \delta y_1}{2} \end{aligned}$$

The equation corresponding to (3) is

$$\frac{d^2 \gamma}{dt^2} = \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \delta \gamma + \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta z'^2 \right) (\gamma_{x_1 + \delta x_1, y_1 + \delta y_1} - \gamma)$$

and
$$\begin{aligned} \gamma_{x_1 + \delta x_1, y_1 + \delta y_1} - \gamma &= a_1 \cos (e_1 x_1 + e_1 \delta x_1 + f_1 y_1 + f_1 \delta y_1 + ct + h) \\ &\quad - a \cos (ex + fy + ct) - b \cos (-ex + fy + ct + g) \\ &= \gamma_1 \cos \delta \rho_1 + \frac{1}{e_1} \frac{d\gamma_1}{dx_1} \sin \delta \rho_1 - \gamma \end{aligned}$$

$$\begin{aligned} \frac{d^2 \gamma}{dt^2} &= -\Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) 2 \sin^2 \frac{\delta \rho}{2} \cdot \gamma + \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \left(\sin \delta \rho \frac{1}{e} \frac{dI}{dx} + \sin \delta R \frac{1}{e} \frac{dR}{dx} \right) \\ &+ \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta z^2 \right) \left(1 - 2 \sin^2 \frac{\delta \rho}{2} \right) \gamma + \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta z^2 \right) \left(\sin \delta \rho \frac{1}{e} \frac{d\gamma}{dx} \right) - \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta z^2 \right) \cdot \gamma \\ &= -\frac{c^2}{2} \gamma + \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \sin \delta \rho \frac{1}{e} \frac{d\gamma}{dx} - \frac{c^2}{2} \gamma + \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta z^2 \right) \sin \delta \rho \frac{1}{e} \frac{d\gamma}{dx} + \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta z^2 \right) (\gamma - \gamma) \end{aligned}$$

Similarly for the lower medium

$$\begin{aligned} \frac{d^2 \gamma'}{dt^2} &= -\frac{c^2}{2} \overline{\gamma + \gamma'} + \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \sin \delta \rho \frac{1}{e} \frac{d\gamma}{dx} \\ &+ \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta z^2 \right) \sin \delta \rho \frac{1}{e} \frac{d\gamma}{dx} + \Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) (\gamma - \gamma') \end{aligned}$$

7. By substituting for $\frac{d^2 \gamma}{dt^2}$ and $\frac{d^2 \gamma'}{dt^2}$ their values, and calling

$$\Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \sin \delta \rho = P$$

$$\Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \sin \delta \rho = P,$$

$$\Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \dots \dots \dots = Q$$

$$\Sigma \left(\phi r + \frac{\phi' r}{r} \delta z^2 \right) \dots \dots \dots = Q,$$

we get

$$-c^2 \gamma = -\frac{c^2}{2} (\gamma + \gamma') + \frac{P}{e} \frac{d\gamma}{dx} + \frac{P}{e} \frac{d\gamma'}{dx} + Q (\gamma - \gamma')$$

$$-c^2 \gamma' = -\frac{c^2}{2} (\gamma + \gamma') + \frac{P}{e} \frac{d\gamma}{dx} + \frac{P}{e} \frac{d\gamma'}{dx} + Q (\gamma - \gamma')$$

Or

$$0 = \frac{c^2}{2} (\gamma - \gamma') + \frac{P}{e} \frac{d\gamma}{dx} + \frac{P}{e} \frac{d\gamma'}{dx} + Q (\gamma - \gamma')$$

$$0 = \frac{c^2}{2} (\gamma - \gamma') + \frac{P}{e} \frac{d\gamma}{dx} + \frac{P}{e} \frac{d\gamma'}{dx} + Q (\gamma - \gamma')$$

By means of these equations we obtain

$$c^2 (\gamma - \gamma') - \overline{Q + Q} \cdot \overline{\gamma - \gamma'} = 0$$

or

$$(c^2 - Q - Q) (\gamma - \gamma') = 0;$$

And

$$\frac{P}{e} \frac{d\gamma}{dx} + \frac{P}{e} \frac{d\gamma'}{dx} + \overline{Q - Q} \cdot (\gamma - \gamma') = 0$$

From the nature of the functions we cannot have $c^2 = Q + Q,$

$$\therefore \gamma - \gamma' = 0$$

is the only mode of satisfying the first equation, and thus the second equation

becomes

$$\frac{P}{e} \frac{d\gamma}{dx} + \frac{P'}{e'} \frac{d\gamma'}{dx} = 0 \dots\dots (A)$$

We need hardly repeat that the latter hypothesis, by which equations (3) and (4) are deduced and combined, is true only for the particular value $x = 0$.

Now, even without retaining the restrictions imposed on the functions in art. 2, we may shew, by the reasoning used there, that

$$\begin{aligned} \frac{P}{e} + \frac{P'}{e'} &= 0 \\ \therefore \frac{d\gamma}{dx} &= \frac{d\gamma'}{dx} \end{aligned}$$

two conditions which completely satisfy the equation (A)

Hence the final result is, that when $x = 0$

$$\begin{aligned} \gamma &= \gamma', \\ \frac{d\gamma}{dx} &= \frac{d\gamma'}{dx} \end{aligned}$$

and the general values of γ are already determined.

Thus the results above obtained, approximately in the case of particles whose action is insensible at sensible distances, is proved true, *without any approximations*, by the reasoning we have employed.

SECTION II.

ON LIGHT CONSISTING OF VIBRATIONS IN THE PLANE OF INCIDENCE.

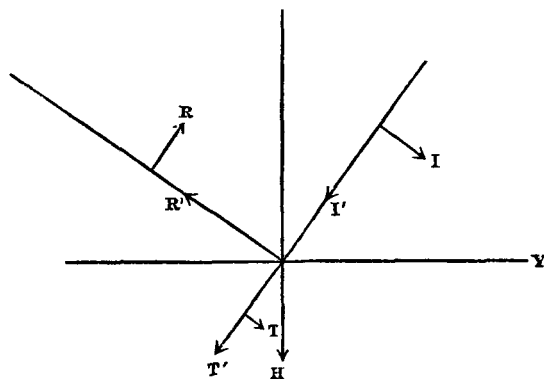
8. We shall assume that it has been demonstrated that light cannot consist of vibrations partly transversal, partly normal, and shall consequently distinguish strictly between a *motion* in the direction of transmission, and a *vibration* in that direction.

At a distance from any break in the state of the molecules, one function will be sufficient to represent the motion of a particle, since any motion not belonging to the type of that function will be transmitted independently of it, and unaffected by it, on the principle of the coexistence of vibrations. When, on the other hand, the state of the particles in the immediate neighbourhood of that under consideration is discontinuous, we cannot assume that a state of motion represented by one type will have no influence on that which is represented by another. On the contrary, we should expect, from the ordinary laws of fluids, that the particular type of the wave itself should undergo a considerable change, and possibly anticipate a reversion of some of the previous axioms by which our calculations were guided. It will be necessary then to retain every term which enters into our expressions, except those only which disappear of themselves by the conditions of symmetry.

Now, before we proceed to apply analogous reasoning to the case of waves whose vibrations take place in the plane of xy , it must be remarked, that, when the motion arrives at the surface, a sudden change takes place. But this sudden change, which occurs necessarily at the first instant the light falls on the surface, will in all the future part of the motion materially affect, not only the vibrations beyond the surface, but those also above it; and the change which takes place is nothing else than that of twisting a vibration which previously had been perpendicular to the direction of motion, so as to cause it no longer to be so. Now, I have shewn in the Transactions of the Cambridge Philosophical Society, vol. vi. p. 180, that a motion perpendicular to the front of the wave cannot be transmitted as a vibration along with the wave. The assumption that it can be so transmitted gives rise to the result that the velocity is an impossible quantity. In other words, some part of the expression which we assumed to be a function of sines and cosines depends on *possible*, as sines and cosines do on impossible, exponentials.

To apply this conclusion to the case in question, we must observe, that, if we reckon along the axis of y , whatever be the motion in question, its value must be a reciprocating one; and further, it is necessary that, whatever value it has for one value of y at a particular time, the same will it have for another value of y at some other time: hence the function which expresses the motion must be a circular function of y and t , but a possible exponential function of x . The motion thus introduced will consequently be a vibration transmitted along the axis of y , and consequently the direction of motion is parallel to the axis of x .

We proceed then to deduce the equations of motion of a particle situated near the common surface of the media, on the hypothesis that the light consists of vibrations in the plane of incidence. As a preliminary step, partly for the purpose of exhibiting the correctness of the method employed, I have deduced the equations of motion of a particle situated at such a distance from the surface that the vibrations transmitted along the axis of x do not affect the forces. Afterwards I have deduced the general equations corresponding to a particle situated at the common surface.



9. We adopt the following notation in addition to that already used :

α, β are the motions parallel to x and y of a particle in the upper medium.

$\alpha, \beta,$ do. do. in the lower.

I, R, T are the incident reflected and refracted vibrations.

$I,$ and $T,$ the corresponding normal motions.

Occasionally δx and δy will be replaced by

$$\delta x' \cos \phi + \delta y' \sin \phi$$

$$\delta y' \cos \phi - \delta x' \sin \phi$$

respectively, when combined with a function depending on the incident wave, and by

$$\delta x'' \cos \phi + \delta y'' \sin \phi$$

$$\delta x'' \sin \phi - \delta y'' \cos \phi$$

when combined with one which depends on the reflected wave.

From the values of δx and δy , it is clear that the axis of x' is the line of transmission at incidence, and that of x'' at reflexion. The values of I, R, are in general not required, but for the purpose of fixing the ideas, they may be conceived to be as follows :

$$I = a \cos (e x + f y + c t)$$

$$R = b \cos (-e x + f y + c t + g)$$

$$T = c \cos (e, x + f y + c t + h)$$

$$I, = A e^{-m x} \cos (f y + c t + \eta)$$

$$T, = C e^{-m, x} \cos (f y + c t + h + \eta,)$$

If it should be thought that these values belong only to a particular case, I would remark that, from the linearity of our equations, the results which we deduce for one circular function, are equally true, *mutatis mutandis*, of a series of such functions.

10. The values of $\alpha, \beta, \alpha, \beta,$ deduced from the figure, are :

$$\alpha = \overline{I - R} \sin \phi + I,$$

$$\beta = (I + R) \cos \phi$$

$$\alpha, = T \sin \phi' + T,$$

$$\beta, = T \cos \phi'$$

The equations of motion in the upper medium are :

$$\frac{d^2 \alpha}{dt^2} = \Sigma \left\{ \phi r + \frac{\phi' r}{r} (\delta x \delta \alpha + \delta y \delta \beta) \right\} \overline{\delta x + \delta \alpha}$$

$$= \Sigma \left(\phi r + \frac{\phi' r}{r} \delta x^2 \right) \delta \alpha + \Sigma \frac{\phi' r}{r} \delta x \delta y \delta \beta$$

$$\frac{d^2 \beta}{dt^2} = \Sigma \left(\phi r + \frac{\phi' r}{r} \delta y^2 \right) \delta \beta + \Sigma \frac{\phi' r}{r} \delta x \delta y \delta \alpha$$

The values thus substituted will, of course, have to be replaced by others, when the particle under consideration is near the common surface of the media. And

$$\delta \alpha = (\delta I - \delta R) \sin \phi + \delta I,$$

but if we adopt the particular values of I, R , &c. which we may do since all values have the same *form*, we have the following results :

$$\begin{aligned} \delta I &= a \cos(e x + f y + c t + e \delta x + f \delta y) - a \cos \overline{e x + f y + c t} \\ &= -a \cos(e x + f y + c t) 2 \sin^2 \left(\frac{e \delta x + f \delta y}{2} \right) - a \sin(e x + f y + c t) \sin(e \delta x + f \delta y) \\ &= -2I \cdot \sin^2 \frac{K x'}{2} + \frac{1}{e} \frac{dI}{dx} \sin K x' \end{aligned}$$

if we denote $e \delta x + f \delta y$ by $K x'$ instead of $k \rho$.

Let us in like manner assume

$$e \delta x - f \delta y = K x''$$

$$e, \delta x + f \delta y = K x,$$

then
$$\delta R = -2R \sin^2 \frac{K x''}{2} + \frac{1}{e} \frac{dR}{dx} \sin K x''$$

$$\delta T = -2T \sin^2 \frac{K x,}{2} + \frac{1}{e,} \frac{dT}{dx} \sin K x,$$

$$\begin{aligned} \delta I, &= A e^{-m x - m \delta x} \cos(f y + c t + \eta + f \delta y) - A e^{-m x} \cos \overline{f y + c t + \eta} \\ &= A e^{-m x} \{ (e^{-m \delta x} \cos f \delta y - 1) \cos \overline{f y + c t + \eta} - e^{-m \delta x} \sin f \delta y \sin \overline{f y + c t + \eta} \} \\ &= -I, (1 - e^{-m \delta x} \cos f \delta y) + \frac{1}{f} \frac{dI,}{dy} e^{-m \delta x} \sin f \delta y \\ \delta T, &= -T, (1 - e^{-m, \delta x} \cos f \delta y) + \frac{1}{f} \frac{dT,}{dy} e^{-m, \delta x} \sin f \delta y \end{aligned}$$

11. Now for a particle at a distance from the common surface δI and δR , vanish ; in such cases the values of $\delta \alpha$ and $\delta \beta$ are

$$\begin{aligned} \delta \alpha &= \sin \phi \left\{ -2I \sin^2 \frac{K x'}{2} + \frac{1}{e} \frac{dI}{dx} \sin K x' + 2R \sin^2 \frac{K x''}{2} - \frac{1}{e} \frac{dR}{dx} \sin K x'' \right\} \\ \delta \beta &= \cos \phi \left\{ -2I \sin^2 \frac{K x'}{2} + \frac{1}{e} \frac{dI}{dx} \sin K x' - 2R \sin^2 \frac{K x''}{2} + \frac{1}{e} \frac{dR}{dx} \sin K x'' \right\} \end{aligned}$$

By substituting these values in the equations in art. 10, and at once omitting terms of the form $\Sigma M \delta x \delta y$, we obtain

$$\begin{aligned} \frac{d^2 \alpha}{dx^2} &= \Sigma \left\{ \phi r + \frac{\phi' r}{r} \overline{\delta x'^2 \cos^2 \phi + \delta y'^2 \sin^2 \phi} \right\} \times \left\{ \sin \phi \left(-2I \sin^2 \frac{K x'}{2} \right) \right\} \\ &+ \Sigma \left\{ \phi r + \frac{\phi' r}{r} \overline{\delta x''^2 \cos^2 \phi + \delta y''^2 \sin^2 \phi} \right\} \left\{ \sin \phi 2R \sin^2 \frac{K x''}{2} \right\} \end{aligned}$$

$$\begin{aligned}
 & + \Sigma \frac{\phi' r}{r} (\delta y'^2 - \delta x'^2) \sin \phi \cos \phi \left(\cos \phi - 2 I \sin^2 \frac{K x'}{2} \right) \\
 & - \Sigma \frac{\phi' r}{r} (\delta y''^2 - \delta x''^2) \sin \phi \cos \phi \left(-2 R \cos \phi \sin^2 \frac{K x''}{2} \right)
 \end{aligned}$$

Now if the law of force be that of the inverse square of the distance,

$$\phi r = \frac{S}{r^3}$$

$$\frac{\phi' r}{r} = -\frac{3 S}{r^5}$$

$$\therefore \phi r + \frac{\phi' r}{r} \delta x'^2 = S \cdot \frac{\delta x'^2 + \delta y'^2 + \delta z'^2 - 3 \delta x'^2}{r^5}$$

And $\Sigma \left(\phi r + \frac{\phi' r}{r} \delta y'^2 \right) 2 \sin^2 \frac{K x'}{2}$ has been already designated c^2

$$\therefore c^2 = 2 S \Sigma \frac{\delta x'^2 + \delta z'^2 - 2 \delta y'^2}{r^5} \sin^2 \frac{K x'}{2}$$

but

$$\Sigma \sin^2 \frac{K x'}{2} \frac{(\delta y'^2 - \delta z'^2)}{r^5} = 0$$

$$\therefore c^2 = 2 S \Sigma \frac{\delta x'^2 - \delta y'^2}{r^5} \sin^2 \frac{K x'}{2}$$

$$= 2 S \Sigma \frac{\delta x''^2 - \delta y''^2}{r^5} \sin^2 \frac{K x''}{2}$$

$$\begin{aligned}
 \text{and } \Sigma \left(\phi r + \frac{\phi' r}{r} \delta x'^2 \right) \sin^2 \frac{K x'}{2} &= 2 S \Sigma \frac{2(\delta y'^2 - \delta x'^2)}{r^5} \sin^2 \frac{K x'}{2} \\
 &= -2 c^2
 \end{aligned}$$

hence by substitution we obtain

$$\begin{aligned}
 \frac{d^2 \alpha}{dt^2} &= -c^2 \sin \phi I (\sin^2 \phi - 2 \cos^2 \phi) + c^2 \sin \phi R (\sin^2 \phi - 2 \cos^2 \phi) - 3 c^2 \sin \phi \cos^2 \phi I + 3 c^2 \sin \phi \cos^2 \phi R \\
 &= -c^2 \sin \phi I + c^2 \sin \phi \cdot R \\
 &= -c^2 (I - R) \sin \phi \\
 &= -c^2 \alpha
 \end{aligned}$$

$$\frac{d^2 \beta}{dt^2} = -c^2 \beta$$

12. This result is obviously correct, and hence we may with confidence apply the same process to the more complicated case, that in which the quantities I , and R , appear, and for which the equations of motion must be found, by taking into account the forces which result from particles on both sides of the surface.

As a preliminary step, we will write down the values of $\delta \alpha$, $\delta \beta$, $\delta \alpha$, and $\delta \beta$. They are

$$\begin{aligned}\delta \alpha &= (\delta I - \delta R) \sin \phi + \delta I, \\ &= -2 I \sin \phi \sin^2 \frac{K x'}{2} + 2 R \sin \phi \sin^2 \frac{K x''}{2} + \frac{d I}{d x} \cdot \frac{1}{e} \sin \phi \sin K x' - \frac{d R}{d x} \cdot \frac{1}{e} \sin \phi \sin K x'' \\ &\quad \left(-I, (1 - e^{-m \delta x} \cos f \delta y) + \frac{d I}{d y} \cdot \frac{1}{f} e^{-m \delta x} \sin f \delta y \right)\end{aligned}$$

$$\delta \beta = (\delta I + \delta R) \cos \phi$$

$$= \cos \phi \left\{ -2 I \sin^2 \frac{K x'}{2} + \frac{1}{e} \frac{d I}{d x} \sin K x' - 2 R \sin^2 \frac{K x''}{2} + \frac{1}{e} \frac{d R}{d x} \sin K x'' \right\}$$

$$\delta \alpha, = \delta T \sin \phi' + \delta T,$$

$$= \sin \phi' \{ c \cdot \cos (e, x, + \delta x, + f y, + \delta y, + c t + g) - c \cos e, x, + f y, + c t + g \}$$

$$+ C e^{-m, x, + \delta x,} \cos (f y, + \delta y, + c t + h + \eta) - C e^{-m, x,} \cos f y, + c t + h + \eta,$$

$$= -2 T \sin \phi' \sin^2 \frac{K x'}{2} + \frac{d T}{d x} \frac{1}{e} \sin \phi' \sin K x' - T, (1 - e^{-m, \delta x} \cos f \delta y) + \frac{d T}{d y} \frac{1}{f} e^{-m, \delta x} \sin f \delta y$$

$$\delta \beta, = \delta T \cos \phi'$$

$$= -2 T \cos \phi' \sin^2 \frac{e, \delta x'}{2} + \frac{1}{e} \frac{d T}{d x} \cos \phi' \sin e, \delta x$$

$$= -2 T \cos \phi' \sin^2 \frac{K x''}{2} + \frac{1}{e} \frac{d T}{d x} \cos \phi' \sin K x''$$

13. The equation of motion in the upper surface parallel to the axis of x is

$$\begin{aligned}\frac{d^2 \alpha}{d t^2} &= \Sigma \left(\phi r + \frac{\phi' r}{r} \delta x^2 \right) \delta \alpha + \Sigma \frac{\phi' r}{r} \delta x \delta y \delta \beta \\ &\quad + \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta x'^2 \right) (\alpha, -\alpha) + \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' (\beta, -\beta)\end{aligned}$$

retaining the limitations to Σ .

$$\text{But } \Sigma \left(\phi r + \frac{\phi' r}{r} \delta x^2 \right) \delta \alpha =$$

$$\begin{aligned}&\Sigma \left(\phi r + \frac{\phi' r}{r} \overline{\delta x'^2 \cos^2 \phi + \delta y'^2 \sin^2 \phi} \right) \sin \phi \times \left\{ -2 I \sin^2 \frac{K x'}{2} + \frac{1}{e} \frac{d I}{d x} \sin K x' \right\} \\ &+ \Sigma \left(\phi r + \frac{\phi' r}{r} \overline{\delta x''^2 \cos^2 \phi + \delta y''^2 \sin^2 \phi} \right) \sin \phi \times \left\{ 2 R \sin^2 \frac{K x''}{2} - \frac{1}{e} \frac{d R}{d x} \sin K x'' \right\} \\ &+ \Sigma \left(\phi r + \frac{\phi' r}{r} \delta x^2 \right) \times \left\{ -I, (1 - e^{-m \delta x} \cos f \delta y) \right\} \\ &= -\frac{e^2}{2} \sin \phi (\sin^2 \phi - 2 \cos^2 \phi) (I - R) \\ &\quad + \frac{M}{e} \sin \phi (\sin^2 \phi - 2 \cos^2 \phi) \left(\frac{d I}{d x} - \frac{d R}{d x} \right) - D I,\end{aligned}$$

where

$$M = \Sigma \left(\phi r + \frac{\phi' r}{r} \delta y^2 \right) \sin K x'$$

$$D = \Sigma \left(\phi r + \frac{\phi' r}{r} \delta x^2 \right) \left(1 - e^{-m \delta x} \cos f \delta y \right)$$

$$\begin{aligned} \Sigma \frac{\phi' r}{r} \delta x \delta y \delta \beta &= \Sigma \frac{\phi' r}{r} (\delta y^2 - \delta x^2) \sin \phi \cos^2 \phi \times \left\{ -2 I \sin^2 \frac{K x'}{2} + \frac{1}{e} \frac{d I}{d x} \sin K x' \right\} \\ &- \Sigma \frac{\phi' r}{r} (\delta y'^2 - \delta x'^2) \sin \phi \cos^2 \phi \times \left\{ -2 R \sin^2 \frac{K x''}{2} + \frac{1}{e} \frac{d R}{d x} \sin K x'' \right\} \\ &= -\frac{3 c^2}{2} \sin \phi \cos^2 \phi (I - R) + 3 \frac{M}{e} \sin \phi \cos^2 \phi \left(\frac{d I}{d x} - \frac{d R}{d x} \right) \end{aligned}$$

By adding this term to that which we have just found, the sum is

$$-\frac{c^2}{2} \sin \phi (I - R) + \frac{M}{e} \sin \phi \left(\frac{d I}{d x} - \frac{d R}{d x} \right) - D I.$$

$$\begin{aligned} &\Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta x'^2 \right) (\alpha', -\alpha) = \\ &\Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta x'^2 \right) \left\{ \overline{T \sin \phi' + T_{x+\delta x, y+\delta y}} - \overline{I - R} \sin \phi - I \right\} \\ &= \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta x'^2 \right) \left\{ \delta T \sin \phi' + \delta T + T \sin \phi' + T - \overline{I - R} \sin \phi - I \right\} \\ &= \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta x'^2 \right) (\alpha, -\alpha) \\ &+ \Sigma \left(\phi r' + \frac{\phi' r'}{r'} (\delta x'^2 \cos \phi' + \delta y'^2 \sin^2 \phi') \right) \sin \phi' \times \left\{ -2 T \sin^2 \frac{K x'}{2} + \frac{1}{e} \frac{d T}{d x} \sin K x' \right\} \\ &+ \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta x'^2 \right) \left\{ -T (1 - e^{-m \delta x} \cos f \delta y) + \frac{1}{f} \frac{d T}{d y} e^{-m \delta x} \sin f \delta y \right\} \\ &= Q (\alpha, -\alpha) - \frac{c^2}{2} T \sin \phi' (\sin^2 \phi' - 2 \cos^2 \phi') + \frac{M}{e} \frac{d T}{d x} \sin \phi' (\sin^2 \phi' - 2 \cos^2 \phi') - D T, \end{aligned}$$

$$\begin{aligned}
\Sigma \frac{\phi' r'}{r'} \delta x' \delta y' (\beta' - \beta) &= \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' \{ \delta \beta + \beta - \beta \} = \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' \delta \beta, \\
&= \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' \delta T \cos \phi' \\
&= \Sigma \frac{\phi' r'}{r'} (\delta y'^2 - \delta x'^2) \sin \phi' \cos^2 \phi' \times \left\{ -2 T \sin^2 \frac{K x'}{2} + \frac{1}{e} \frac{dT}{dx} \sin K x' \right\} \\
&= -\frac{3 c^2}{2} T \sin \phi' \cos^2 \phi' + \frac{3 M}{e} \frac{dT}{dx} \sin \phi' \cos^2 \phi'
\end{aligned}$$

By adding this term to that just found, we get

$$Q, (\alpha, -\alpha) - \frac{c^2}{2} T \sin \phi' + \frac{M}{e} \frac{dT}{dx} \sin \phi' - D, T,$$

Hence

$$\frac{d^2 \alpha}{d t^2} = -\frac{c^2}{2} \{ \overline{I - R} \sin \phi + T \sin \phi' \} + Q, (\alpha, -\alpha) + \frac{M}{e} \sin \phi \left(\frac{dI}{dx} - \frac{dR}{dx} \right) + \frac{M}{e} \sin \phi' \frac{dT}{dx} - D I, -D, T,$$

14. From this equation we obtain, by interchanging the quantities $(I - R) \sin \phi$, $T \sin \phi'$ &c.

$$\begin{aligned}
\frac{d^2 \alpha}{d t^2} &= -\frac{c^2}{2} (\overline{I - R} \sin \phi + T \sin \phi') \\
&+ Q (\alpha - \alpha) + \frac{M}{e} \sin \phi \left(\frac{dI}{dx} - \frac{dR}{dx} \right) + \frac{M}{e} \sin \phi' \frac{dT}{dx} \\
&- D, T, -D I,
\end{aligned}$$

By subtraction

$$\begin{aligned}
\frac{d^2 \alpha}{d t^2} - \frac{d^2 \alpha}{d t^2} &= (Q, + Q) (\alpha, -\alpha) \\
&= -(Q, + Q) (\alpha - \alpha)
\end{aligned}$$

Now $Q, + Q$ differs from c^2 by a finite quantity: hence this equation can only be satisfied by making

$$\begin{aligned}
\alpha, - \alpha &= 0 \\
\frac{d^2 \alpha}{d t^2} - \frac{d^2 \alpha}{d t^2} &= 0
\end{aligned}$$

the second of which equations is a consequence of the first.

By adding the two equations we get

$$\begin{aligned}
\frac{d^2 \alpha}{d t^2} + \frac{d^2 \alpha}{d t^2} &= -c^2 (\overline{I - R} \sin \phi + T \sin \phi') \\
&+ (Q - Q) (\alpha - \alpha) - 2 D, I, -2 D, T, \\
&+ \frac{2 M}{e} \sin \phi \left(\frac{dI}{dx} - \frac{dR}{dx} \right) + \frac{2 M}{e} \sin \phi' \frac{dT}{dx}
\end{aligned}$$

Now the sum of the two quantities which constitute the first line is

$$-c^2 (\bar{I} - \bar{R} \sin \phi + T \sin \phi') - c^2 (I + T)$$

And, from the nature of the functions, the last two lines of the above equation cannot, when $x=0$, give any part of the quantity $-c^2 (\alpha + \alpha')$, for the one involves sines of the same quantities whose cosines constitute the other; hence we must have separately equal to 0 the two following expressions, viz.

$$(c^2 - 2D) I + \bar{c}^2 - 2\bar{D}, T, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{and} \quad \frac{M}{e} \sin \phi \left(\frac{dI}{dx} - \frac{dR}{dx} \right) + \frac{M'}{e'} \sin \phi' \frac{dT}{dx} \quad . \quad . \quad . \quad . \quad (2)$$

The former equation gives $I + T = 0$, for D , and \bar{D} are the same thing. Hence

$$\alpha + \alpha' = \bar{I} - \bar{R} \sin \phi + T \sin \phi'$$

$$\text{and} \quad \frac{d^2(\alpha + \alpha')}{dt^2} = -c^2 (\alpha + \alpha')$$

as it ought to be.

On the second of the above equations we shall make some remarks after we have deduced the equations for the motion parallel to the surface.

15. It would be rather difficult to write down the equation for β from the equation for α , I shall therefore briefly deduce it.

$$\begin{aligned} \frac{d^2 \beta}{dt^2} = & \Sigma \left(\phi r + \frac{\phi' r}{r} \delta y^2 \right) \delta \beta + \Sigma \frac{\phi' r}{r} \delta x \delta y \delta \alpha \\ & + \Sigma \left(\phi r' + \frac{\phi' r'}{r'} \delta y^2 \right) (\beta' - \beta) + \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' (\alpha' - \alpha) \end{aligned}$$

$$\text{Now} \quad \Sigma \left(\phi r + \frac{\phi' r}{r} \delta y^2 \right) \delta \beta =$$

$$\begin{aligned} & \left\{ \Sigma \phi r + \frac{\phi' r}{r} (\delta x'^2 \sin^2 \phi + \delta y'^2 \cos^2 \phi) \right\} \cos \phi \\ & \times \left\{ -2I \sin^2 \frac{Kx'}{2} + \frac{1}{e} \frac{dI}{dx} \sin Kx' - 2R \sin^2 \frac{Kx''}{2} + \frac{1}{e} \frac{dR}{dx} \sin Kx'' \right\} \\ & = -\frac{c^2}{2} \cos \phi (I + R) (\cos^2 \phi - 2 \sin^2 \phi) + \frac{M}{e} \left(\frac{dI}{dx} + \frac{dR}{dx} \right) \cos \phi (\cos^2 \phi - 2 \sin^2 \phi) \end{aligned}$$

Again, if we denote $\Sigma \frac{\phi' r}{r} \delta x \delta y e^{-m \delta x} \sin f \delta y$ by F , we obtain

$$\begin{aligned} \Sigma \frac{\phi' r}{r} \delta x \delta y \delta \alpha = & -\frac{3c^2}{2} \sin^2 \phi \cos \phi (I + R) \\ & + \frac{3M}{e} \sin^2 \phi \cos \phi \left(\frac{dI}{dx} + \frac{dR}{dx} \right) + \frac{F}{f} \frac{dI}{dy} \end{aligned}$$

by adding this term to the former we obtain

$$-\frac{c^2}{2} \cos \phi (I + R) + \frac{M}{e} \cos \phi \left(\frac{dI}{dx} + \frac{dR}{dx} \right) + \frac{F}{f} \frac{dI}{dy}$$

Again

$$\begin{aligned} & \Sigma \left(\phi' r' + \frac{\phi' r'}{r'} \delta y'^2 \right) (\beta' - \beta) \\ &= \Sigma \left(\phi' r' + \frac{\phi' r'}{r'} \delta y'^2 \right) (\delta \beta, + \beta, - \beta) \\ &= Q, (\beta, - \beta) - \frac{c^2}{2} T \cos \phi' (\cos^2 \phi' - 2 \sin^2 \phi') + \frac{M,}{e,} \frac{dT}{dx} \cos \phi' (\cos^2 \phi' - 2 \sin^2 \phi') \end{aligned}$$

and

$$\begin{aligned} & \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' (\alpha' - \alpha) = \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' \delta \alpha' \\ &= -\frac{3c^2}{2} T \cos \phi' \sin^2 \phi' + \frac{3M,}{e,} \frac{dT}{dx} \sin^2 \phi' \cos \phi' + \frac{F,}{f} \frac{dT,}{dy} \end{aligned}$$

which being added to the previous term gives

$$Q, (\beta, - \beta) - \frac{c^2}{2} T \cos \phi' + \frac{M,}{e,} \frac{dT}{dx} \cos \phi' + \frac{F,}{f} \frac{dT,}{dy}$$

Hence

$$\begin{aligned} \frac{d^2 \beta}{dt^2} &= -\frac{c^2}{2} (\bar{I} + \bar{R} \cos \phi + T \cos \phi') \\ &+ Q, (\beta, - \beta) + \frac{M}{e} \cos \phi \left(\frac{dI}{dx} + \frac{dR}{dx} \right) + \frac{M,}{e,} \cos \phi' \frac{dT}{dx} \\ &+ \frac{F}{f} \frac{dI,}{dy} + \frac{F,}{f} \frac{dT,}{dy} \end{aligned}$$

Similarly

$$\begin{aligned} \frac{d^2 \beta,}{dt^2} &= -\frac{c^2}{2} (\bar{I} + \bar{R} \cos \phi + T \cos \phi') \\ &+ Q (\beta - \beta,) + \frac{M}{e} \cos \phi \left(\frac{dI}{dx} + \frac{dR}{dx} \right) + \frac{M,}{e,} \cos \phi' \frac{dT}{dx} \\ &+ \frac{F}{f} \frac{dI,}{dy} + \frac{F,}{f} \frac{dT,}{dy} \end{aligned}$$

16. By the same mode which we exhibited for α and α , we can shew that

$$\frac{d^2 \beta}{dt^2} - \frac{d^2 \beta,}{dt^2} = -\overline{Q + Q,} \overline{\beta - \beta,}$$

and

$$\therefore \beta = \beta,$$

also

$$\begin{aligned} \frac{d^2 \beta}{dt^2} + \frac{d^2 \beta,}{dt^2} &= -c^2 (\beta + \beta,) \\ &+ \frac{2M}{e} \cos \phi \left(\frac{dI}{dx} + \frac{dR}{dx} \right) + \frac{2M,}{e,} \cos \phi' \frac{dT}{dx} \\ &+ \frac{2F}{f} \frac{dI,}{dy} + \frac{2F,}{f} \frac{dT,}{dy} \end{aligned}$$

and since

$$\frac{d^2\beta}{dt^2} + \frac{d^2\beta'}{dt^2} = -c^2(\beta + \beta')$$

from the nature of the case, it follows that

$$\begin{aligned} \frac{M}{e} \cos \phi \left(\frac{dI}{dx} + \frac{dR}{dx} \right) + \frac{M'}{e'} \cos \phi' \frac{dT}{dx} \\ + \frac{F}{f} \frac{dI'}{dy} + \frac{F'}{f'} \frac{dT'}{dy} = 0 \dots\dots\dots (3) \end{aligned}$$

It only remains that we find the values of M, M', F, F' , and substitute them in the five equations

$$(I - R) \sin \phi + I' = T \sin \phi' + T', \dots (1)$$

$$(I + R) \cos \phi = T \cos \phi' \dots\dots\dots (2)$$

$$I' + T' = 0 \dots\dots\dots (3)$$

$$\frac{M}{e} \sin \phi \left(\frac{dI}{dx} - \frac{dR}{dx} \right) + \frac{M'}{e'} \sin \phi' \frac{dT}{dx} = 0 \dots\dots\dots (4)$$

and

$$\frac{M}{e} \cos \phi \left(\frac{dI}{dx} + \frac{dR}{dx} \right) + \frac{M'}{e'} \cos \phi' \frac{dT}{dx} + \frac{F}{f} \frac{dI'}{dy} + \frac{F'}{f'} \frac{dT'}{dy} = 0 \dots\dots\dots (5)$$

Now, we have already shewn that

$$\frac{M}{e} + \frac{M'}{e'} = 0$$

and in precisely the same manner it appears that

$$\frac{F}{f} + \frac{F'}{f'} = 0$$

17. By substituting in equation (4) of the last article, we deduce

$$\cos \phi (I' - R') - \cos \phi' T' = 0$$

where I', R', T' are the differential coefficients of I, R , and T .

But if we differentiate (2), we obtain the same result; hence equation (4) is a result of (2), and cannot be employed in our calculation.

Now

$$\begin{aligned} e &= \frac{\cos \phi}{\lambda} & e' &= \frac{\cos \phi'}{\lambda'} = \frac{\mu \cos \phi'}{\lambda} = \frac{\sin \phi}{\lambda \sin \phi'} \cos \phi' \\ f &= \frac{\sin \phi}{\lambda} \end{aligned}$$

Hence equation (4) becomes

$$\frac{M}{e} \left\{ \frac{\cos^2 \phi}{\lambda} (I - R) - \frac{\sin \phi \cos^2 \phi'}{\lambda \sin \phi'} T \right\} + \frac{F}{f} \left\{ \frac{\sin \phi}{\lambda} I' - \frac{\sin \phi}{\lambda} T' \right\} = 0$$

Or
$$(I-R) \frac{\cos^2 \phi}{\sin \phi} - T \frac{\cos^2 \phi'}{\sin \phi'} = -\frac{e F}{f M} (I-T)$$

$$= -\frac{e F}{f M} \cdot 2 I, \text{ by means of (3).}$$

Now
$$\frac{M}{e} = \frac{1}{e} \Sigma \left(\phi r + \frac{\phi' r}{r} \delta y^2 \right) \sin K x'$$

$$= \frac{S}{e} \Sigma \cdot \frac{\delta x^2 + \delta z^2 - 2 \delta y^2}{r^5} \sin \delta x'$$

$$-\frac{F}{f} = +\frac{S}{f} \frac{3 \delta x \delta y}{r^5} e^{-m \delta x} \sin f \delta y$$

and the quantities on the right hand sides of these two equations, are the coefficients respectively of terms which result from forces arising from a motion perpendicular to that of transmission, but extending only half through the system. There are, in fact, two terms arising from this cause, the one corresponding to the vibratory motion each, and having its value c^2 in both, and the other the term in question.

Hence we conclude, that

$$\frac{M}{e} = -\frac{F}{f}$$

Our equation (4) is by this means reduced to

$$\frac{\cos^2 \phi}{\sin \phi} (I-R) - \frac{\cos^2 \phi'}{\sin \phi'} T = 2 I,$$

and
$$(I-R) \sin \phi = T \sin \phi' - 2 I, \text{ by (1).}$$

By addition

$$(I-R) \left(\frac{\cos^2 \phi}{\sin \phi} + \sin \phi \right) = T \left(\frac{\cos^2 \phi'}{\sin \phi'} + \sin \phi' \right)$$

or
$$\frac{(I-R)}{\sin \phi} = \frac{T}{\sin \phi'}$$

or
$$(I-R) \sin \phi' = T \sin \phi$$

and by (2)
$$(I+R) \cos \phi = T \cos \phi'$$

$$\therefore (I-R) \sin 2 \phi' = (T+R) \sin 2 \phi$$

$$I (\sin 2 \phi' - \sin 2 \phi) = R (\sin 2 \phi' + \sin 2 \phi)$$

$$R = -I \cdot \frac{\sin 2 \phi - \sin 2 \phi'}{\sin 2 \phi + \sin 2 \phi'}$$

$$= -I \frac{\tan \phi - \tan \phi'}{\tan \phi + \tan \phi'}$$

Again, by eliminating R, we obtain

$$I \sin \phi' \cos \phi + I \sin \phi' \cos \phi = T \sin \phi \cos \phi + T \sin \phi' \cos \phi'$$

$$\begin{aligned} T &= 2 \cdot I \cdot \frac{\sin \phi' \cos \phi}{\sin \phi \cos \phi + \sin \phi' \cos \phi'} \\ &= 2 I \cdot \frac{\cos \phi}{\cos \phi'} \left\{ \frac{\sin \phi'}{\sin \phi \frac{\cos \phi}{\cos \phi'} + \sin \phi'} \right\} \\ &= I \cdot \frac{\cos \phi}{\cos \phi'} \left\{ 1 - \frac{\sin \phi \frac{\cos \phi}{\cos \phi'} - \sin \phi'}{\sin \phi \frac{\cos \phi}{\cos \phi'} + \sin \phi'} \right\} \\ &= I \cdot \frac{\cos \phi}{\cos \phi'} \left\{ 1 - \frac{\sin 2 \phi - \sin 2 \phi'}{\sin 2 \phi + \sin 2 \phi'} \right\} \\ &= I \cdot \frac{\cos \phi}{\cos \phi'} \left\{ 1 - \frac{\tan(\phi - \phi')}{\tan \phi + \phi'} \right\} \end{aligned}$$

These are precisely FRESNEL's results; in fact, the equation (2) corresponds with his empirical formula.

In conclusion, I have only to observe, that some of the equations involve what appears almost too accurate a substitution to be called an approximation, but which may in some extreme cases give rise to considerable deviation from the resulting formulæ. It will not, however, repay us for the labour of entering into the discussion of such points; suffice it to say, that the more *deviation* a ray suffers, the greater is the difference between the assumed and the real value of some of the forces. Except, however, the deviation be very great indeed, they cannot differ widely from each other.

EDINBURGH, *February* 4. 1839.