13.

Nova Theoremata de functionum Abelianarum cuiusque ordinis valoribus quibus pro complementis argumentorum atque indicum dimidiis induuntur.

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Postquam ad instar functionum ellipticarum, ab illustrissimis ***Jacobi*** et ***Abel*** inventarum, ille vir celeberrimus inversis integralium ***Abelianorum*** functionibus, quae plures variabiles involvunt, introductis analyseos fines amplius protulit, quae in theoria ellipticarum soluta vident, eorum similia problemata de his functionibus naturae multo sublimioris ac fere inauditae Geometris tractanda se offerunt. Quae disquisitiones de functionibus ***Abelianis*** sive ultraellipticis novis quodammodo analyseos fundamentis superstruenda videntur. Qui enim per Theorema ***Abelianum***, quippe quod pro unico aditu aperto ad huius theoriae fundamenta habent, calculo integrali adhibito penetrare velint in haec mysteria recondita, ii maximi calculi impedimenta sibi obvenire videbunt. Quae quomodocunque se habeant, in hac dissertatione unum e numero illorum problematum per calculum algebraicum haud inelegantem, calculo integrali advocato, tractavimus. Nimirum hic agitur de hoc problemate: Si $e_i = \pm 1$ et brevitatis gratia ponitur:

$$\varDelta z = -(z-m_1)(z-m_2)\dots(z-m_{2n+2}),$$

ubi differentiae:

$$m_2-m_1,\quad m_3-m_2,\quad \dots\dots\quad m_{2n+2}-m_{2n+1},$$

positivis valoribus gaudent, argumenta

$$u,\; u',\; u'',\; \dots\dots\; u^{(n-1)},$$

definiantur per aequationes has:

$$\int_{m_1}^{y_1}\frac{e_1\,dy}{\sqrt{(\varDelta y)}}+\int_{m_3}^{y_2}\frac{e_2\,dy}{\sqrt{(\varDelta y)}}+\dots+\int_{m_{2n-1}}^{y_n}\frac{e_n\,dy}{\sqrt{(\varDelta y)}} = 2u,$$

$$\int_{m_1}^{y_1}\frac{e_1\,y\,dy}{\sqrt{(\varDelta y)}}+\int_{m_3}^{y_2}\frac{e_2\,y\,dy}{\sqrt{(\varDelta y)}}+\dots+\int_{m_{2n-1}}^{y_n}\frac{e_n\,y\,dy}{\sqrt{(\varDelta y)}} = 2u',$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\int_{m_1}^{y_1}\frac{e_1\,y^{n-1}\,dy}{\sqrt{(\varDelta y)}}+\int_{m_3}^{y_2}\frac{e_2\,y^{n-1}\,dy}{\sqrt{(\varDelta y)}}+\dots+\int_{m_{2n-1}}^{y_n}\frac{e_n\,y^{n-1}\,dy}{\sqrt{(\varDelta y)}} = 2u^{(n-1)},$$

atque introducantur functiones $\lambda_1(u, u', u'', \ldots. u^{(n-1)})$, $\lambda_2(u, u', u'', \ldots. u^{(n-1)})$ etc. .... atque $G(u, u', u'', \ldots. u^{(n-1)})$ his aequationibus determinatae:

$$\lambda_1(u, u', u'', \ldots. u^{(n-1)}) = (m_1 - y_1)(m_1 - y_2) \ldots. (m_1 - y_n),$$
$$\lambda_2(u, u', u'', \ldots. u^{(n-1)}) = (m_2 - y_1)(m_2 - y_2) \ldots. (m_2 - y_n),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\lambda_{2n+2}(u, u', u'', \ldots. u^{(n-1)}) = (m_{2n+2} - y_1)(m_{2n+2} - y_2)\ldots.(m_{2n+2} - y_n),$$
$$G(u, u', u'', \ldots. u^{(n-1)})$$
$$= \int_{m_1}^{y_1} e_1 \frac{\Phi y\, dy}{(\alpha - y)\sqrt{(\Delta y)}} + \int_{m_3}^{y_2} e_2 \frac{\Phi y\, dy}{(\alpha - y)\sqrt{(\Delta y)}} + \ldots. + \int_{m_{2n-1}}^{y_n} e_n \frac{\Phi y\, dy}{(\alpha - y)\sqrt{(\Delta y)}},$$

ubi Φz denotat functionem rationalem ipsius z integram, et α quantitatem constantem. Iam, si valores argumentorum $u, u', u'', \ldots. u^{(n-1)}$ pro

$$y_1 = m_2, \quad y_2 = m_4, \quad y_3 = m_6, \quad \ldots. \quad y_n = m_{2n},$$

respective transeunt in $M, M', M'', \ldots. M^{(n-1)}$, quaeruntur et functionum:

$$\lambda_1(M - u, M' - u', M'' - u'', \ldots. M^{(n-1)} - u^{(n-1)}),$$
$$\lambda_2(M - u, M' - u', M'' - u'', \ldots. M^{(n-1)} - u^{(n-1)}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\lambda_{2n+2}(M - u, M' - u', M'' - u'', \ldots. M^{(n-1)} - u^{(n-1)}),$$
$$G(M - u, M' - u', M'' - u'', \ldots. M^{(n-1)} - u^{(n-1)}),$$

per ipsas quantitates $y_1, y_2, \ldots. y_n$ expressiones, nec non valores ipsorum:

$$\lambda_1(\tfrac{1}{2}M, \tfrac{1}{2}M', \ldots. \tfrac{1}{2}M^{(n-1)}),$$
$$\lambda_2(\tfrac{1}{2}M, \tfrac{1}{2}M', \ldots. \tfrac{1}{2}M^{(n-1)}),$$
$$\cdots\cdots\cdots\cdots$$
$$\lambda_{2n+2}(\tfrac{1}{2}M, \tfrac{1}{2}M', \ldots. \tfrac{1}{2}M^{(n-1)}),$$
$$G(\tfrac{1}{2}M, \tfrac{1}{2}M', \ldots. \tfrac{1}{2}M^{(n-1)}).$$

In his disquisitionibus versante me, illustrissimus *Jacobi,* ingenio augustissimo ductus, quas ipsas functiones *Abelianas* plures variabiles involventes olim in Analysin introduxerat, e novis unius tantum variabilis functionibus theorematis *Abeliani* ope algebraice adnotavit componi (id quod in diario ab illustrissimo *Liouville* edito dec. 1843 pag. 505 invenis), ita ut in has simpliciores functiones, quippe in quas ceterae reducantur, ante omnes inquirendum esse videatur; tamen quin hanc elaborationem Geometris communicarem, haud dubitavi, et per se eorum attentione haud prorsus indignam, et ut singularem theorematis *Abeliani* applicationem.

1.

Initium, ut problematis natura melius perspiciatur, a casu simplicissimo facere velimus $n=1$, ubi functio λu congruit cum ipso $\sin^2 \mathrm{am}(u, \varkappa)$, atque expressiones ipsorum:

$$\sin \mathrm{am}(K-u), \qquad \sin \mathrm{am}\, \tfrac{1}{2} K,$$

determinandae sunt. Quem ad valorem notissimum eruendum, solvere placet hoc generalius problema algebraicum:

Si differentiae: $m_2 - m_1$, $m_3 - m_2$, $m_4 - m_3$, positivis valoribus gaudent, in expressione secundi ordinis:

$$1. \quad c(z-m_3)(z-m_4)+(z-m_1)(z-m_2)$$

quantitas c ita est determinanda, ut ipsa fiat quadratum formae:

$$2. \quad (1+c)(z-x_1)^2 = c(z-m_3)(z-m_4)+(z-m_1)(z-m_2)$$

ipsaque quantitas x_1 quaerenda est.

Quem ad finem in aequatione (2.) ipsiusque differentiali secundum ipsum z sumto, si substituitur: $z=x_1$ prodeunt formulae:

$$c(x_1-m_3)(x_1-m_4) + (x_1-m_1)(x_1-m_2) = 0,$$
$$c[(x_1-m_3)+(x_1-m_4)]+(x_1-m_1)+(x_1-m_2) = 0,$$

et inde, ipso c eliminato, aequatio ad ipsum x_1 determinandum

$$3. \quad \frac{1}{x_1-m_4}+\frac{1}{x_1-m_3} = \frac{1}{x_1-m_2}+\frac{1}{x_1-m_1},$$

quae docet alterum ipsius x_1 valorem

$$4. \quad x_1' = \frac{m_3\sqrt{[(m_4-m_1)(m_4-m_2)]}-m_4\sqrt{[(m_3-m_1)(m_3-m_2)]}}{\sqrt{[(m_4-m_1)(m_4-m_2)]}-\sqrt{[(m_3-m_1)(m_3-m_2)]}}$$

in intervallo $m_1 \ldots . m_2$, et alterum:

$$5. \quad x_1'' = \frac{m_3\sqrt{[(m_4-m_1)(m_4-m_2)]}+m_4\sqrt{[(m_3-m_1)(m_3-m_2)]}}{\sqrt{[(m_4-m_1)(m_4-m_2)]}+\sqrt{[(m_3-m_1)(m_3-m_2)]}}$$

in intervallo $m_3 \ldots . m_4$ contineri. Quantitas $\sqrt{c}$ in priori casu valore:

$$6. \quad \sqrt{c'} = \frac{\sqrt{[(m_3-m_1)(m_4-m_2)]}-\sqrt{[(m_3-m_2)(m_4-m_1)]}}{m_4-m_3},$$

in posteriorique valore:

$$7. \quad \sqrt{c''} = \frac{\sqrt{[(m_3-m_1)(m_4-m_2)]}+\sqrt{[(m_3-m_2)(m_4-m_1)]}}{m_4-m_3}$$

induitur.

Iam vero aequationis quadraticae:

$$C(z-m_4)(z-m_3)+(z-m_2)(z-m_1) = 0,$$

radices sint y_1 et Y_1, ita ut habeantur aequationes:

$$C(y_1-m_4)(y_1-m_3) = -(y_1-m_2)(y_1-m_1),$$
$$C(Y_1-m_4)(Y_1-m_3) = -(Y_1-m_2)(Y_1-m_1),$$

quibus logarithmice differentiatis, prodeunt formulae differentiales:

$$\frac{1}{C}\cdot\frac{dC}{dy_1} = \frac{1}{(y_1-m_1)} + \frac{1}{(y_1-m_2)} - \frac{1}{y_1-m_3} - \frac{1}{y_1-m_4},$$
$$\frac{1}{C}\cdot\frac{dC}{dY_1} = \frac{1}{Y_1-m_1} + \frac{1}{Y_1-m_2} - \frac{1}{Y_1-m_3} - \frac{1}{Y_1-m_4}.$$

Inde docemur quantitatem C, radicibus y_1 et Y_1 respective ab m_1 usque ad x_1' et ab m_2 usque ad x_1' continuo progredientibus, ipsam a nihilo usque ad c' continuo crescere, nec non brevitatis gratia posito:

$$\Delta z = -(z-m_1)(z-m_2)(z-m_3)(z-m_4)$$

simul haberi:

$$8.\quad \sqrt{C} = \frac{\sqrt{(\Delta y_1)}}{(y_1-m_3)(y_1-m_4)} = \frac{\sqrt{(\Delta Y_1)}}{(Y_1-m_3)(Y_1-m_4)};$$

aeque ac, dum radices y_1 et Y_1 respective ab m_3 et m_4 usque ad x_1'' continuo pergant, ipsam C ab infinito usque ad c'' continuo decrescere, simulque fore:

$$9.\quad \sqrt{C} = -\frac{\sqrt{(\Delta y_1)}}{(y_1-m_3)(y_1-m_4)} = -\frac{\sqrt{(\Delta Y_1)}}{(Y_1-m_3)(Y_1-m_4)}.$$

Aequatio vero identica:

$$10.\quad C(z-m_4)(z-m_3) + (z-m_2)(z-m_1) = (C+1)(z-y_1)(z-Y_1),$$

has suppeditat:

$$11.\quad \begin{cases} (m_4-m_2)(m_4-m_1) = (C+1)(y_1-m_4)(Y_1-m_4), \\ (m_3-m_2)(m_3-m_1) = (C+1)(y_1-m_3)(Y_1-m_3), \\ C(m_4-m_2)(m_3-m_2) = (C+1)(y_1-m_2)(Y_1-m_2), \\ C(m_4-m_1)(m_3-m_1) = (C+1)(y_1-m_1)(Y_1-m_1), \end{cases}$$

unde prodeunt formulae:

$$12.\quad \frac{(m_3-m_2)(m_3-m_1)}{(m_4-m_2)(m_4-m_1)} = \frac{(m_3-y_1)(m_3-Y_1)}{(m_4-y_1)(m_4-Y_1)},$$

$$13.\quad \sqrt{C} = \sqrt{\left(\frac{(m_4-m_2)(m_4-m_1)}{(m_4-m_1)(m_3-m_1)}\right)}\sqrt{\left(\frac{(y_1-m_1)(Y_1-m_1)}{(m_4-y_1)(m_4-Y_1)}\right)}.$$

Aequatione (10.) et aequatione utraque priori (11.) logarithmice differentiatis, emanant hae formulae:

$$\frac{dC(z-m_4)(z-m_3)}{C(z-m_4)(z-m_3)+(z-m_2)(z-m_1)} - \frac{dC}{1+C} = -\frac{dy_1}{z-y_1} - \frac{dY_1}{z-Y_1},$$
$$\frac{dC}{1+C} = \frac{dy_1}{m_4-y_1} + \frac{dY_1}{m_4-Y_1},$$
$$\frac{dC}{1+C} = \frac{dy_1}{m_3-y_1} + \frac{dY_1}{m_3-Y_1};$$

quarum prima et secunda per $\frac{1}{(z-m_4)(m_4-m_3)\sqrt{C}}$ multiplicatis,

prima et tertia per $\frac{1}{(z-m_3)(m_3-m_4)\sqrt{C}}$ multiplicatis,

additioneque facta, prodit haec denique aequatio:

$$-\frac{dC}{\sqrt{C}\{C(z-m_4)(z-m_3)+(z-m_2)(z-m_1)\}}$$
$$=\frac{dy_1}{(z-y_1)(m_4-y_1)(m_3-y_1)\sqrt{C}}+\frac{dY_1}{(z-Y_1)(m_4-Y_1)(m_3-Y_1)\sqrt{C}}.$$

In altera huius aequationis parte loco ipsius $\sqrt{C}$ introducantur valores e formulis (8.) et (9.), atque in utraque integratio instituatur, quo facto habentur formulae:

$$14.\quad \int_{m_1}^{y_1}\frac{dy_1}{(z-y_1)\sqrt{(\Delta y_1)}}+\int_{m_2}^{Y_1}\frac{dY_1}{(z-Y_1)\sqrt{(\Delta Y_1)}}=-\frac{2}{\sqrt{(-\Delta z)}}\operatorname{arc\,tang}\frac{\sqrt{C}(z-m_3)(z-m_4)}{\sqrt{(-\Delta z)}},$$

$$15.\quad \int_{m_3}^{y_1}\frac{dy_1}{(z-y_1)\sqrt{(\Delta y_1)}}+\int_{m_4}^{Y_1}\frac{dY_1}{(z-Y_1)\sqrt{(\Delta Y_1)}}=\quad\frac{2}{\sqrt{(-\Delta z)}}\operatorname{arc\,tang}\frac{\sqrt{C}(z-m_3)(z-m_4)}{\sqrt{(-\Delta z)}},$$

illa pro prioribus, haec pro posterioribus limitibus antea propositis valens. Inter limites utriusque aequationis y_1, Y_1 constat aequatio algebraica (12.), nec non $\sqrt{C}$ determinatur ope formulae (13.).

Aequationis (14.) utroque termino secundum descendentes ipsius z potestates evoluto prodeunt hae:

$$\int_{m_1}^{y_1}\frac{dy}{\sqrt{(\Delta y)}}+\int_{m_2}^{Y_1}\frac{dy}{\sqrt{(\Delta y)}}=0,\qquad \int_{m_1}^{y_1}\frac{y\,dy}{\sqrt{(\Delta y)}}+\int_{m_2}^{Y_1}\frac{y\,dy}{\sqrt{(\Delta y)}}=-2\operatorname{arc\,tang}\sqrt{C},$$

$$\int_{m_1}^{y_1}\frac{y^{\varkappa}dy}{\sqrt{(\Delta y)}}+\int_{m_2}^{Y_1}\frac{y^{\varkappa}dy}{\sqrt{(\Delta y)}}=-2\left[\frac{z^{\varkappa}}{\sqrt{(-\Delta z)}}\operatorname{arc\,tang}\frac{\sqrt{C}(z-m_3)(z-m_4)}{\sqrt{(-\Delta z)}}\right]_{z^{-1}},$$

nec non generalior:

$$\int_{m_1}^{y_1}\frac{\Phi y\,dy}{(\alpha-y)\sqrt{(\Delta y)}}+\int_{m_2}^{Y_1}\frac{\Phi y\,dy}{(\alpha-y)\sqrt{(\Delta y)}}$$
$$=+2\left[\frac{\Phi z}{(z-\alpha)\sqrt{(-\Delta z)}}\operatorname{arc\,tang}\frac{\sqrt{C}(z-m_3)(z-m_4)}{\sqrt{(-\Delta z)}}\right]_{z^{-1}}$$
$$-2\frac{\Phi\alpha}{\sqrt{(-\Delta\alpha)}}\operatorname{arc\,tang}\frac{\sqrt{C}(\alpha-m_3)(\alpha-m_4)}{\sqrt{(-\Delta\alpha)}},$$

ubi Φz functionem ipsius z rationalem integram, α quantitatem constantem quamlibet denotat, nec non denotatio usitata pro coëfficiente evolutionis adhibita est.

Exempli gratia posito:

$$m_4 = \infty, \quad m_3 = \frac{1}{\varkappa^2}, \quad m_2 = 1, \quad m_1 = 0,$$

$$\int_0^{y_1} \frac{dy}{\sqrt{[y(1-y)(1-\varkappa^2 y)]}} = 2u, \quad \int_0^{Y_1} \frac{dy}{\sqrt{[y(1-y)(1-\varkappa^2 y)]}} = 2U,$$

$$\int_0^1 \frac{dy}{\sqrt{[y(1-y)(1-\varkappa^2 y)]}} = 2K, \quad y_1 = \sin^2 \operatorname{am} u, \quad Y_1 = \sin^2 \operatorname{am} U,$$

$$\int_0^{y_1} \frac{\sqrt{(1-\varkappa^2 y)}dy}{\sqrt{[y(1-y)]}} = 2E(u), \quad \int_0^{Y_1} \frac{\varkappa^2(\sin\operatorname{am} a \cos\operatorname{am} a\, \Delta\operatorname{am} a) y\, dy}{(1-\varkappa^2\sin^2\operatorname{am} a . y)\sqrt{[y(1-y)(1-\varkappa^2 y)]}} = 2\Pi(u,a),$$

ex antecedentibus sequuntur formulae notissimae:

$$\sin\operatorname{am}(K-u) = \frac{\cos\operatorname{am} u}{\sqrt{(1-\varkappa^2\sin^2\operatorname{am} u)}},$$

$$\sin\operatorname{am}\tfrac{1}{2}K = \frac{1}{\sqrt{(1+\varkappa_1)}},$$

$$E(u)+E(K-u)-E(K) = \varkappa^2\sin\operatorname{am} u \sin\operatorname{am}(K-u),$$

$$\Pi(u,a)+\Pi(K-u,a)-\Pi(K,a) = \tfrac{1}{2}\log\left(\frac{1-\varkappa^2\sin\operatorname{am} u\sin\operatorname{am}(K-u)\sin\operatorname{am} a\sin\operatorname{am}(K-a)}{1+\varkappa^2\sin\operatorname{am} u\sin\operatorname{am}(K-u)\sin\operatorname{am} a\sin\operatorname{am}(K-a)}\right),$$

$$2E(\tfrac{1}{2}K) - E(K) = 1-\varkappa_1,$$

$$2\Pi(\tfrac{1}{2}K, a) - \Pi(K,a) = \tfrac{1}{2}\log\left(\frac{1-(1-\varkappa_1)\sin\operatorname{am} a\sin\operatorname{am}(K-a)}{1+(1-\varkappa_1)\sin\operatorname{am} a\sin\operatorname{am}(K-a)}\right),$$

$$\sin\operatorname{am}(\tfrac{1}{2}K + iK_1) = \frac{1}{\sqrt{(1-\varkappa_1)}}.$$

ubi ponitur:

$$\sqrt{(1-\varkappa^2)} = \varkappa_1, \quad \int_0^1 \frac{dy}{\sqrt{[y(1-y)(1-\varkappa_1^2 y)]}} = 2K_1.$$

2.

Easdem disquisitiones de *Abelianis* integralibus instituentibus nobis, primum problema simile algebraicum speciale solvere, atque deinde ad analyticum generalius aggredi placet. Iam vero methodus illud solvendi, in articulo antecedenti adhibita, quae hic ad aequationem quadraticam ducit, in simili problemate ad integralia ultraelliptica pertinente, aequationem sublimioris gradus provocat, quam ad systemata aequationum minoris gradus reducere convenit. Quam ob causam extemplo alia via ad systemata haec ipsa pervenire malim eamque in casu iam exposito simplicissimo persequi.

Posito:

$$17.\quad \frac{m_3-z}{m_4-z}=v^2,\quad \frac{m_3-m_1}{m_4-m_1}=\eta_1^2,\quad \frac{m_3-m_2}{m_4-m_2}=\eta_2^2,$$

$$18.\quad \sqrt{(c+1)}\{v^2(m_4-x_1)-(m_3-x_1)\}=U,\quad \sqrt{c}.(m_4-m_3)v=V,$$

aequatio identica:

$$c(z-m_4)(z-m_3)+(z-m_2)(z-m_1)=(1+c)(z-x_1)^2,$$

in hanc abit:

$$19.\quad (m_4-m_1)(m_4-m_2)(v^2-\eta_1^2)(v^2-\eta_2^2)=U^2-V^2.$$

Inde coniicis, quia U est functio par, et V functio impar ipsius v, illa secundi, haec primi gradus, fore:

$$20.\quad U+V=c_1(v+\eta_1)(v+\eta_2),\quad U-V=c_1(v-\eta_1)(v-\eta_2),$$

igiturque formulis (18.) advocatis:

$$U=\tfrac{1}{2}c_1\{(v+\eta_1)(v+\eta_2)+(v-\eta_1)(v-\eta_2)\}=\sqrt{(1+c)}\{v^2(m_4-x_1)-(m_3-x_1)\},$$
$$V=\tfrac{1}{2}c_1\{(v+\eta_1)(v+\eta_2)-(v-\eta_1)(v-\eta_2)\}=\sqrt{c}\,\{m_4-m_3\}\,v.$$

Ibi posito $v^2=\infty$, $v^2=1$, prodeunt formulae:

$$21.\quad \frac{\sqrt{c}(m_4-m_3)}{\eta_1+\eta_2}=c_1,\qquad 22.\quad \frac{\sqrt{(1+c)}(m_4-m_3)}{1+\eta_1\eta_2}=c_1,$$

nec non posito: $v^2=\eta_1^2$, $v^2=\eta_2^2$ post faciles reductiones:

$$23.\quad \begin{cases} m_4-x_1=\dfrac{m_4-m_3}{1+\eta_1\eta_2}, & m_3-x_1=-\dfrac{m_4-m_3}{1+\eta_1\eta_2}\eta_1\eta_2,\\[2ex] m_2-x_1=\dfrac{m_4-m_3}{1+\eta_1\eta_2}\cdot\dfrac{\eta_2(\eta_1+\eta_2)}{\eta_2^2-1}, & m_1-x_1=\dfrac{m_4-m_3}{1+\eta_1\eta_2}\cdot\dfrac{\eta_1(\eta_1+\eta_2)}{\eta_1^2-1}.\end{cases}$$

E formulis (18.), (19.) et (20.), posito $v=1$, sequitur, valorem ipsius U^2-V^2 pro $v=1$ fore:

$$=(m_4-m_3)^2=(m_4-m_1)(m_4-m_2)(1-\eta_1^2)(1-\eta_2^2)=c_1^2(1-\eta_1^2)(1-\eta_2^2)$$

unde valor ipsius c_1, quem, quia quantitates η_1^2 et η_2^2 unitate minores sint, positivum esse formula (22.) docet, deducitur:

$$c_1=\frac{m_4-m_3}{\sqrt{[(1-\eta_1^2)(1-\eta_2^2)]}}=\sqrt{[(m_4-m_1)(m_4-m_2)]},$$

quo in formulis (21.) et (22.) substituto, habentur formulae:

$$24.\quad \sqrt{c}=\frac{\eta_1+\eta_2}{\sqrt{[(1-\eta_1^2)(1-\eta_2^2)]}},\qquad 25.\quad \sqrt{(1+c)}=\frac{1+\eta_1\eta_2}{\sqrt{[(1-\eta_1^2)(1-\eta_2^2)]}}.$$

Quia igitur summa $\eta_1+\eta_2$, nec non differentia:

$$\eta_1^2-\eta_2^2=\frac{(m_4-m_3)(m_2-m_1)}{(m_4-m_1)(m_3-m_2)},$$

positivis valoribus gaudent, etiam differentia $\eta_1-\eta_2$, simul cum ipso η_1 positiva sit necesse est. Itaque ponere licet:

$$\eta_1 = \sqrt{\left(\frac{m_3 - m_1}{m_4 - m_1}\right)}, \qquad \eta_2 = \pm\sqrt{\left(\frac{m_3 - m_2}{m_4 - m_2}\right)}.$$

Formulae (23.) vero docent, valorem x_1 pro superiori signo ipsius η_2 in intervallo $m_3 - m_4$ contineri, pro inferiori in intervallo $m_1 - m_2$, nec non fore:

$$x_1 = \frac{m_3 + m_4\,\eta_1\eta_2}{1 + \eta_1\eta_2},$$

quae formulae cum formulis (4.), (5.), (6.), (7.) optime congruunt.

Si expressionem:

$$c(z - m_4)(z - m_3) + (z - m_2)(z - m_1)$$

brevitatis gratia per (4,3), similiterque per (3,2), (2,1), (1,4), (4,2), (3,1) designas expressiones similes, quarum prior terminus respective est:

$$c(z-m_3)(z-m_2),\quad c(z-m_2)(z-m_1),\quad c(z-m_1)(z-m_4),\quad c(z-m_4)(z-m_2),$$
$$c(z-m_3)(z-m_1),$$

similem calculum in his quinque ceteris formis instituere superfluum esset.

Substitutionis enim linearis ope huius:

$$Z = p\,\frac{z-n}{z-m},$$

ubi quantitas m his respective satisfacit conditionibus:

$$\begin{array}{ll} m_4 > m \geqq m_3, & \\ m_3 > m \geqq m_2, & \qquad p(n-m) > 0, \\ m_2 > m \geqq m_1, & \end{array}$$

formae respective (3,2), (2,1), (1,4) ad formam fundamentalem (4,3) revocantur. Nimirum hac substitutione habetur, si h quilibet numerorum 1, 2, 3, 4, est, et ponitur:

$$M_h = p\cdot\frac{n - m_h}{m - m_h},$$

$$m_h - z = (m_h - m)\frac{Z - M_h}{Z - p},$$

$$\frac{dZ}{dz} = \frac{p(n-m)}{(z-m)^2} = \frac{(Z-p)^2}{p(n-m)}.$$

Inde concluditur argumento z ab m usque ad $-\infty$, et ab ∞ usque ad m continuo pergente, argumentum Z ab ∞ usque ad p, et ab p usque ad $-\infty$ continuo decrescere. Hinc patet, si fuerit:

$$m_4 > m \geqq m_3,$$

fore:

$$M_3 > M_2 > M_1 > M_4,$$

si fuerit:

$$m_3 > m \geqq m_2,$$

fore:

$$M_2 > M_1 > M_4 > M_3,$$

atque si fuerit:

$$m_2 > m \geqq m_1,$$

fore:

$$M_1 > M_4 > M_3 > M_2.$$

Unde sequitur formas (3,2), (2,1), (1,4), in novis signis forma (4,3) indui. —

Formae denique (4,2) et (3,1) realem problematis solutionem non admittunt, quippe quae, si quantitas c realis est, duplici factore gaudere nequeunt. Aequatio enim formae:

$$(4,2) = 0$$

exempli gratia unam singulam habet radicem aut in intervallo $m_3 - m_4$ aut in intervallo $m_2 - m_3$, prout quantitas c positivo vel negativo valore gaudet.

3.

Problema algebraicum simile, ad integralia *Abeliana* primi ordinis pertinens ita pronuntiatur.

Si differentiae quantitatum realium $m_1, m_2, \dots m_6$

$$m_6 - m_5, \quad m_5 - m_4, \quad \dots\dots \quad m_2 - m_1$$

positivae sunt, quantitates c, a, x_1, x_2 ita sunt determinandae, ut expressio biquadratica:

$$26. \qquad c(z-a)^2(z-m_6)(z-m_5)+(z-m_4)(z-m_3)(z-m_2)(z-m_1),$$

formam:

$$= (c+1)(z-x_1)^2(z-x_2)^2,$$

induat.

Expressione (26.) cum ipsius differentiali pro $z = x_1$, et $z = x_2$ evanescente, habentur formulae:

$$27. \quad \begin{cases} \dfrac{2}{x_1-a} = \dfrac{1}{x_1-m_1}+\dfrac{1}{x_1-m_2}+\dfrac{1}{x_1-m_3}+\dfrac{1}{x_1-m_4}-\dfrac{1}{x_1-m_5}-\dfrac{1}{x_1-m_6}, \\[2ex] \dfrac{2}{x_2-a} = \dfrac{1}{x_2-m_1}+\dfrac{1}{x_2-m_2}+\dfrac{1}{x_2-m_3}+\dfrac{1}{x_2-m_4}-\dfrac{1}{x_2-m_5}-\dfrac{1}{x_2-m_6}. \end{cases}$$

Posteriori a priori subtracta, divisioneque per $(x_2 - x_1)$ facta, haec prodit, signo summatorio adhibito, formula:

$$\frac{2}{(x_1-a)(x_2-a)} = \sum_1^4{}^h \left(\frac{1}{(x_1-m_h)(x_2-m_h)}\right) - \frac{1}{(x_1-m_5)(m_2-m_5)} - \frac{1}{(x_1-m_6)(x_2-m_6)},$$

quae, quantitate a ope formularum (27.) eliminata, hanc suppeditat aequationem:

$$28.\quad \left\{\sum_1^4{}_h\left(\frac{1}{x_1-m_h}\right)-\frac{1}{x_1-m_5}-\frac{1}{x_1-m_6}\right\}\left\{\sum_1^4{}_h\left(\frac{1}{x_2-m_h}\right)-\frac{1}{x_2-m_5}-\frac{1}{x_2-m_6}\right\}$$
$$=2\sum_1^4{}_h\left(\frac{1}{(x_1-m_h)(x_2-m_h)}\right)-\frac{2}{(x_1-m_5)(x_2-m_5)}-\frac{2}{(x_1-m_6)(x_2-m_6)}.$$

Iam vero ex aequatione proposita:

$$29.\quad c(z-a)^2(z-m_6)(z-m_5)+(z-m_4)(z-m_3)(z-m_2)(z-m_1)$$
$$=(c+1)(z-x_1)^2(z-x_2)^2,$$

emanat formula:

$$\frac{(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)}{(m_5-m_1)(m_5-m_2)(m_5-m_3)(m_5-m_4)}=\frac{(m_6-x_1)^2(m_6-x_2)^2}{(m_5-x_1)^2(m_5-x_2)^2},$$

cuius ope ipso x_2 ex aequatione (28.) eliminato, aequatio sedecimi gradus ad ipsum x_1 determinandum oritur. Haec in octo aequationes quadraticas discerpitur hoc modo. Ponatur:

$$\frac{m_5-z}{m_6-z}=v^2,\qquad \frac{m_5-m_h}{m_6-m_h}=\eta_h^2,$$

ubi h est quilibet quatuor numerorum 1, 2, 3, 4. Inde prodeunt formulae pro qualibet quantitate y valentes:

$$31.\quad z-y=\frac{v^2(m_6-y)-(m_5-y)}{v^2-1},\qquad m_h-y=\frac{\eta_h^2(m_6-y)-(m_5-y)}{\eta_h^2-1};$$

nec non aequatio (29.) in hanc abit:

$$(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)(v^2-\eta_1^2)(v^2-\eta_2^2)(v^2-\eta_3^2)(v^2-\eta_4^2)=U^2-V^2,$$

ubi ponitur:

$$\sqrt{(1+c)}\{v^2(m_6-x_1)-(m_5-x_1)\}\{v^2(m_6-x_2)-(m_5-x_2)\}=U,$$
$$\sqrt{c}\,\{v^2(m_6-a)-(m_5-a)\}(m_6-m_5)v=V.$$

Inde eodem modo ac antea coniicitur, fore:

$$U+V=c_1(v+\eta_1)(v+\eta_2)(v+\eta_3)(v+\eta_4),$$
$$U-V=c_1(v-\eta_1)(v-\eta_2)(v-\eta_3)(v-\eta_4),$$

nec non brevitatis gratia posito:

$$\psi(z)=(z+\eta_1)(z+\eta_2)(z+\eta_3)(z+\eta_4),$$

$$32.\begin{cases}U=\frac{1}{2}c_1(\psi(v)+\psi(-v))=\sqrt{(c+1)}\{v^2(m_6-x_1)-(m_5-x_1)\}\{v^2(m_6-x_2)-(m_5-x_2)\},\\ V=\frac{1}{2}c_1(\psi(v)-\psi(-v))=\sqrt{c}\,(m_6-m_5)v.\{v^2(m_6-a)-(m_5-a)\}.\end{cases}$$

Inde, posito $v^2=1$, $v^2=\infty$, $v^2=0$, prodeunt formulae:

$$33.\quad \sqrt{c}=\tfrac{1}{2}c_1.\frac{\psi(1)-\psi(-1)}{(m_6-m_5)^2},\qquad 34.\quad \sqrt{(1+c)}=\tfrac{1}{2}c_1.\frac{\psi(1)+\psi(-1)}{(m_6-m_5)^2},$$

porro

$$35.\quad (m_6-a) = \frac{c_1}{\sqrt{c}}\cdot\frac{\eta_1+\eta_2+\eta_3+\eta_4}{m_6-m_5} = \frac{2(m_6-m_5)(\eta_1+\eta_2+\eta_3+\eta_4)}{\psi(1)-\psi(-1)},$$

$$36.\quad (m_6-x_1)(m_6-x_2) = \frac{c_1}{\sqrt{(1+c)}} = \frac{2(m_6-m_5)^2}{\psi(1)+\psi(-1)},$$

$$37.\quad m_5-a = -\frac{c_1}{\sqrt{c}}\cdot\frac{\eta_1\eta_2\eta_3\eta_4}{m_6-m_5}\left\{\frac{1}{\eta_1}+\frac{1}{\eta_2}+\frac{1}{\eta_3}+\frac{1}{\eta_4}\right\}$$

$$= -\frac{2(m_6-m_5)\eta_1\eta_2\eta_3\eta_4}{\psi(1)-\psi(-1)}\left\{\frac{1}{\eta_1}+\frac{1}{\eta_2}+\frac{1}{\eta_3}+\frac{1}{\eta_4}\right\},$$

$$38.\quad (m_5-x_1)(m_5-x_2) = \frac{c_1}{\sqrt{(1+c)}}\eta_1\eta_2\eta_3\eta_4 = \frac{2(m_6-m_5)^2\eta_1\eta_2\eta_3\eta_4}{\psi(1)+\psi(-1)},$$

posito vero $v^2=\eta_h^2$, hae formulae:

$$39.\quad \begin{cases} m_h-a = -\dfrac{\psi(\eta_h)}{\eta_h}\cdot\dfrac{m_6-m_h}{\psi(1)-\psi(-1)}, \\ (m_h-x_1)(m_h-x_2) = \psi\eta_h\cdot\dfrac{(m_6-m_h)^2}{\psi(1)+\psi(-1)}. \end{cases}$$

Ex aequationibus (33.) et (34.) sequitur fore:

$$\frac{(m_6-m_5)^4}{(1-\eta_1^2)(1-\eta_2^2)(1-\eta_3^2)(1-\eta_4^2))} = c_1^2,$$

sive ope formularum (30.) posterioris:

$$c_1^2 = (m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4).$$

Iam vero formula (34.), quia quantitates η_h^2 unitate minores sunt, docet quantitatem c_1 semper positivam esse, quae cum ita sint, erit:

$$40.\quad c_1 = \sqrt{[(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)]},$$

quo valore in formulis (33.) et (34.) substituto, habentur formulae:

$$41.\quad \begin{cases} \sqrt{(1+c)} = \dfrac{\sqrt{[(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)]}}{2(m_6-m_5)^2}\{\psi(1)+\psi(-1)\}, \\ \sqrt{c} = \dfrac{\sqrt{[(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)]}}{2(m_6-m_5)^2}\{\psi(1)-\psi(-1)\}, \end{cases}$$

in quibus simul cum formulis (35.), ... (39.) problematis propositi solutio continetur. Inde praeter determinationem quantitatum a, x_1, x_2, functiones $z-a$, $(z-x_1)(z-x_2)$ multis in formis exprimere licet, quarum principales hic proponantur. Priori enim per $P(z)$, posteriori per $\varphi(z)$ denotata, e formulis (31.) et (32.) prodeunt hae:

$$P(z) =$$

$$\frac{c_1}{2\sqrt{c}(m_6-m_5)}\cdot\frac{\psi(v)-\psi(-v)}{v(v^2-1)} = \frac{-2}{\psi(1)-\psi(-1)}\{(z-m_5)C_1+(z-m_6)C_3\},$$

$$\varphi(z) =$$

$$\frac{c_1}{2\sqrt{(1+c)}}\cdot\frac{\psi(v)+\psi(-v)}{(v^2-1)^2} = \frac{2}{\psi(1)+\psi(-1)}\{(z-m_5)^2+(z-m_5)(z-m_6)C_2+(z-m_6)^2C_4\},$$

ubi per C_1, C_2, C_3, C_4 designantur respective summae unionum, binionum, ternionum, quaternionumque quantitatum η_1, η_2, η_3, η_4 sine repetitione. Deinde, si per b_1 et b_2 duas quaslibet quatuor quantitatum m_1, m_2, m_3, m_4, hiscum cohaerentes ipsarum η_1^2, η_2^2, η_3^2, η_4^2 duas per β_1^2 et β_2^2 nec non productum:

$$(z-b_1)(z-b_2) = Fz,$$

denotamus, ex identicis aequationibus:

$$42.\quad \begin{cases} \dfrac{Pz}{Fz} = \dfrac{Pb_1}{(b_1-b_2)(z-b_1)} + \dfrac{Pb_2}{(b_2-b_1)(z-b_2)}, \\ \dfrac{\varphi z}{Fz} = 1 + \dfrac{\varphi b_1}{(b_1-b_2)(z-b_1)} + \dfrac{\varphi b_2}{(b_2-b_1)(z-b_2)}, \end{cases}$$

aequationibus (39.) advocatis has nanciscimur functionum Pz et φz formas:

$$Pz = -\frac{(z-b_1)(z-b_2)}{\psi(1)-\psi(-1)}\left\{\frac{m_6-b_1}{b_1-b_2}\cdot\frac{\psi\beta_1}{\beta_1}\cdot\frac{1}{z-b_1} + \frac{m_6-b_2}{b_2-b_1}\cdot\frac{\psi\beta_2}{\beta_2}\cdot\frac{1}{z-b_2}\right\},$$

$$\varphi z = (z-b_1)(z-b_2) + \frac{1}{\psi(1)+\psi(-1)}\left\{\frac{(m_6-b_1)^2}{b_1-b_2}\psi\beta_1.(z-b_2) + \frac{(m_6-b_2)^2}{b_2-b_1}\psi\beta_2.(z-b_1)\right\}.$$

Iam igitur quantitas a determinatur ut radix aequationis:

$$43.\quad (z-m_5)C_1 + (z-m_6)C_3 = 0,$$

sive huius:

$$44.\quad \frac{(m_6-b_1)}{z-b_1}\cdot\frac{\psi\beta_1}{\beta_1} - \frac{m_6-m_2}{z-b_2}\cdot\frac{\psi\beta_2}{\beta_2} = 0,$$

atque x_1, x_2 ut radices aequationis quadraticae:

$$45.\quad (z-m_5)^2 + (z-m_5)(z-m_6)C_2 + (z-m_6)^2 C_4 = 0,$$

vel huius:

$$46.\quad \psi(1)+\psi(-1) + \frac{(m_6-b_1)^2}{b_1-b_2}\cdot\frac{\psi\beta_1}{z-b_1} + \frac{(m_6-b_2)^2}{b_2-b_1}\cdot\frac{\psi\beta_2}{z-b_2} = 0.$$

Adnotare adhuc placet, valores φb_1, φb_2 etiam ut valores quantitatum incognitarum systematis singularis duarum aequationum linearium dari. Si enim per c_1 et c_2 denotantur ceterae duae quantitatum quatuor m_1, m_2, m_3, m_4, exceptis b_1 et b_2, atque per γ_1^2, γ_2^2 iiscum cohaerentes quatuor quantitatum η_1^2, η_2^2, η_3^2, η_4^2, ope formulae ex aequationibus (39.) prodeuntis:

$$Pm_h = -\frac{\psi(1)+\psi(-1)}{\psi(1)-\psi(-1)}\cdot\frac{\varphi m_h}{\eta_h(m_6-m_h)},$$

e formulis identicis (42.) emanant hae aequationes:

$$\frac{\varphi c_1}{Fc_1} = \frac{m_6-c_1}{(m_6-b_1)(c_1-b_1)}\cdot\frac{\varphi b_1}{F'b_1}\cdot\frac{\gamma_1}{\beta_1} + \frac{m_6-c_1}{(m_6-b_2)(c_1-b_2)}\cdot\frac{\varphi b_2}{F'b_2}\cdot\frac{\gamma_1}{\beta_2},$$

$$\frac{\varphi c_2}{Fc_2} = \frac{m_6-c_2}{(m_6-b_1)(c_2-b_1)}\cdot\frac{\varphi b_1}{F'b_1}\cdot\frac{\gamma_2}{\beta_1} + \frac{m_6-c_2}{(m_6-b_2)(c_2-b_2)}\cdot\frac{\varphi b_2}{F'b_2}\cdot\frac{\gamma_2}{\beta_2},$$

$$\frac{\varphi c_1}{Fc_1} = 1 + \frac{1}{c_1 - b_1} \cdot \frac{\varphi b_1}{F' b_1} + \frac{1}{c_1 - b_2} \cdot \frac{\varphi b_2}{F' b_2},$$

$$\frac{\varphi c_2}{Fc_2} = 1 + \frac{1}{c_2 - b_1} \cdot \frac{\varphi b_1}{F' b_1} + \frac{1}{c_2 - b_2} \cdot \frac{\varphi b_2}{F' b_2},$$

quibus apte collatis, ob formulam identicam:

$$m_6 - m_h = \frac{m_6 - m_5}{1 - \eta_h^2},$$

prodit systema harum aequationum:

$$1 + \frac{\varphi b_1}{(m_6 - b_1) F' b_1} \cdot \frac{\gamma_1 + \frac{1}{\beta_1}}{\gamma_1 + \beta_1} + \frac{\varphi b_2}{(m_6 - b_2) F' b_2} \cdot \frac{\gamma_1 + \frac{1}{\beta_2}}{\gamma_1 + \beta_2} = 0,$$

$$1 + \frac{\varphi b_1}{(m_6 - b_1) F' b_1} \cdot \frac{\gamma_2 + \frac{1}{\beta_1}}{\gamma_2 + \beta_1} + \frac{\varphi b_2}{(m_6 - b_2) F' b_2} \cdot \frac{\gamma_2 + \frac{1}{\beta_2}}{\gamma_2 + \beta_2} = 0.$$

Inde formularum (39.) secunda adhibita, sequitur si habeantur aequationes:

$$47. \quad \begin{cases} 1 + z_1 . \dfrac{\gamma_1 + \frac{1}{\beta_1}}{\gamma_1 + \beta_1} + z_2 . \dfrac{\gamma_1 + \frac{1}{\beta_2}}{\gamma_1 + \beta_2} = 0, \\[2ex] 1 + z_1 . \dfrac{\gamma_2 + \frac{1}{\beta_1}}{\gamma_2 + \beta_2} + z_2 . \dfrac{\gamma_2 + \frac{1}{\beta_2}}{\gamma_2 + \beta_2} = 0, \end{cases}$$

fore:

$$48. \quad \begin{cases} z_1 = \dfrac{1 - \beta_2^2}{\beta_2^2 - \beta_1^2} \cdot \dfrac{\psi(\beta_1)}{\psi(1) + \psi(-1)} = \dfrac{\varphi b_1}{(b_1 - b_2)(m_6 - b_1)}, \\[2ex] z_2 = \dfrac{1 - \beta_1^2}{\beta_1^2 - \beta_2^2} \cdot \dfrac{\psi(\beta_2)}{\psi(1) + \psi(-1)} = \dfrac{\varphi b_2}{(b_2 - b_1)(m_6 - b_2)}, \end{cases}$$

ubi ponitur:

$$\psi z = (z + \beta_1)(z + \beta_2)(z + \gamma_1)(z + \gamma_2).$$

Id quod facilis calculus comprobat.

4.

Iam in naturam quantitatum c, a, x_1, x_2 pro diversis quantitatum η_1, η_2, η_3, η_4 valoribus inquirere placet. Formularum (39.) prior docet, quantitatem a realem, atque formulae (41.), ipsum c adeo positivum esse. Inde iam coniicis, radices duplices x_1, x_2 aequationis:

$$47. \quad c(z-a)^2(z-m_6)(z-m_5) + (z-m_4)(z-m_3)(z-m_2)(z-m_1) = 0$$

imaginariis valoribus nisi coniugatis gaudere non posse. Si enim haberetur $x_1 = m + ni$, quia aequatio realibus coëfficientibus gaudet, $x_2 = m - ni$ ponatur, necesse esset. Deinde patet valores ipsorum x_1 et x_2 in ullo intervallorum horum:

$$-\infty \ldots . m_1, \quad m_2 \ldots . m_3, \quad m_4 \ldots . m_5, \quad m_6 \ldots . \infty,$$

contineri non posse, quippe pro eiusmodi valore ipsius z prior aequationis (29.) terminus, semper positivum valorem induens, evanescere nequit. Quantitatis denique a valor in nullo intervallorum horum:

$$m_1 \ldots . m_2, \quad m_3 \ldots . m_4,$$

contineri potest. Si enim exempli gratia prior casus locum haberet, ita ut differentiae:

$$a - m_1, \quad m_2 - a$$

positivae essent, prior terminus aequationis (29.) pro $z = a$ negativo, posterior vero positivo valore indueretur, id quod fieri nequit.

Quae considerationes ceteris articuli praecedentis formulis optime comprobantur in diversis, quas facere licet de quantitatibus η_h suppositionibus. Quas ut inveniamus, adnotetur, differentias

$$\eta_1^2 - \eta_2^2, \quad \eta_2^2 - \eta_3^2, \quad \eta_3^2 - \eta_4^2$$

ob ipsorum m_1, m_2, m_3, m_4 naturam positivas esse, formulis (30.) comprobari. Deinde e formularum (41.) secunda conditio inter quantitates η_h necessaria,

$$(1+\eta_1)(1+\eta_2)(1+\eta_3)(1+\eta_4) - (1-\eta_1)(1-\eta_2)(1-\eta_3)(1-\eta_4) > 0,$$

emanat, quae quum, simul η_1 cum $-\eta_1$, η_2 cum $-\eta_2$, η_3 cum $-\eta_3$, η_4 cum $-\eta_4$ commutatis, constare nequeat, harum quantitatum quod attinet ad signa, nonnisi octo suppositiones constitui posse patet. Octo expressiones diversae functionis φz inde orientes tales erunt, ut, ipsarum producto posito $= 0$, aequatio sedecimi gradus emergat, quam, in articulo 3. memoratam, hoc modo in octo aequationes quadraticas resolutam videas.

Animadvertendum est, et ut in naturam octo classium penetrare liceat, et quia in problemate analytico postea adhibetur, expressiones:

$$m_6 - a, \quad m_5 - a, \quad m_4 - a, \quad m_3 - a, \quad m_2 - a, \quad m_1 - a,$$

respective simul cum expressionibus:

$$\eta_1 + \eta_2 + \eta_3 + \eta_4, \quad -\eta_1\eta_2\eta_3\eta_4\left(\frac{1}{\eta_1} + \frac{1}{\eta_2} + \frac{1}{\eta_3} + \frac{1}{\eta_4}\right), \quad -\eta_1\eta_2\eta_3, \quad -\eta_1\eta_2\eta_3, \quad -\eta_1, \quad -\eta_1$$

nec non expressiones:

$$\varphi m_6, \quad \varphi m_5, \quad \varphi m_4, \quad \varphi m_3, \quad \varphi m_2, \quad \varphi m_1,$$

respective simul cum expressionibus:

$$1, \quad \eta_1\eta_2\eta_3\eta_4, \quad \eta_1\eta_2\eta_3\eta_4, \quad \eta_1\eta_2, \quad \eta_1\eta_2, \quad 1$$

positivas vel negativas esse. Id quod hoc modo demonstratur, ut in expressionibus (39.) quantitatis $\psi\eta_h$ factor quilibet

$$\eta_h+\eta_\varkappa$$

transformetur, prout $h \lessgtr \varkappa$ est, in formas:

$$\eta_h\left(1+\frac{\eta_\varkappa}{\eta_h}\right) \quad \text{vel} \quad \eta_\varkappa\left(1+\frac{\eta_h}{\eta_\varkappa}\right),$$

quae, cum respective $\left(\frac{\eta_\varkappa}{\eta_h}\right)^2$ vel $\left(\frac{\eta_h}{\eta_\varkappa}\right)^2$ unitate minores sint, respective simul cum η_h vel $\eta_\varkappa$ positivis negativisve valoribus gaudent. Quibus propositionibus adiutus, in quibusnam intervallis pro octo diversis de quantitatibus η_h suppositionibus, valores quantitatum a, x_1, x_2, contineantur concludis. Id quod ex hac tabula desumere licet.

Casus primus:

$$\eta_1=\sqrt{\left(\frac{m_5-m_1}{m_6-m_1}\right)},\quad \eta_2=-\sqrt{\left(\frac{m_5-m_2}{m_6-m_2}\right)},\quad \eta_3=\sqrt{\left(\frac{m_5-m_3}{m_6-m_3}\right)},\quad \eta_4=-\sqrt{\left(\frac{m_5-m_4}{m_6-m_4}\right)},$$

a continetur in intervallo $m_2 - m_3$,
x_1 - - - - - $m_1 - m_2$,
x_2 - - - - - $m_3 - m_4$.

Casus sècundus:

$$\eta_1=\pm\sqrt{\left(\frac{m_5-m_1}{m_6-m_1}\right)},\quad \eta_2=\mp\sqrt{\left(\frac{m_5-m_2}{m_6-m_2}\right)},\quad \eta_3=\mp\sqrt{\left(\frac{m_5-m_3}{m_6-m_3}\right)},\quad \eta_4=\pm\sqrt{\left(\frac{m_5-m_4}{m_6-m_4}\right)},$$

ubi superiora vel inferiora signa simul eligenda sunt, ita ut differentia $(1+\eta_1)(1+\eta_2)(1+\eta_3)(1+\eta_4)-(1-\eta_1)(1-\eta_2)(1-\eta_3)(1-\eta_4)$ posito valore induatur;

a continetur in intervallo $m_4 \ldots\ldots \infty$ pro superioribus signis,
a - - - - - $-\infty \ldots\ldots m_1$ pro inferioribus signis,
x_1 - - - - - $m_1 \ldots\ldots m_2$,
x_2 - - - - - $m_3 \ldots\ldots m_4$.

Casus tertius:

$$\eta_1=\sqrt{\left(\frac{m_5-m_1}{m_6-m_1}\right)},\quad \eta_2=-\sqrt{\left(\frac{m_5-m_2}{m_6-m_2}\right)},\quad \eta_3=\sqrt{\left(\frac{m_5-m_3}{m_6-m_3}\right)},\quad \eta_4=\sqrt{\left(\frac{m_5-m_4}{m_6-m_4}\right)},$$

a continetur in intervallo $m_2 \ldots\ldots m_3$,
x_1 - - - - - $m_1 \ldots\ldots m_2$,
x_2 - - - - - $m_5 \ldots\ldots m_6$.

Casus quartus:

$$\eta_1=\pm\sqrt{\left(\frac{m_5-m_1}{m_6-m_1}\right)},\quad \eta_2=\mp\sqrt{\left(\frac{m_5-m_2}{m_6-m_2}\right)},\quad \eta_3=\mp\sqrt{\left(\frac{m_5-m_3}{m_6-m_3}\right)},\quad \eta_4=\mp\sqrt{\left(\frac{m_5-m_4}{m_6-m_4}\right)},$$

ubi superiora vel inferiora signa simul ponenda, ita ut differentia:

$$(1+\eta_1)(1+\eta_2)(1+\eta_3)(1+\eta_4)-(1-\eta_1)(1-\eta_2)(1-\eta_3)(1-\eta_4)$$

positiva sit;

a continetur in intervallo $m_5 \ldots\ldots \infty$ pro superioribus signis,
a - - - - - - $-\infty \ldots\ldots m_1$ pro inferioribus signis,
x_1 - - - - - - $m_1 \ldots\ldots m_2$,
x_2 - - - - - - $m_5 \ldots\ldots m_6$.

Casus quintus:

$$\eta_1=\sqrt{\left(\frac{m_5-m_1}{m_6-m_1}\right)},\quad \eta_2=\sqrt{\left(\frac{m_5-m_2}{m_6-m_2}\right)},\quad \eta_3=-\sqrt{\left(\frac{m_5-m_3}{m_6-m_3}\right)},\quad \eta_4=\sqrt{\left(\frac{m_5-m_4}{m_6-m_4}\right)},$$

a continetur in intervallo $m_2 \ldots\ldots m_3$,
x_1 - - - - - $m_3 \ldots\ldots m_4$,
x_2 - - - - - $m_5 \ldots\ldots m_6$.

Casus sextus:

$$\eta_1=\sqrt{\left(\frac{m_5-m_1}{m_6-m_1}\right)},\quad \eta_2=\sqrt{\left(\frac{m_5-m_2}{m_6-m_2}\right)},\quad \eta_3=\sqrt{\left(\frac{m_5-m_3}{m_6-m_3}\right)},\quad \eta_4=-\sqrt{\left(\frac{m_5-m_4}{m_6-m_4}\right)},$$

a continetur in intervallo $m_4 \ldots\ldots m_6$,
x_1 - - - - - $m_3 \ldots\ldots m_4$,
x_2 - - - - - $m_5 \ldots\ldots m_6$.

Casus septimus:

$$\eta_1=\sqrt{\left(\frac{m_5-m_1}{m_6-m_1}\right)},\quad \eta_2=\sqrt{\left(\frac{m_5-m_2}{m_6-m_2}\right)},\quad \eta_3=-\sqrt{\left(\frac{m_5-m_3}{m_6-m_3}\right)},\quad \eta_4=-\left(\frac{m_5-m_4}{m_6-m_4}\right),$$

a continetur in intervallo $m_2 \ldots\ldots m_3$,

x_1 et x_2 continentur utrumque in uno trium intervallorum $m_1 \ldots\ldots m_2$, $m_3 \ldots\ldots m_4$, $m_5 - m_6$, aut gaudent valoribus imaginariis coniugatis.

Casus octavus:

$$\eta_1=\sqrt{\left(\frac{m_5-m_1}{m_6-m_1}\right)},\quad \eta_2=\sqrt{\left(\frac{m_5-m_2}{m_6-m_2}\right)},\quad \eta_3=\sqrt{\left(\frac{m_5-m_3}{m_6-m_3}\right)},\quad \eta_4=\sqrt{\left(\frac{m_5-m_4}{m_6-m_4}\right)},$$

a continetur in intervallo $m_5 \ldots\ldots m_6$,
x_1 - - - - - $m_5 \ldots\ldots a$,
x_2 - - - - - $a \ldots\ldots m_6$.

Sufficiat casum ultimum accuratius exponere, ubi a ut radix aequationis (43.), cuius prior terminus pro $z=m_5$ negativo, et pro $z=m_6$ positivo valore induitur, in intervallo $m_5 \ldots\ldots m_6$ contineatur necesse est; nec non x_1 et x_2 ut radices aequationis (45.), cuius prior terminus pro $z=m_5$ et $z=m_6$ positivis valoribus gaudet, nec non pro $z = a$ valore hoc induitur negativo:

$$\frac{(a-m_5)^2}{C_3^2}\{C_3^2 - C_1 C_2 C_3 + C_1^2 C_4\},$$

respective in intervallis $m_5 \dots a$ et $a \dots m_6$ iacent.

In antecedentibus completa continetur solutio problematis, expressionem formae:

$$c(z-a)^2(z-m_6)(z-m_5)+(z-m_4)(z-m_3)(z-m_2)(z-m_1),$$

quam hic rursus per [6, 5] denotare placet, in formam:

$$(c+1)(z-x_1)^2(z-x_2)^2$$

redigendi. Quinque expressionis fundamentalis formas, quae, simili denotatione adhibita, erunt:

[5, 4], [4, 3], [3, 2], [2, 1], [1, 6],

singulas similiter tractare licet, quippe quae similem redactionem admittunt. Eandem vero ope substitutionis:

$$Z = p\left(\frac{n-z}{m-z}\right),$$

ubi quantitas m respective relationibus:

$$m_6 > m \geqq m_5, \quad m_5 > m \geqq m_4, \quad m_4 > m \geqq m_3, \quad m_3 > m \geqq m_2,$$
$$m_2 > m \geqq m_1$$

satisfacit, nec non habetur: $p(n-m) > 0$, effici posse patet, quae illas respective formas ad formam [6, 5] pro argumento Z revocat.

Contra formas [6, 4] cum quinque ipsi cognatis:

[5, 3], [4, 2], [3, 1], [2, 6], [1, 5]

atque [6, 3] cum duabus [5, 2], [4, 1], quippe in quibus problemata similia reali solutione carent, omittere placet. Si enim quantitates c et a reales sunt, aequatio:

[6, 4] $= 0$ in intervallis $m_3 - m_4$, vel $m_4 - m_5$,

et aequatio [6, 3] $= 0$ in intervallis $m_3 - m_4$, vel $m_2 - m_3$,

prout valor ipsius c positivus est vel negativus, impari numero radicum realium gaudent. Id quod cum forma $(c+1)(z-x_1)^2(z-x_2)^2$ congruere nequit. Idem de formis his cognatis mutatis mutandis observatur.

5.

Iam ad partem analyticam harum de integralibus functionibusque ***Abelianis*** primi ordinis disquisitionum transeuntes, ponamus aequationem eiusdem formae generalem:

$$1. \quad C(z-A)^2(z-m_6)(z-m_5)+(z-m_4)(z-m_3)(z-m_2)(z-m_1) = 0$$

radices quatuor y_1, y_2, Y_1, Y_2, habere, ita ut aequatio valeat identica:

$$2.\quad C(z-A)^2(z-m_6)(z-m_5)+(z-m_4)(z-m_3)(z-m_2)(z-m_1) = (C+1)(z-y_1)(z-y_2)(z-Y_1)(z-Y_2).$$

Sint quantitates e_1, e_2, E_1, E_2, tales unitates positivae vel negativae, ut habeantur formulae ex aequatione (1.) ortae:

$$3.\quad \begin{cases} \sqrt{C}(y_1-A)=\dfrac{e_1\sqrt{(\Delta y_1)}}{(y_1-m_5)(y_1-m_6)}, & \sqrt{C}(y_2-A)=\dfrac{e_2\sqrt{(\Delta y_2)}}{(y_2-m_5)(y_2-m_6)}, \\ \sqrt{C}(Y_1-A)=\dfrac{E_1\sqrt{(\Delta Y_1)}}{(Y_1-m_5)(Y_1-m_6)}, & \sqrt{C}(Y_2-A)=\dfrac{E_2\sqrt{(\Delta Y_2)}}{(Y_2-m_5)(Y_2-m_6)}, \end{cases}$$

ubi brevitatis gratia ponitur:

$$-(z-m_1)(z-m_2)(z-m_3)(z-m_4)(z-m_5)(z-m_6) = \Delta z.$$

Iam secundum theorema *Abelianum* inter quatuor quantitates y_1, y_2, Y_1, Y_2, quas duabus aequationibus algebraicis inter se coniungi patet, plures constant aequationes transcendentes. Eas hoc loco sequenti methodo evolvere placet. Aequatio (2.) has suppeditat formulas:

$$4.\quad (A-m_1)(A-m_2)(A-m_3)(A-m_4)=(C+1)(A-y_1)(A-y_2)(A-Y_1)(A-Y_2),$$

$$5.\quad (m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)=(C+1)(m_6-y_1)(m_6-y_2)(m_6-Y_1)(m_6-Y_2),$$

$$6.\quad (m_5-m_1)(m_5-m_2)(m_5-m_3)(m_5-m_4)=(C+1)(m_5-y_1)(m_5-y_2)(m_5-Y_1)(m_5-Y_2),$$

$$7.\quad C(m_\varkappa-A)^2(m_\varkappa-m_6)(m_\varkappa-m_5)=(C+1)(m_\varkappa-y_1)(m_\varkappa-y_2)(m_\varkappa-Y_1)(m_\varkappa-Y_2),$$

designante $\varkappa$ indices 1, 2, 3, 4. Eadem aequatione (2.) identica secundum z differentiata, et deinde $z=A$ posito, formula (4.) advocata, prodit haec:

$$8.\quad \frac{1}{A-m_1}+\frac{1}{A-m_2}+\frac{1}{A-m_3}+\frac{1}{A-m_4}=\frac{1}{A-y_1}+\frac{1}{A-y_2}+\frac{1}{A-Y_1}+\frac{1}{A-Y_2}.$$

Iam vero e logarithmica differentiatione aequationum (4.), (5.), (6.), (2.) quantitatibus C, A, y_1, y_2, Y_1, Y_2, ut variabilibus assumtis, formulae (8.) ope, prodeunt hae formulae differentiales:

$$9.\quad \frac{dC}{1+C}=\frac{dy_1}{A-y_1}+\frac{dY_1}{A-Y_1}+\frac{dy_2}{A-y_2}+\frac{dY_2}{A-Y_2},$$

$$10.\quad \frac{dC}{1+C}=\frac{dy_1}{m_6-y_1}+\frac{dY_1}{m_6-Y_1}+\frac{dy_2}{m_6-y_2}+\frac{dY_2}{m_6-Y_2},$$

$$11.\quad \frac{dC}{1+C}=\frac{dy_1}{m_5-y_1}+\frac{dY_1}{m_5-Y_1}+\frac{dy_2}{m_5-y_2}+\frac{dY_2}{m_5-Y_2},$$

$$12.\quad \frac{(z-A)(z-m_6)(z-m_5)\{(z-A)dC-2CdA\}}{C(z-A)^2(z-m_6)(z-m_5)+(z-m_4)(z-m_3)(z-m_2)(z-m_1)}-\frac{dC}{1+C}$$
$$=-\frac{dy_1}{z-y_1}-\frac{dY_1}{z-Y_1}-\frac{dy_2}{z-y_2}-\frac{dY_2}{z-Y_2}.$$

quarum postrema brevitatis gratia posito:

$$U = \frac{\sqrt{C}(z-A)(z-m_6)(z-m_5)}{\sqrt{(-\Delta z)}}$$

in hanc abit

$$13.\quad 2\,U\,d.(\text{arc tang}\,U) - \frac{dC}{1+C} = -\frac{dy_1}{z-y_1} - \frac{dY_1}{z-Y_1} - \frac{dy_2}{z-y_2} - \frac{dY_2}{z-Y_2}.$$

Summa aequationum (9.) et (13.) per $\frac{1}{(z-A)(A-m_6)(A-m_5)\sqrt{C}}$,

\- - - - - - (10.) et (13.) per $\frac{1}{(z-m_6)(m_6-m_5)(m_6-A)\sqrt{C}}$,

\- - - - - - (11.) et (13.) per $\frac{1}{(z-m_5)(m_5-A)(m_5-m_6)\sqrt{C}}$

multiplicata, triumque horum productorum additione facta ob aequationem identicam:

$$\frac{1}{(z-A)(A-m_6)(A-m_5)} + \frac{1}{(z-m_6)(m_6-m_5)(m_6-A)} + \frac{1}{(z-m_5)(m_5-A)(m_5-m_6)}$$
$$= \frac{1}{(z-A)(z-m_6)(z-m_5)},$$

emanat haec denique formula:

$$-\frac{2d.\,\text{arc tang}\,U}{\sqrt{(-\Delta z)}}$$
$$= \frac{dy_1}{\sqrt{C}(z-y_1)(y_1-A)(y_1-m_6)(y_1-m_5)} + \frac{dY_1}{\sqrt{C}(z-Y_1)(Y_1-A)(Y_1-m_6)(Y_1-m_5)}$$
$$+ \frac{dy_2}{\sqrt{C}(z-y_2)(y_2-A)(y_2-m_6)(y_2-m_5)} + \frac{dY_2}{\sqrt{C}(z-Y_2)(Y_2-A)(Y_2-m_6)(Y_2-m_5)},$$

quae, quatuor formulis (3.) adhibitis, integrationeque instituta inde a valoribus respective:

$$U^0,\quad y_1^0,\quad Y_1^0,\quad y_2^0,\quad Y_2^0,$$

qui simul cum C^0, A^0, cohaerentes quantitatum

$$U,\quad y_1,\quad Y_1,\quad y_2,\quad Y_2,\quad C,\quad A,$$

valores, aequationi (2.) satisfacientes sunt, suppeditat hanc relationem, pro quolibet ipsius z valore comprobatam:

$$14.\quad \int_{y_1^0}^{y_1} \frac{c_1\,dy}{(z-y)\sqrt{(\Delta y)}} + \int_{Y_1^0}^{Y_1} \frac{E_1\,dy}{(z-y)\sqrt{(\Delta y)}} + \int_{y_2^0}^{y_2} \frac{e_2\,dy}{(z-y)\sqrt{(\Delta y)}} + \int_{Y_2^0}^{Y_2} \frac{E_2\,dy}{(z-y)\sqrt{(\Delta y)}}$$
$$= \frac{2}{\sqrt{(-\Delta z)}} \left\{ \text{arc tang} \frac{\sqrt{C^0}(z-A^0)(z-m_6)(z-m_5)}{\sqrt{(-\Delta z)}} - \text{arc tang} \frac{\sqrt{C}(z-A)(z-m_6)(z-m_5)}{\sqrt{(-\Delta z)}} \right\}.$$

Utroque termino per z^x multiplicato, in evolutione secundum descendentes ipsius z potestates facta, coëfficientem potestatis z^{-1} sumere, eamque, ut fieri solet, denotare placet; quo facto habetur altera relatio:

$$15.\quad \int_{y_1^0}^{y_1} \frac{e_1\, y^\varkappa\, dy}{\sqrt{(\varDelta y)}} + \int_{Y_1^0}^{Y_1} \frac{E_1\, y^\varkappa\, dy}{\sqrt{(\varDelta y)}} + \int_{y_2^0}^{y_2} \frac{e_2\, y^\varkappa\, dy}{\sqrt{(\varDelta y)}} + \int_{Y_2^0}^{Y_2} \frac{E_2\, y^\varkappa\, dy}{\sqrt{(\varDelta y)}}$$

$$= \left[\frac{2 z^\varkappa}{\sqrt{(-\varDelta z)}} (\text{arc tang } U^0 - \text{arc tang } U)\right]_{z^{-1}}.$$

Inde deducis has aequationes:

$$16.\quad \Sigma \int_{y^0}^{y} \frac{e\, dy}{\sqrt{(\varDelta y)}} = 0,$$

$$17.\quad \Sigma \int_{y^0}^{y} \frac{e y\, dy}{\sqrt{(\varDelta y)}} = 0,$$

$$18.\quad \Sigma \int_{y^0}^{y} \frac{e y^2\, dy}{\sqrt{(\varDelta y)}} = 2(\text{arc tang}\sqrt{C^0} - \text{arc tang}\sqrt{C}),$$

$$19.\quad \Sigma \int_{y^0}^{y} \frac{e y^3\, dy}{\sqrt{(\varDelta y)}} = (m_1+m_2+m_3+m_4+m_5+m_6)(\text{arc tang}\sqrt{C^0} - \text{arc tang}\sqrt{C})$$
$$+ (m_1+m_2+m_3+m_4-m_5-m_6)\left(\frac{\sqrt{C^0}}{1+C^0} - \frac{\sqrt{C}}{1+C}\right)$$
$$-2\left\{\frac{A^0\sqrt{C^0}}{1+C^0} - \frac{A\sqrt{C}}{1+C}\right\},$$

$$20.\quad \Sigma \int_{y^0}^{y} \frac{e\, \Phi y\, dy}{\sqrt{(\varDelta y)}} = \left[\frac{2\Phi z}{\sqrt{(-\varDelta z)}} (\text{arc tang } U^0 - \text{arc tang } U)\right]_{z^{-1}},$$

$$21.\quad \Sigma \int_{y^0}^{y} \frac{e\, \Phi y\, dy}{(\alpha-y)\sqrt{(\varDelta y)}} = \left[\frac{2\Phi z}{(\alpha-z)\sqrt{(-\varDelta z)}} (\text{arc tang } U^0 - \text{arc tang } U)\right]_{z^{-1}}$$
$$+ \frac{2\Phi\alpha}{\sqrt{(-\varDelta\alpha)}} (\text{arc tang } U_\alpha^0 - \text{arc tang } U_\alpha)$$

ubi Φy functionem rationalem integram ipsius y denotat, atque brevitatis gratia formula summatoria:

$$22.\quad \Sigma e f y = e_1 f y_1 + E_1 f Y_1 + e_2 f y_2 + E_2 f Y_2$$

et denotationes:

$$U_\alpha = \frac{\sqrt{C}(\alpha-A)(\alpha-m_6)(\alpha-m_5)}{\sqrt{(-\varDelta\alpha)}}, \qquad U_\alpha^0 = \frac{\sqrt{C^0}(\alpha-A^0)(\alpha-m_6)(\alpha-m_5)}{\sqrt{(-\varDelta\alpha)}}$$

adhibentur.

Iam restat relationes algebraicas inter quantitates y_1, Y_1, y_2, Y_2, determinare. Si quantitates y_1 et y_2 una cum signis e_1 et e_2 ut datas, atque ceteras Y_1 et Y_2 una cum E_1, E_2, C et A ut determinandas consideras, e formulis (2.) et (3.) producis hanc, pro quolibet ipsius v valore,

$$23.\quad (y_1-y_2)\sqrt{C(v-A)} = \frac{e_1\sqrt{(\Delta y_1)}(v-y_2)}{(y_1-m_6)(y_1-m_5)} - \frac{e_2\sqrt{(\Delta y_2)}(v-y_1)}{(y_2-m_6)(y_2-m_5)};$$

unde has derivas:

$$24.\quad \sqrt{C} = \frac{1}{y_1-y_2}\left(\frac{e_1\sqrt{(\Delta y_1)}}{(y_1-m_6)(y_1-m_5)} - \frac{e_2\sqrt{(\Delta y_2)}}{(y_2-m_6)(y_2-m_5)}\right),$$

$$25.\quad A = \frac{e_1 y_2\sqrt{(\Delta y_1)}(y_2-m_6)(y_2-m_5) - e_2 y_1\sqrt{(\Delta y_2)}(y_1-m_6)(y_1-m_5)}{e_1\sqrt{(\Delta y_1)}(y_2-m_6)(y_2-m_5) - e_2\sqrt{(\Delta y_2)}(y_1-m_6)(y_1-m_5)},$$

quibus adhibitis, e formulis (5.), (6.), (7.) formulae hae prodeunt:

$$26.\quad \left\{\begin{array}{ll} (m_6-Y_1)(m_6-Y_2) & \dfrac{(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)}{(m_6-y_1)(m_6-y_2)} \\ :(m_5-Y_1)(m_5-Y_2) & :\dfrac{(m_5-m_1)(m_5-m_2)(m_5-m_3)(m_5-m_4)}{(m_5-y_1)(m_5-y_2)} \\ :(m_\varkappa-Y_1)(m_\varkappa-Y_2) \quad = & :\left\{\dfrac{e_1\sqrt{\left[\Delta y_1\cdot\left(\frac{m_\varkappa-y_2}{m_\varkappa-y_1}\right)\right]}}{(m_5-y_1)(m_6-y_1)} - \dfrac{e_2\sqrt{\left[\Delta y_2\cdot\left(\frac{m_\varkappa-y_1}{m_\varkappa-y_2}\right)\right]}}{(m_5-y_2)(m_6-y_2)}\right\}^2\dfrac{(m_5-m_\varkappa)(m_6-m_\varkappa)}{(y_1-y_2)^2} \\ :1 & :1+\dfrac{1}{(y_1-y_2)^2}\left\{\dfrac{e_1\sqrt{(\Delta y_1)}}{(m_5-y_1)(m_6-y_1)} - \dfrac{e_2\sqrt{(\Delta y_2)}}{(m_5-y_2)(m_6-y_2)}\right\}^2, \end{array}\right.$$

in quibus loco quantitatum y_1, y_2, Y_1, Y_2 etiam valores initiales y_1^0, y_2^0, Y_1^0, Y_2^0 substituere licet. Ad computationem signorum E_1 et E_2 adhibeantur formulae ex (3.) et (23.) sponte prodeuntes:

$$E_1 = \frac{(Y_1-m_6)(Y_1-m_5)}{y_1-y_2}\left\{\frac{e_1\sqrt{(\Delta y_1)}(Y_1-y_2)}{(y_1-m_6)(y_1-m_5)} - \frac{e_2\sqrt{(\Delta y_2)}(Y_1-y_1)}{(y_2-m_6)(y_2-m_5)}\right\},$$

$$E_2 = \frac{(Y_2-m_6)(Y_2-m_5)}{y_1-y_2}\left\{\frac{e_1\sqrt{(\Delta y_1)}(Y_2-y_2)}{(y_1-m_6)(y_1-m_5)} - \frac{e_2\sqrt{(\Delta y_2)}(Y_2-y_1)}{(y_2-m_6)(y_2-m_5)}\right\}.$$

Adiiciatur vero adhuc alia expressio functionis $\sqrt{C(v-A)}$ e formulis (5.) et (7.) deducta, omnesque quatuor quantitates y_1, y_2, Y_1, Y_2 continens. Inde enim, si $\varkappa$ et λ quilibet numerorum 1, 2, 3, 4 sunt, per $e^{(\varkappa)}$ et $e^{(\lambda)}$ positiva vel negativa denotatur unitas, atque brevitatis gratia ponitur:

$$fz = (z-m_1)(z-m_2)(z-m_3)(z-m_4),$$
$$\Pi z = (z-y_1)(z-Y_1)(z-y_2)(z-Y_2),$$

prodeunt formulae:

$$27.\quad \left\{\begin{array}{l} \sqrt{C(m_\varkappa-A)} = e^{(\varkappa)}\sqrt{\left(\dfrac{fm_6}{\Pi m_6}\right)}\sqrt{\left(\dfrac{\Pi m_\varkappa}{(m_6-m_\varkappa)(m_5-m_\varkappa)}\right)}, \\ \sqrt{C(m_\lambda-A)} = e^{(\lambda)}\sqrt{\left(\dfrac{fm_6}{\Pi m_6}\right)}\sqrt{\left(\dfrac{\Pi m_\lambda}{(m_6-m_\lambda)(m_5-m_\lambda)}\right)}, \end{array}\right.$$

nec non inde haec:

$$28. \quad \sqrt{C(v-A)}$$

$$= \sqrt{\left(\frac{fm_6}{\Pi m_6}\right)}\left\{\frac{e^{(\varkappa)}\sqrt{(\Pi m_\varkappa)}}{\sqrt{(m_6-m_\varkappa)(m_5-m_\varkappa)}}\cdot\frac{v-m_\lambda}{m_\varkappa-m_\lambda}+\frac{e^{(\lambda)}\sqrt{(\Pi m_\lambda)}}{\sqrt{(m_6-m_\lambda)(m_5-m_\lambda)}}\cdot\frac{v-m_\varkappa}{m_\lambda-m_\varkappa}\right\}.$$

Pro casu fundamentali hic eum assumere placet, ubi quantitates y_1 et y_2 respective in intervallis $m_1 \dots m_2$ et $m_3 \dots m_4$ continentur. Quo posito aequatio (2.) docet ipsum C positivo valore gaudere, quippe quod si esset negativum, prior eiusdem aequationis terminus pro $z=y_1$, evanescere non posset. Hinc sequitur quantitatem Y_1 in intervallo $m_1 \dots m_2$, et quantitatem Y_2 in intervallo $m_3 \dots m_4$ contineri. Deinde patet, si quantitas A in intervallo $m_1 \dots m_2$ iacet, sive si habetur aeqnatio: $e^{(1)}e^{(2)}=-1$, fore: $e_1=-E_1$, $e_2=E_2$, eodem modo, si habetur: $e^{(3)}e^{(4)}=-1$, fore: $e_1=E_1$, $e_2=-E_2$. Contra si quantitas A nec in intervallo $m_1 \dots m_2$, nec in intervallo $m_3 \dots m_4$ continetur, sive si habetur $e^{(1)}e^{(2)}=1$, $e^{(3)}e^{(4)}=1$ erit: $e_1=E_1$, $e_2=E_2$. Inverse e signis e_1, e_2, E_1, E_2, signa $e^{(1)}$, $e^{(2)}$, $e^{(3)}$, $e^{(4)}$, determinari possunt.

Deinde denotatis minori quantitatum y_1 et Y_1 per v_0,
maiori - - - - - - - per v_1,
minori quantitatum y_2 et Y_2 per v_2,
maiori - - - - - - - per v_3,

nec non signis e_1, E_1, e_2, E_2 iis correspondentibus per:

ε_0, ε_1, ε_2, ε_3,

habetur hoc lemma:

„Si quatuor radices v_0, v_1, v_2, v_3 aequationis:

$$29. \quad C(z-A)^2(z-m_6)(z-m_5)+(z-m_1)(z-m_2)(z-m_3)(z-m_4)=0,$$

ex ordine scriptae tales sunt, ut differentiae:

$$v_0-m_1,\quad m_2-v_1,\quad v_2-m_3,\quad m_4-v_3,$$

positivis valoribus gaudeant, aequationis eiusdem formae:

$$30. \quad C^0(z-A)^2(z-m_6)(z-m_5)+(z-m_4)(z-m_3)(z-m_2)(z-m_1)=0$$

radices quatuor v_0^0, v_1^0, v_2^0, v_3^0, si C^0 quantitas positiva ipso C minor est, tales erunt, ut differentiae:

$$v_0^0-m_1.\quad v_0-v_0^0,\quad v_1^0-v_1,\quad m_2-v_1^0,$$
$$v_2^0-m_3,\quad v_2-v_2^0,\quad v_3^0-v_3,\quad m_4-v_3^0,$$

et ipsae positivae sint."

Demonstratio. Aequationis prior terminus:

pro $z = m_1$ positivo valore,
- $z = v_0$ negativo valore,
- $z = v_1$ negativo valore,
- $z = m_2$ positivo valore,
- $z = m_3$ positivo valore,
- $z = v_2$ negativo valore,
- $z = v_3$ negativo valore,
- $z = m_4$ positivo valore

gaudet, unde sequitur q. e. d.

Adiicere placet, posito:

$$\Pi z = (z - v_0)(z - v_1)(z - v_2)(z - v_3),$$
$$\Pi_0 z = (z - v_0^0)(z - v_1^0)(z - v_2^0)(z - v_3^0),$$

pro quolibet ipsius C^0 valore positivo, ipsam C haud superante,

quantitates: $\Pi_0'(v_0^0)$ et $\Pi_0'(v_2^0)$ negativas et
quantitates: $\Pi_0'(v_1^0)$ et $\Pi_0'(v_3^0)$ positivas esse.

Inde deducitur hoc theorema:

„Quantitate C^0 a nihilo, usque ad valorem C talem, ut radices aequationis:

$$C(z-A)^2(z-m_6)(z-m_5) + (z-m_4)(z-m_3)(z-m_2)(z-m_1) = 0,$$

v_0, v_1, in intervallo $m_1 \dots m_2$, atque v_2 et v_3 in intervallo $m_3 \dots m_4$ iaceant, continuo crescente, simul radices aequationis:

$$C^0(z-A)^2(z-m_6)(z-m_5) + (z-m_4)(z-m_3)(z-m_2)(z-m_1) = 0,$$

ab m_1 usque ad v_0 crescendo,
ab m_2 - - v_1 decrescendo,
ab m_3 - - v_2 crescendo,
ab m_4 - - v_3 decrescendo,

continuo progrediuntur."

Demonstratio. Ipso C^0 ut variabili, et z loco radicis aequationis (30.) assumtis habetur per differentiationem:

$$-\frac{dC^0}{dz} = (1 + C^0)\frac{\Pi_0' z}{(z-A)^2(z-m_6)(z-m_5)}.$$

Iam igitur, ex antecedentibus sequitur, valores ipsorum:

$$\frac{dC^0}{dv_0^0}, \quad \frac{dC^0}{dv_2^0}$$

positivos finitos, nec non ipsorum:

$$\frac{dC^0}{dv_1^0}, \quad \frac{dC^0}{dv_3^0}$$

negativos finitos manere, quoad quantitas positiva C a quantitate C^0 haud superetur. Unde sequitur q. e. d.

Quia expressiones quatuor:

$$\frac{dC^0}{dv_0^0},\quad \frac{dC^0}{dv_1^0},\quad \frac{dC^0}{dv_2^0},\quad \frac{dC^0}{dv_3^0},$$

dum quantitates C^0 a nihilo usque ad C pergit, signum non mutant, et hanc ob rem, nec evanescere possunt, nec in infinitum abire, valor C talis sit necesse est, ut ab omnibus expressionis:

$$-\frac{(z-m_4)(z-m_3)(z-m_2)(z-m_1)}{(z-A)^2(z-m_6)(z-m_5)}$$

maximis et minimis positivis superetur, si quantitas A constantem valorem obtinet. At adeo, nihil impedit, quo minus, eadem quantitate simul cum C^0 apte se variante, considerationes antecedentes de radicum quatuor continuitate valeant. Assumantur enim hunc ad finem quantitatis variabilis C^0 valores supremi, ipsi variabiles et ita decrescentes, ut superent nullum maximorum minimorumve positivorum, quae functio:

$$-\frac{(z-m_4)(z-m_3)(z-m_2)(z-m_1)}{(z-A)^2(z-m_6)(z-m_5)}$$

pro singulo quoque ipsius A valore, inter duos eiusdem limites iacente, assequitur. Adiiciendum est generaliter signa ε_0, ε_1, ε_2, ε_3, quantitatibus continuo progredientibus v_0^0, v_1^0, v_2^0, v_3^0, manere, nisi quantitas A per valorem ullum harum radicum permigrat. Id quod e formulis (3.) sponte prodit, nec nisi pro $C^0=0$ sive pro $v_0^0=m_1$, $v_1^0=m_2$, $v_2^0=m_3$, $v_3^0=m_4$, fieri potest. —

Quae cum ita sint, in aequationibus (16.), (17.), (18.), (19.), (20), (21.) inferiores integralium limites apte commutari possunt cum ipsis: m_1, m_2, m_3, m_4, unde emanent hae aequationes:

31. $$\int_{m_1}^{v_0}\frac{\varepsilon_0\,dy}{\sqrt{(\Delta y)}}+\int_{m_2}^{v_1}\frac{\varepsilon_1\,dy}{\sqrt{(\Delta y)}}+\int_{m_3}^{v_2}\frac{\varepsilon_2\,dy}{\sqrt{(\Delta y)}}+\int_{m_4}^{v_3}\frac{\varepsilon_3\,dy}{\sqrt{(\Delta y)}}=0,$$

32. $$\int_{m_1}^{v_0}\frac{\varepsilon_0\,y\,dy}{\sqrt{(\Delta y)}}+\int_{m_2}^{v_1}\frac{\varepsilon_1\,y\,dy}{\sqrt{(\Delta y)}}+\int_{m_3}^{v_2}\frac{\varepsilon_2\,y\,dy}{\sqrt{(\Delta y)}}+\int_{m_4}^{v_3}\frac{\varepsilon_3\,dy}{\sqrt{(\Delta y)}}=0,$$

33. $$\int_{m_1}^{v_0}\frac{\varepsilon_0\,\Phi y\,dy}{\sqrt{(\Delta y)}}+\int_{m_2}^{v_1}\frac{\varepsilon_1\,\Phi y\,dy}{\sqrt{(\Delta y)}}+\int_{m_3}^{v_2}\frac{\varepsilon_2\,\Phi y\,dy}{\sqrt{(\Delta y)}}+\int_{m_4}^{v_3}\frac{\varepsilon_3\,\Phi y\,dy}{\sqrt{(\Delta y)}}$$

$$=-2\left[\frac{\Phi z}{\sqrt{(-\Delta z)}}\operatorname{arc\,tang}\frac{\sqrt{C}(z-A)(z-m_6)(z-m_5)}{\sqrt{(-\Delta z)}}\right]_{z^{-1}},$$

$$34.\quad \int_{m_1}^{v_0}\frac{\varepsilon_0\,\Phi y\,dy}{(\alpha-y)\sqrt{(\varDelta y)}}+\int_{m_2}^{v_1}\frac{\varepsilon_1\,\Phi y\,dy}{(\alpha-y)\sqrt{(\varDelta y)}}+\int_{m_3}^{v_2}\frac{\varepsilon_2\,\Phi y\,dy}{(\alpha-y)\sqrt{(\varDelta y)}}+\int_{m_4}^{v_3}\frac{\varepsilon_3\,\Phi y\,dy}{(\alpha-y)\sqrt{(\varDelta y)}}$$
$$=-2\left[\frac{\Phi z}{(\alpha-z)\sqrt{(-\varDelta z)}}\,\text{arc tang}\,\frac{\sqrt{C}(z-A)(z-m_6)(z-m_5)}{\sqrt{(-\varDelta z)}}\right]_{z^{-1}}$$
$$-2\,\frac{\varphi\alpha}{\sqrt{(-\varDelta\alpha)}}\,\text{arc tang}\,\frac{\sqrt{C}(\alpha-A)(\alpha-m_6)(\alpha-m_5)}{\sqrt{(-\varDelta\alpha)}},$$

ubi secundum formulam (28.) expressio $\sqrt{C}(z-A)$ determinatur aequatione:

$$\sqrt{C}(z-A)=$$
$$\sqrt{\left(\frac{fm_6}{\Pi m_6}\right)}\left\{e^{(1)}\sqrt{\left(\frac{\Pi m_1}{(m_6-m_1)(m_5-m_1)}\right)}\frac{z-m_3}{m_1-m_3}+e^{(3)}\sqrt{\left(\frac{\Pi m_3}{(m_6-m_3)(m_5-m_3)}\right)}\frac{z-m_1}{m_3-m_1}\right\}.$$

Ad formulas in articulo praecedenti expositas in functiones, quas vocant, ***Abelianas*** primi ordinis transferendas, eos casus pro principalibus habere placet, ubi est

$$\varepsilon_0=\varepsilon_1,\quad \varepsilon_2=\varepsilon_3,$$

in quos ob earundem functionum periodicitatem ceteros revocare licet. Quo posito habentur aequationes:

$$e'=e^{(2)}=\varepsilon_0=\varepsilon_1,\qquad e^{(3)}=e^{(4)}=\varepsilon_2=\varepsilon_3.$$

Adiicere placet, duos priores casus articuli (4.) in his suppositionibus modo propositis ipsos contineri, ita ut in priori illorum casuum poni possit:

$$v_0=v_1=x_1,\qquad v_2=v_3=x_2,\qquad A=\alpha,\qquad C=c,$$
$$\varepsilon_0=\varepsilon_1=-1,\qquad \varepsilon_2=\varepsilon_3=+1,$$

atque in posteriori, prout superiora vel inferiora signa ibi valent:

$$\varepsilon_0=\varepsilon_1=\mp1,\qquad \varepsilon_2=\varepsilon_3=\mp1.$$

Signa enim

$$e^{(1)},\quad e^{(2)},\quad e^{(3)},\quad e^{(4)},$$

quibus signa:

$$\varepsilon_0,\quad \varepsilon_1,\quad \varepsilon_2,\quad \varepsilon_3$$

aequalia sunt, congruunt respective cum signis ipsorum:

$$-\eta,\quad -\eta_1,\quad -\eta_1\eta_2\eta_3,\quad -\eta_1\eta_2\eta_3,$$

id quod in art. 4. demonstravimus. Quae cum ita sint, emanant haec duo theoremata.

Theorema I.

„Si, posito:

$$\int_{m_1}^{v_0}\frac{\varepsilon_0\,a\,dy}{\sqrt{(\varDelta y)}}+\int_{m_3}^{v_2}\frac{\varepsilon_2\,a\,dy}{\sqrt{(\varDelta y)}}=2u,$$
$$\int_{m_1}^{v_0}\frac{\varepsilon_0\,a'y\,dy}{\sqrt{(\varDelta y)}}+\int_{m_3}^{v_2}\frac{\varepsilon_2\,a'y\,dy}{\sqrt{(\varDelta y)}}=2u',$$

„ubi a et a' quantitates constantes quaelibet sunt, limites superiores v_0 et v_2 „considerantur ut functiones argumentorum u et u', et brevitatis gratia haec „denotatio adhibetur

$$(m_h - v_0)(m_h - v_2) = \lambda_h(u, u'),$$

„ipso h quemlibet numerorum 1, 2, 3, 4, 5, 6 denotante, nec non ponitur:

$$\int_{m_1}^{v_0} \frac{\varepsilon_0\, a'' y^2\, dy}{\sqrt{(\varDelta y)}} + \int_{m_3}^{v_2} \frac{\varepsilon_2\, a'' y^2\, dy}{\sqrt{(\varDelta y)}} = 2E_2(u, u'),$$

$$\int_{m_1}^{v_0} \frac{\varepsilon_0\, \Phi y\, dy}{(\alpha - y)\sqrt{(\varDelta y)}} + \int_{m_3}^{v_2} \frac{\varepsilon_2\, \Phi y\, dy}{(\alpha - y)\sqrt{(\varDelta y)}} = 2G(u, u'),$$

„ubi a'' et α quantitates constantes quaelibet, et Φy functio rationalis ipsius y „integra est, si deinde argumenta u et u', pro $v_0 = m_2$ et $v_2 = m_4$ transeunt „in valores M et M', ita ut habeantur aequationes:

$$\int_{m_1}^{m_2} \frac{\varepsilon_0\, a\, dy}{\sqrt{(\varDelta y)}} + \int_{m_3}^{m_4} \frac{\varepsilon_2\, a\, dy}{\sqrt{(\varDelta y)}} = 2M,$$

$$\int_{m_1}^{m_2} \frac{\varepsilon_0\, a' y\, dy}{\sqrt{(\varDelta y)}} + \int_{m_3}^{m_4} \frac{\varepsilon_2\, a' y\, dy}{\sqrt{(\varDelta y)}} = 2M',$$

$$\lambda_2(M, M') = 0, \qquad \lambda_4(M, M') = 0,$$

„hae formulae memorabiles habentur, ex antecedentibus sponte prodeuntes:

$$\begin{array}{l} \lambda_6(u,u').\lambda_6(M-u, M'-u') \\ :\lambda_5(u,u').\lambda_5(M-u, M'-u') \\ :\lambda_\varkappa(u,u').\lambda_\varkappa(M-u, M'-u') \\ :1 \end{array} = \begin{array}{l} (m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4) \\ :(m_5-m_1)(m_5-m_2)(m_5-m_3)(m_5-m_4) \\ :\left\{\frac{\varepsilon_0\sqrt{(\varDelta v_0)}(m_\varkappa - v_2)}{(m_5-v_0)(m_6-v_0)} - \frac{\varepsilon_2\sqrt{(\varDelta v_2)}(m_\varkappa - v_0)}{(m_5-v_2)(m_6-v_2)}\right\}^2 \frac{(m_5-m_\varkappa)(m_6-m_\varkappa)}{(v_0-v_2)^2} \\ :1+\left\{\frac{\varepsilon_0\sqrt{(\varDelta v_0)}}{(m_5-v_0)(m_6-v_0)} - \frac{\varepsilon_2\sqrt{(\varDelta v_2)}}{(m_5-v_2)(m_6-v_2)}\right\}^2 \frac{1}{(v_0-v_2)^2}, \end{array}$$

$$E_2(u, u') + E_2(M-u, M'-u') - E_2(M, M') =$$

$$\text{arc tang}\sqrt{\left(\frac{(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)}{(m_3-m_1)^2\lambda_6(u,u')\lambda_6(M-u, M'-u')}\right)}\left\{\varepsilon_0\sqrt{\left(\frac{\lambda_1(u,u')\lambda_1(M-u, M'-u')}{m_6-m_1 \cdot m_5-m_1}\right)} - \varepsilon_2\sqrt{\left(\frac{\lambda_3(u,u')\lambda_3(M-u, M'-u')}{(m_6-m_3)(m_5-m_3)}\right)}\right\},$$

$$G(u, u') + G(M-u, M'-u') - G(M, M') =$$

$$-\left[\frac{\Phi z}{(\alpha-z)\sqrt{(-\varDelta z)}}\,\text{arc tang}\,\frac{\sqrt{C}(z-A)(z-m_6)(z-m_5)}{\sqrt{(-\varDelta z)}}\right]_{z=1} - \frac{\Phi\alpha}{\sqrt{(-\varDelta\alpha)}}\,\text{arc tang}\left(\frac{\sqrt{C}(\alpha-A)(\alpha-m_6)(\alpha-m_5)}{\sqrt{(-\varDelta\alpha)}}\right),$$

„ubi ponitur:

$$\sqrt{C}(z-A) =$$

$$-\frac{1}{m_3-m_1}\cdot\frac{\sqrt{[(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)]}}{\sqrt{(\lambda_6(u,u')\lambda_6(M-u, M'-u'))}}\left\{\varepsilon_0\sqrt{\left(\frac{\lambda_1(u,u')\lambda_1(M-u, M'-u')}{(m_6-m_1)(m_5-m_1)}\right)}(z-m_3)\right.$$

$$\left. - \varepsilon_2\sqrt{\left(\frac{\lambda_3(u,u')\lambda_3(M-u, M'-u')}{(m_6-m_3)(m_5-m_3)}\right)}(z-m_1)\right\}."$$

Theorema II.

„Iisdem denotationibus adhibitis, nec non brevitatis gratia posito:

$$\psi z = \left[z-\varepsilon_0\sqrt{\left(\frac{m_5-m_1}{m_6-m_1}\right)}\right]\left[z+\varepsilon_0\sqrt{\left(\frac{m_5-m_2}{m_6-m_2}\right)}\right]\left[z+\varepsilon_2\sqrt{\left(\frac{m_5-m_3}{m_6-m_3}\right)}\right]\left[z-\varepsilon_2\sqrt{\left(\frac{m_5-m_4}{m_6-m_4}\right)}\right],$$

„ubi signa ε_0, ε_2 conditioni $\psi(1) > \psi(-1)$ satisfaciunt: habentur relationes „memorabiles:

$$\begin{aligned}
&\sqrt{[\lambda_6(\tfrac{1}{2}M,\tfrac{1}{2}M')]} && \sqrt{2}\,(m_6-m_5)\\
&:\sqrt{[\lambda_5(\tfrac{1}{2}M,\tfrac{1}{2}M')]} && :\sqrt{2}\,(m_6-m_5)\sqrt[4]{\left(\frac{(m_5-m_1)(m_5-m_2)(m_5-m_3)(m_5-m_4)}{(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)}\right)}\\
&:\sqrt{[\lambda_4(\tfrac{1}{2}M,\tfrac{1}{2}M')]} && :(m_6-m_4)\sqrt[2]{\left[\psi\left(-\varepsilon_2\sqrt{\left(\frac{m_5-m_4}{m_6-m_4}\right)}\right)\right]}\\
&:\sqrt{[\lambda_3(\tfrac{1}{2}M,\tfrac{1}{2}M')]} = && :(m_6-m_3)\sqrt[2]{\left[\psi\left(+\varepsilon_2\sqrt{\left(\frac{m_5-m_3}{m_6-m_3}\right)}\right)\right]}\\
&:\sqrt{[\lambda_2(\tfrac{1}{2}M,\tfrac{1}{2}M')]} && :(m_6-m_2)\sqrt[2]{\left[\psi\left(+\varepsilon_0\sqrt{\left(\frac{m_5-m_2}{m_6-m_2}\right)}\right)\right]}\\
&:\sqrt{[\lambda_1(\tfrac{1}{2}M,\tfrac{1}{2}M')]} && :(m_6-m_1)\sqrt[2]{\left[\psi\left(-\varepsilon_0\sqrt{\left(\frac{m_5-m_1}{m_6-m_1}\right)}\right)\right]}\\
&:1 && :\sqrt{[\psi(1)+\psi(-1)]},
\end{aligned}$$

$$2E_2(\tfrac{1}{2}M,\tfrac{1}{2}M')-E_2(M,M')$$
$$=\text{arc tang}\frac{\sqrt{[(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)]}}{(m_6-m_5)^2}\cdot\frac{\psi(1)-\psi(-1)}{2},$$

$$2G(\tfrac{1}{2}M,\tfrac{1}{2}M')-G(M,M')$$
$$=-\left[\frac{\Phi(z)}{(\alpha-z)}\cdot\frac{\text{arc tang}\,\chi(z)}{\sqrt{(-\Delta z)}}\right]_{z^{-1}}-\frac{\Phi(\alpha)}{\sqrt{(-\Delta\alpha)}}\,\text{arc tang}\,\chi(\alpha),$$

„ubi ponitur:

$$\chi(z)=\frac{\sqrt{[(m_6-m_1)(m_6-m_2)(m_6-m_3)(m_6-m_4)]}}{\sqrt{(-\Delta z)}}\left(\frac{z-m_6}{m_6-m_5}\right)^2(z-m_5)\left\{\frac{\psi\left(\sqrt{\left(\frac{z-m_5}{z-m_6}\right)}\right)-\psi\left(-\sqrt{\left(\frac{z-m_5}{z-m_6}\right)}\right)}{2\sqrt{\left(\frac{z-m_5}{z-m_6}\right)}}\right\}."$$

Ceteros casus §i. 4. similiter ad formulas in articulo praecedenti expositas, quamquam in promptu est, alio tamen loco una cum simili interpretatione analytica applicare velimus. —

Si in utroque theoremate antecedente loco quantitatum a, a', a'' ponitur $\sqrt{m_6}$, atque loco functionis $\Phi(z)$ vel $\sqrt{m_6}(a-z)\Phi_1(z)$, vel $\sqrt{m_6}\,\Phi_2(z)$, ubi $\Phi_1(z)$ et $\Phi_2(z)$ ipsius z functiones rationales integrae sunt, illa ordinis tertii, haec ordinis secundi, pro valore ipsius m_6 in infinitum abeunte, haec de tribus functionum **Abelianarum** primi ordinis generibus theoremata emanant.

Theorema III.

„Si denotatio introducitur haec:

$$(y-m_1)(y-m_2)(y-m_3)(y-m_4)(y-m_5) = Dy,$$

$$\int_{m_1}^{v_0}\frac{\varepsilon_0\, dy}{\sqrt{(Dy)}}+\int_{m_3}^{v_2}\frac{\varepsilon_2\, dy}{\sqrt{(Dy)}} = 2u,$$

$$\int_{m_1}^{v_0}\frac{\varepsilon_0\, y\, dy}{\sqrt{(Dy)}}+\int_{m_3}^{v_2}\frac{\varepsilon_2\, y\, dy}{\sqrt{(Dy)}} = 2u',$$

„atque limites v_0, v_2 considerantur ut functiones argumentorum u, u' tales, ut „generaliter ponatur:

$$(m_\varkappa - v_0)(m_\varkappa - v_2) = \lambda_\varkappa(u, u'),$$

$$\int_{m_1}^{v_0}\frac{\varepsilon_0\, \Phi_1(y)\, dy}{\sqrt{(Dy)}}+\int_{m_3}^{v_2}\frac{\varepsilon_2\, \Phi_2(y)\, dy}{\sqrt{(Dy)}} = 2E(u, u'),$$

$$\int_{m_1}^{v_0}\frac{\varepsilon_0\, \Phi_2(y)\, dy}{(\alpha-y)\sqrt{(Dy)}}+\int_{m_3}^{v_2}\frac{\varepsilon_2\, \Phi_2(y)\, dy}{(\alpha-y)\sqrt{(Dy)}} = 2G(u, u'),$$

„ubi per $\varkappa$ quilibet numerorum 1, 2, ... 5, et per $\Phi_1(y)$, $\Phi_2(y)$ functio- „nes integrae rationales illa tertii, haec secundi ordinis designantur, habentur „aequationes:

$$\lambda_5(u, u')\ \lambda_5(M-u, M'-u') = (m_5-m_1)(m_5-m_2)(m_5-m_3)(m_5-m_4),$$

$$\lambda_\varkappa(u, u')\ \lambda_\varkappa(M-u, M'-u') = \frac{m_5-m_\varkappa}{(v_2-v_0)^2}\left\{\frac{m_\varkappa-v_2}{m_5-v_0}\sqrt{(Dv_0)}-\frac{m_\varkappa-v_0}{m_5-v_2}\sqrt{(Dv_2)}\right\}^2,$$

$$E(u, u')+E(M-u, M'-u')-E(M, M')$$
$$= +\left[\frac{\Phi(z)}{\sqrt{(Dz)}}\left(\chi(z)+\frac{\{\chi(z)\}^3}{3}\right)\right]_{z^{-1}},$$

$$G(u, u')+G(M-u, M'-u')-G(M, M') = +\frac{\Phi_2(\alpha)}{\sqrt{(-D\alpha)}}\operatorname{arc\,tang}(\sqrt{(-1)}\,\chi\alpha)$$
$$= -\frac{1}{2}\,\frac{\Phi_2(\alpha)}{\sqrt{(D\alpha)}}\log\left(\frac{1-\chi\alpha}{1+\chi\alpha}\right),$$

„ubi quantitates M, M', χz determinantur aequationibus:

$$\int_{m_1}^{m_2}\frac{\varepsilon_0\, dy}{\sqrt{(Dy)}}+\int_{m_3}^{m_4}\frac{\varepsilon_2\, dy}{\sqrt{(Dy)}} = 2M, \qquad \int_{m_1}^{m_2}\frac{\varepsilon_0\, y\, dy}{\sqrt{(Dy)}}+\int_{m_3}^{m_4}\frac{\varepsilon_2\, y\, dy}{\sqrt{(Dy)}} = 2M',$$

$$\chi(z) = +\frac{z-m_5}{m_3-m_1}\cdot\frac{1}{\sqrt{(Dz)}}\left\{\varepsilon_0(z-m_3)\sqrt{\left(\frac{\lambda_1(u,u')\,\lambda_1(M-u, M'-u)}{m_5-m_1}\right)}\right.$$
$$\left.-\varepsilon_2(z-m_1)\sqrt{\left(\frac{\lambda_3(u,u')\,\lambda_3(M-u, M'-u')}{m_5-m_3}\right)}\right\}.“$$

Theorema IV.

„Iisdem denotationibus adhibitis, nec non brevitatis gratia posito: $(z-\varepsilon_0\sqrt{(m_5-m_1)})(z+\varepsilon_0\sqrt{(m_5-m_2)})(z+\varepsilon_2\sqrt{(m_5-m_3)})(z-\varepsilon_2\sqrt{(m_5-m_4)}) = \psi(z)$, „ubi signa ε_0, ε_2 conditioni

$$\psi(1) > \psi(-1)$$

„satisfaciunt, valores functionum $\lambda(u,u')$, $E(u,u')$, $G(u,u')$, pro $u=\frac{1}{2}M$, „$u'=\frac{1}{2}M'$ dantur his formulis:

$$\lambda_5(\tfrac{1}{2}M,\tfrac{1}{2}M') = \sqrt{\{(m_5-m_1)(m_5-m_2)(m_5-m_3)(m_5-m_4)\}},$$
$$\lambda_4(\tfrac{1}{2}M,\tfrac{1}{2}M') = \tfrac{1}{2}\psi(-\varepsilon_2\sqrt{(m_5-m_4)}),$$
$$\lambda_3(\tfrac{1}{2}M,\tfrac{1}{2}M') = \tfrac{1}{2}\psi(\varepsilon_2\sqrt{(m_5-m_3)}),$$
$$\lambda_2(\tfrac{1}{2}M,\tfrac{1}{2}M') = \tfrac{1}{2}\psi(\varepsilon_0\sqrt{(m_5-m_2)}),$$
$$\lambda_1(\tfrac{1}{2}M,\tfrac{1}{2}M') = \tfrac{1}{2}\psi(-\varepsilon_0\sqrt{(m_5-m_1)}),$$
$$2E(\tfrac{1}{2}M,\tfrac{1}{2}M')-E(M,M') = \left[\frac{\varphi_1(z)}{\sqrt{(Dz)}}(\chi(z)+\tfrac{1}{3}\{\chi(z)\}^3)\right]_{z^{-1}},$$
$$2G(\tfrac{1}{2}M,\tfrac{1}{2}M')-G(M,M') = \frac{\Phi_2(\alpha)}{\sqrt{(-D\alpha)}}\operatorname{arc\,tang}(\sqrt{(-1)}\,\chi(\alpha))$$
$$= -\frac{1}{2}\,\frac{\Phi_2\,\alpha}{\sqrt{(D\alpha)}}\log\frac{1-\chi(\alpha)}{1+\chi(\alpha)},$$

„ubi ponitur:

$$\chi z = \frac{(z-m_5)}{\sqrt{(-Dz)}}\left\{\frac{\psi(\sqrt{(-1)}\sqrt{(z-m_5)})-\psi(-\sqrt{(-1)}\sqrt{(z-m_5)})}{2\sqrt{(z-m_5)}}\right\};$$

„ita ut exempli gratia, si quantitas z quantitatem m_5 superat, habeatur:

$$\chi z = \frac{(z-m_5)^2}{\sqrt{(Dz)}}\{\varepsilon_0(\sqrt{(m_5-m_1)}-\sqrt{(m_5-m_2)})-\varepsilon_2(\sqrt{(m_5-m_3)}-\sqrt{(m_5-m_4)})\}$$
$$-\frac{z-m_5}{\sqrt{(Dz)}}\left(\frac{\varepsilon_0}{\sqrt{(m_5-m_1)}}-\frac{\varepsilon_0}{\sqrt{(m_5-m_2)}}\right.$$
$$\left.-\frac{\varepsilon_2}{\sqrt{(m_5-m_3)}}+\frac{\varepsilon_2}{\sqrt{(m_5-m_4)}}\right)\sqrt{(m_5-m_1)(m_5-m_2)(m_5-m_3)(m_5-m_4)}.$$

6.

In sequentibus easdem disquisitiones de integralibus atque functionibus *Abelianis* generalis cuiuslibet ordinis instituturi, rursus initium facere velimus a problematis algebraici solutione. Quod problema generale hoc est:

Si quantitates datae $m_1, m_2, \ldots m_{2n+2}$ tales sunt, ut differentiae m_2-m_1, $m_3-m_2, \ldots m_{2n+2}-m_{2n+1}$, positivis valoribus gaudeant, quantitates $c, a, a_1, a_2, \ldots a_{n-2}$, ita sunt determinandae, ut expressio:

$$1.\quad c(z-a)^2(z-a_1)^2\ldots(z-a_{n-2})^2(z-m_{2n+2})(z-m_{2n+1})$$
$$+(z-m_{2n})(z-m_{2n-1})\ldots(z-m_1),$$

functionis integrae determinandae

2. $\sqrt{(1+c)(z-x)(z-x_2)(z-x_4)\dots(z-x_{2n-2})}$,

quadratum fiat.

Quod ad solvendum brevitatis gratia introducantur signa haec:

$$3.\quad \begin{cases} f(z) = (z-m_1)(z-m_2)\dots(z-m_{2n}), \\ \varrho(z) = (z-a)(z-a_1)\dots(z-a_{n-2}), \\ \varphi(z) = (z-x)(z-x_2)\dots(z-x_{2n-2}), \end{cases}$$

ita ut habeatur aequatio identica:

$$4.\quad c(\varrho(z))^2(z-m_{2n+2})(z-m_{2n+1})+f(z) = (1+c)(\varphi(z))^2.$$

In cuius utroque termino earundem ipsius z potestatum coëfficientes comparando, $2n$ prodeunt aequationes inter $2n$ quantitates:

$$c,\ a,\ a_1,\ a_2,\ \dots\ a_{n-2},\qquad x,\ x_2,\ \dots\ x_{2n-2}.$$

Unde docemur problema propositum esse determinatum. Et adeo facillime demonstratur, quantitatem $\frac{1+c}{c}$ aeque ac omnes functionum $\varphi(z)$ et $\varrho(z)$ coëfficientes per radicem aequationis 2^{2n-1}ti gradus rationaliter exprimi, cuius coëfficientes functiones rationales quantitatum $m_1, m_2, \dots m_{2n+2}$ sunt; ita ut quantitates $a, a_1, \dots a_{n-2}$ radices aequationis simili natura gaudentis $(n-1)2^{2n-1}$ti gradus, et quantitates $x, x_2, \dots x_{2n-2}$ radices fiant aequationis similis $n.2^{2n-1}$ti gradus.

Nimirum in aequatione (4.) posito $z = m_h$, ubi h quemlibet denotat numerorum 1, 2, ... $2n$, prodeunt $2n$ aequationes formae:

$$5.\quad (m_{2n+2}-m_h)(m_{2n+1}-m_h)(\varrho(m_h))^2-\left(\frac{1+c}{c}\right)(\varphi(m_h))^2 = 0$$

unde ceteras $2n-1$ coëfficientes rationaliter exprimere licet per $\frac{1+c}{c}$. Ex $2n$ aequationibus vero, ex aequationibus (5.) emanantibus, formae:

$$6.\quad \sqrt{\left(\frac{1+c}{c}\right)}\varphi(m_h) = \pm\sqrt{((m_{2n+2}-m_h)(m_{2n+1}-m_h))}(\varrho(m_h))$$

sive per eliminationem, seu potius advocato theoremate notissimo illustrissimi *Cauchy*, quo functio:

$$\sqrt{\left(\frac{1+c}{c}\right)}\frac{\varphi(z)}{\varrho(z)}$$

ex $2n$ ipsius valoribus, pro totidem ipsius z valoribus, determinatur, et quantitatis $\sqrt{\left(\frac{1+c}{c}\right)}$ et ceterarum $2n-1$ coëfficientium expressiones, $2n$ radicalia involventes, prodeunt. Iam ipsius $\frac{1+c}{c}$ expressio inde deducta, radicalium signa quomodocunque assumendo, nonnisi 2^{2n-1} valores diversos induens, aequationis 2^{2n-1}ti gradus radix est, cuius aequationis coëfficientes radicalia non involvunt. Simili natura ceterae coëfficientes gaudent, et adeo, quippe quae rationaliter per $\frac{1+c}{c}$ exprimuntur, ut functiones rationales radicis eiusdem illius aequationis 2^{2n-1}ti gradus determinantur.

Solutione problematis propositi, quae ex antecedentibus deducitur, formulas perlongas atque impeditas suppeditante, e quibus de ipsius natura iudicium repeti nequit, eas in alias elegantiores transmutare iuvat, quas sequenti brevi methodo adipiscimur.

Denotationibus:

$$7.\qquad \frac{m_{2n+1}-z}{m_{2n+2}-z} = v^2,\qquad \frac{m_{2n+1}-m_h}{m_{2n+2}-m_h} = \eta_h^2,$$

introductis, pro quolibet quantitatis y valore formulae habentur:

$$8.\qquad z-y = \frac{v^2(m_{2n+2}-y)-(m_{2n+1}-y)}{v^2-1},$$

$$9.\qquad m_h-y = \frac{\eta_h^2(m_{2n+2}-y)-(m_{2n+1}-y)}{\eta_h^2-1},$$

$$10.\qquad 1-\eta_h^2 = \frac{m_{2n+2}-m_{2n+1}}{m_{2n+2}-m_h}.$$

Ibi loco ipsius y quantitatem a et omnes quantitates m substituendo, functiones $\varrho(z)$, $\varphi(z)$ et aequatio (4.) in has formas transmutantur:

$$11.\qquad \varrho(z) = \frac{\{(m_{2n+2}-a)v^2-(m_{2n+1}-a)\}\{(m_{2n+2}-a_1)v^2-(m_{2n+1}-a_1)\}\dots\{(m_{2n+2}-a_{n-2})v^2-(m_{2n+1}-a_{n-2})\}}{(v^2-1)^{n-1}},$$

$$12.\qquad \varphi(z) = \frac{\{(m_{2n+2}-x)v^2-(m_{2n+1}-x)\}\{(m_{2n+2}-x_2)v^2-(m_{2n+1}-x_2)\}\dots\{(m_{2n+2}-x_{2n-2})v^2-(m_{2n+1}-x_{2n-2})\}}{(v^2-1)^n},$$

$$13.\qquad V^2+(f(m_{2n+2}))\,\psi(v)\,\psi(-v) = U^2,$$

ubi brevitatis gratia ponitur:

$$14.\left\{\begin{aligned} V &= \sqrt{c}\,(m_{2n+2}-m_{2n+1})v\{(m_{2n+2}-a)v^2-(m_{2n+1}-a)\}\{(m_{2n+2}-a_1)v^2-(m_{2n+1}-a_1)\}\dots \\ &\qquad\dots\{(m_{2n+2}-a_{n-2})v^2-(m_{2n+1}-a_{n-2})\}, \\ U &= \sqrt{(1+c)}\,\{(m_{2n+2}-x)v^2-(m_{2n+1}-x)\}\{(m_{2n+2}-x_2)v^2-(m_{2n+1}-x_2)\}\dots \\ &\qquad\dots\{(m_{2n+2}-x_{2n-2})v^2-(m_{2n+1}-x_{2n-2})\}, \end{aligned}\right.$$

$$\psi(v) = (v+\eta_1)(v+\eta_2)\dots(v+\eta_{2n}).$$

Hinc facile concluditur functiones U et V his formis gaudere:

$$15.\quad U = c_1\frac{\psi(v)+\psi(-v)}{2},\qquad V = c_1\frac{\psi(v)-\psi(-v)}{2}.$$

Quantitas constans c_1, simul cum ipso c atque valoribus $\varrho(m_{2n+2})$, $\varrho(m_{2n+1})$, $\varrho(m_h)$, $\varphi(m_{2n+2})$, $\varphi(m_{2n+1})$, $\varphi(m_h)$ determinatur ponendo in utroque formularum (11.) termino:

$$v^2 = 1,\quad v^2 = \infty,\quad v^2 = \frac{1}{\infty},\quad v^2 = \eta_h^2.$$

Inde enim prodeunt formulae:

$$16.\quad \sqrt{c} = \frac{c_1}{2}\cdot\frac{\psi(1)-\psi(-1)}{(m_{2n+2}-m_{2n+1})^n},$$

$$17.\quad \sqrt{(1+c)} = \frac{c_1}{2}\cdot\frac{\psi(1)+\psi(-1)}{(m_{2n+2}-m_{2n+1})^n},$$

$$18.\quad \varrho(m_{2n+2}) = \frac{c_1}{\sqrt{c}}\cdot\frac{\eta_1+\eta_2+\dots+\eta_{2n}}{m_{2n+2}-m_{2n+1}},$$

$$19.\quad \varphi(m_{2n+2}) = \frac{c_1}{\sqrt{(1+c)}},$$

$$20.\quad \varrho(m_{2n+1}) = (-1)^{n-1}\frac{c_1}{\sqrt{c}}\cdot\frac{\eta_1\eta_2\dots\eta_{2n}}{m_{2n+2}-m_{2n+1}}\left\{\frac{1}{\eta_1}+\frac{1}{\eta_2}+\dots+\frac{1}{\eta_{2n}}\right\},$$

$$21.\quad \varphi(m_{2n+1}) = (-1)^n\frac{c_1}{\sqrt{(1+c)}}\eta_1\eta_2\dots\eta_{2n},$$

$$22.\quad \varrho(m_h) = (-1)^{n-1}\frac{c_1}{2\sqrt{c}}\cdot\frac{(m_{2n+2}-m_h)^{n-1}}{(m_{2n+2}-m_{2n+1})^n}\cdot\left(\frac{\psi(\eta_h)}{\eta_h}\right),$$

$$23.\quad \varphi(m_h) = (-1)^n\frac{c_1}{2\sqrt{(1+c)}}\cdot\frac{(m_{2n+2}-m_h)^n}{(m_{2n+2}-m_{2n+1})^n}(\psi(\eta_h)).$$

Formulae (16.) et (17.), advocata formula (10.), suppeditant hunc ipsius c_1^2 valorem:

$$c_1^2 = (m_{2n+2}-m_1)(m_{2n+2}-m_2)\dots(m_{2n+2}-m_{2n}) = f(m_{2n+2})$$

nec non formula (17.) docet, quia quantitates η_h^2 unitate minores sunt, ipsum c_1 esse positivum; quibus collatis hae denique emanant formulae elegantes, ad determinationem ipsius c atque functionum ϱz et φz utiles:

$$24.\quad c_1 = \sqrt{(f(m_{2n+2}))},$$

$$25.\quad \sqrt{c} = \frac{\sqrt{(f(m_{2n+2}))}}{2}\cdot\frac{\psi(1)-\psi(-1)}{(m_{2n+2}-m_{2n+1})^n},$$

$$26.\quad \sqrt{(1+c)} = \frac{\sqrt{(f(m_{2n+2}))}}{2}\cdot\frac{\psi(1)+\psi(-1)}{(m_{2n+2}-m_{2n+1})^n},$$

$$27.\quad \varrho(m_{2n+2}) = 2.\frac{(m_{2n+2}-m_{2n+1})^{n-1}}{\psi(1)-\psi(-1)}\{\eta_1+\eta_2+\ldots+\eta_{2n}\},$$

$$28.\quad \varrho(m_{2n+1}) = (-1)^{n-1}2.\frac{(m_{2n+2}-m_{2n+1})^{n-1}}{\psi(1)-\psi(-1)}\eta_1\eta_2\ldots\eta_{2n}\left\{\frac{1}{\eta_1}+\frac{1}{\eta_2}+\ldots+\frac{1}{\eta_{2n}}\right\},$$

$$29.\quad \varrho(m_h) = (-1)^{n-1}\frac{(m_{2n+2}-m_h)^{n-1}}{\psi(1)-\psi(-1)}\cdot\frac{\psi(\eta_h)}{\eta_h},$$

$$30.\quad \varphi(m_{2n+2}) = 2.\frac{(m_{2n+2}-m_{2n+1})^n}{\psi(1)+\psi(-1)},$$

$$31.\quad \varphi(m_{2n+1}) = (-1)^n 2.\frac{(m_{2n+2}-m_{2n+1})^n}{\psi(1)+\psi(-1)}\eta_1\eta_2\ldots\eta_{2n},$$

$$32.\quad \varphi(m_h) = (-1)^n\frac{(m_{2n+2}-m_h)^n}{\psi(1)+\psi(-1)}\psi(\eta_h).$$

Iam formulas (29.) et (32.) ex aequatione (4.) adhuc alio modo deducere placet, nimirum systema peculiare aequationum linearium resolvendo. Aequatio enim (6.) inde deducta, denotatione (7.) adhibita, in hanc abit:

$$33.\quad \varrho(m_h) = \pm\sqrt{\left(\frac{1+c}{c}\right)}\cdot\frac{\varphi(m_h)}{\eta_h(m_{2n+2}-m_h)},$$

ubi radicalium η_h signa talia assumantur, ut pro *omnibus* ipsius h valoribus aut superius aut inferius signum valeat.

Sint n quantitates e numero $2n$ quantitatum

$$m_1,\ m_2,\ \ldots\ m_{2n},$$

ex arbitrio electae:

$$b_1,\ b_2,\ \ldots\ b_n,$$

ceteraeque:

$$c_1,\ c_2,\ \ldots\ c_n,$$

atque denotentur generaliter expressiones:

$$\pm\sqrt{\left(\frac{m_{2n+1}-b_\varkappa}{m_{2n+2}-b_\varkappa}\right)}\quad\text{et}\quad\pm\sqrt{\left(\frac{m_{2n+1}-c_\lambda}{m_{2n+2}-c_\lambda}\right)},$$

per $\beta_\varkappa$ et γ_λ, ubi numerorum $1, 2, \ldots n$ quilibet designantur per $\varkappa$ et λ. Iam si in aequationibus identicis:

$$34.\quad \frac{\varrho(z)}{F(z)} = \sum_1^n{}_\varkappa \frac{\varrho(b_\varkappa)}{F'(b_\varkappa)}\cdot\frac{1}{z-b_\varkappa},\qquad \frac{\varphi(z)}{F(z)} = 1+\sum_1^n{}_\varkappa\frac{\varphi(b_\varkappa)}{F'(b_\varkappa)}\cdot\frac{1}{z-b_\varkappa},$$

ubi ponitur:

$$35.\quad F(z) = (z-b_1)(z-b_2)\ldots(z-b_n),$$

substituitur, $z=c_\lambda$, formulae (33.) ope prodeunt aequationes:

$$\frac{\varphi(c_\lambda)}{F'(c_\lambda)} = \sum_1^n{}^{\varkappa} \frac{\gamma_\lambda}{\beta_\varkappa} \cdot \frac{m_{2n+2}-c_\lambda}{m_{2n-2}-b_\varkappa} \cdot \frac{\varphi(b_\varkappa)}{F'(b_\varkappa)} \cdot \frac{1}{c_\lambda - b_\varkappa},$$

$$\frac{\varphi(c_\lambda)}{F'(c_\lambda)} = 1 + \sum_1^n{}^{\varkappa} \frac{\varphi(b_\varkappa)}{F'(b_\varkappa)} \cdot \frac{1}{c_\lambda - b_\varkappa},$$

quarum differentia, formula (10.) et sequentibus, quae inde derivantur

$$\frac{m_{2n+2}-c_\lambda}{m_{2n+2}-b_\varkappa} = \frac{1-\beta_\varkappa^2}{1-\gamma_\lambda^2}, \qquad \frac{m_{2n+2}-b_\varkappa}{c_\lambda - b_\varkappa} = \frac{1-\gamma_\lambda^2}{\beta_\varkappa^2 - \gamma_\lambda^2},$$

adhibitis, suppeditatur haec aequatio, pro $\lambda = 1$, $\lambda = 2$, ... $\lambda = n$ valens:

$$0 = 1 + \sum_1^n{}^{\varkappa} \frac{\gamma_\lambda + \frac{1}{\beta_\varkappa}}{\gamma_\lambda + \beta_\varkappa} \cdot \frac{\varphi(b_\varkappa)}{(m_{2n+2}-b_\varkappa)F'(b_\varkappa)}.$$

Quae cum ita sint, posito generaliter brevitatis gratia:

$$36. \qquad z_\varkappa = \frac{\varphi(b_\varkappa)}{(m_{2n+2}-b_\varkappa)F'(b_\varkappa)},$$

determinationem ipsius $\varphi b_\varkappa$ reductam invenis ad resolutionem huius aequationum linearium systematis:

$$37. \begin{cases} 0 = 1 + \dfrac{\gamma_1 + \frac{1}{\beta_1}}{\gamma_1 + \beta_1} z_1 + \dfrac{\gamma_1 + \frac{1}{\beta_2}}{\gamma_1 + \beta_2} z_2 + \dots + \dfrac{\gamma_1 + \frac{1}{\beta_n}}{\gamma_1 + \beta_n} z_n, \\ 0 = 1 + \dfrac{\gamma_2 + \frac{1}{\beta_1}}{\gamma_2 + \beta_1} z_1 + \dfrac{\gamma_2 + \frac{1}{\beta_2}}{\gamma_2 + \beta_2} z_2 + \dots + \dfrac{\gamma_2 + \frac{1}{\beta_n}}{\gamma_2 + \beta_n} z_n, \\ \dots\dots\dots\dots\dots\dots \\ 0 = 1 + \dfrac{\gamma_n + \frac{1}{\beta_1}}{\gamma_n + \beta_1} z_1 + \dfrac{\gamma_n + \frac{1}{\beta_2}}{\gamma_n + \beta_2} z_2 + \dots + \dfrac{\gamma_n + \frac{1}{\beta_n}}{\gamma_n + \beta_n} z_n. \end{cases}$$

Quae hoc modo instituitur. E systemate ipso sponte prodit expressionem:

$$1 + \frac{z + \frac{1}{\beta_1}}{z + \beta_1} z_1 + \frac{z + \frac{1}{\beta_2}}{z + \beta_2} z_2 + \dots + \frac{z + \frac{1}{\beta_n}}{z + \beta_n} z_n$$

evanescere, pro: $z = \gamma_1$, $z = \gamma_2$, ... $z = \gamma_n$, atque in infinitum abire, pro: $z = -\beta_1$, $z = -\beta_2$, ... $z = -\beta_n$, unde concludis ipsam identicam esse cum expressione:

$$= (1 + z_1 + z_2 + \dots + z_n) \frac{(z-\gamma_1)(\gamma-\gamma_2)\dots(z-\gamma_n)}{(z+\beta_1)(z+\beta_2)\dots(z+\beta_n)}.$$

Utraque ipsius forma in fractiones simplices genuinas dissoluta, numeratores denominatoris $z + \beta_\varkappa$ comparando nanciscimur formulam:

$$-\frac{z_\varkappa}{1 + z_1 + z_2 + \dots + z_n}$$

$$= \frac{1}{1-\beta_\varkappa^2} \cdot \frac{\beta_\varkappa(\beta_\varkappa + \gamma_1)(\beta_\varkappa + \gamma_2)\dots(\beta_\varkappa + \gamma_n)}{(\beta_\varkappa - \beta_{\varkappa+1})(\beta_\varkappa - \beta_{\varkappa+2})\dots(\beta_\varkappa - \beta_n)(\beta_\varkappa - \beta_{\varkappa-1})(\beta_\varkappa - \beta_{\varkappa-2})\dots(\beta_\varkappa - \beta_1)},$$

quae numeratore denominatoreque per:

$$(\beta_\varkappa+\beta_{\varkappa+1})(\beta_\varkappa+\beta_{\varkappa+2})\ldots\ldots(\beta_\varkappa+\beta_{\varkappa-2})(\beta_\varkappa+\beta_{\varkappa-1})\ldots\ldots$$

multiplicatis, denotationeque

$$\chi(z)=(z-\beta_1^2)(z-\beta_2^2)\ldots\ldots(z-\beta_n^2),$$
$$\psi(z)=(z+\beta_1)(z+\beta_2)\ldots\ldots(z+\beta_n)(z+\gamma_1)(z+\gamma_2)\ldots\ldots(z+\gamma_n),$$

adhibita in hanc abit:

$$38.\qquad -\frac{z_\varkappa}{1+z_1+z_2+\ldots+z_n}=\tfrac{1}{2}\cdot\frac{\psi(\beta_\varkappa)}{\chi'(\beta_\varkappa^2)}\cdot\frac{1}{1-\beta_\varkappa^2}.$$

Si loco ipsius $\varkappa$ hic ponitur ex ordine: 1, 2, ... n, atque expressiones inde prodeuntes inter se et cum unitate additione coniunguntur, habetur haec formula:

$$39.\qquad \frac{1}{1+z_1+z_2+\ldots+z_n}=1+\tfrac{1}{2}\sum_1^n{}_\varkappa\frac{\psi(\beta_\varkappa)}{\chi'(\beta_\varkappa^2)}\cdot\frac{1}{1-\beta_\varkappa^2}.$$

Iam vero expressione:

$$\tfrac{1}{2}\cdot\frac{\psi(z)+\psi(-z)}{(z^2-\beta_1^2)(z^2-\beta_2^2)\ldots(z^2-\beta_n^2)}$$

in fractiones simplices resoluta, ac deinde posito $z=1$, patet fore:

$$1+\tfrac{1}{2}\sum_1^n{}_\varkappa\frac{\psi(\beta_\varkappa)}{\chi'(\beta_\varkappa^2)}\cdot\frac{1}{1-\beta_\varkappa^2}=\tfrac{1}{2}\cdot\frac{\psi(1)+\psi(-1)}{\chi(1)},$$

qua aequatione cum formulis (38.) et (39.) collata, emanat valor ipsius z quaesitus:

$$40.\qquad z_\varkappa=-\frac{\chi(1)}{\psi(1)+\psi(-1)}\cdot\frac{\psi(\beta_\varkappa)}{\chi'(\beta_\varkappa^2)}\cdot\frac{1}{1-\beta_\varkappa^2}.$$

Inde, revocato ipsius $z_\varkappa$ valore (36.), ope formulae e (10.) derivatae:

$$\frac{1-\beta_\lambda^2}{\beta_\varkappa^2-\beta_\lambda^2}=-\frac{m_{2n+2}-b_\varkappa}{b_\varkappa-b_h},$$

deducitur valor ipsius $\varphi b_\varkappa$ cum ipso (32.) congruens

$$41.\qquad \varphi(b_\varkappa)=(-1)^n\frac{(m_{2n+2}-b_\varkappa)^n}{\psi(1)+\psi(-1)}\psi(\beta_\varkappa).$$

Iam e formulis (33.) et (36.) sequitur haec:

$$42.\qquad \frac{\varrho(b_\varkappa)}{F'(b_\varkappa)}=\pm\sqrt{\left(\frac{1+c}{c}\right)}\cdot\frac{z_\varkappa}{\beta_\varkappa},$$

quae in aequatione identica:

$$\sum_1^n{}_\varkappa\frac{\varrho(b_\varkappa)}{F'(b_\varkappa)}=1$$

introducta, formula (40.) advocata, hanc suppeditat:

$$43.\qquad \sqrt{\left(\frac{1+c}{c}\right)}\frac{\chi(1)}{\psi(1)+\psi(-1)}\sum_1^n{}_\varkappa\left(\frac{\psi(\beta_\varkappa)}{\chi'(\beta_\varkappa^2)}\cdot\frac{1}{\beta_\varkappa(1-\beta_\varkappa^2)}\right)=\mp1.$$

Expressione vero hac:

$$\frac{\psi(z)-\psi(-z)}{(z^2-\beta_1^2)(z^2-\beta_2^2)\dots(z^2-\beta_n^2)}$$

in fractiones simplices resoluta, positoque $z=1$, prodit aequatio identica:

$$\frac{\psi(1)-\psi(-1)}{(1-\beta_1^2)(1-\beta_2^2)\dots(1-\beta_n^2)} = \sum_1^n{}_x \frac{\psi(\beta_x)}{\chi'(\beta_x^2)}\cdot\frac{1}{\beta_x(1-\beta_x^2)},$$

cuius ope formula (43.) in hanc abit:

$$44.\quad \sqrt{\left(\frac{1+c}{c}\right)} = \mp\frac{\psi(1)+\psi(-1)}{\psi(1)-\psi(-1)}.$$

Hanc ob rem, si radicalia $\beta_1, \beta_2, \dots \beta_n, \gamma_1, \gamma_2, \dots \gamma_n$ quae cum radicalibus $\eta_1, \eta_2, \dots \eta_{2n}$ congruunt, talia sunt, ut differentia $\psi(1)-\psi(-1)$ positivo valore gaudeat, in aequationibus (44.) et (33.) inferius signum eligendum est, unde emanant formulae cum ipsis (25.), (26.) et (29.) congruentes.

Quia ad functionis $\varrho(z)$ determinationem n, functionis $\varphi(z)$ autem $n+1$ ipsarum valores pro datis ipsius z valoribus sufficiunt, e systemate formularum (27.), ... (32.) permultas harum functionum formas componere licet. Quarum nonnisi principales hic proponere placet.

E formulis (11.), (12.), (14.), (15.) sponte prodeunt hae formae:

$$\varrho(z) = \frac{c_1}{2\sqrt{c}}\cdot\frac{1}{m_{2n+2}-m_{2n+1}}\cdot\frac{\psi(v)-\psi(-v)}{v(v^2-1)^{n-1}},$$

$$\varphi(z) = \frac{c_1}{2\sqrt{(1+c)}}\cdot\frac{\psi(v)+\psi(-v)}{(v^2-1)^n},$$

quae ope formularum (7.), (24.), (25.), (22.) in has abeunt:

$$45.\quad \varrho(z) =$$
$$\frac{(-1)^{n-1}2}{\psi(1)-\psi(-1)}\cdot\left\{(m_{2n+1}-z)^{n-1}C_1+(m_{2n+1}-z)^{n-2}(m_{2n+2}-z)C_3+\dots+(m_{2n+2}-z)^{n-1}C_{2n-1}\right\},$$

$$46.\quad \varphi(z) =$$
$$\frac{(-1)^n 2}{\psi(1)+\psi(-1)}\cdot\left\{(m_{2n+1}-z)^n+(m_{2n+1}-z)^{n-1}(m_{2n+2}-z)C_2+\dots+(m_{2n+2}-z)^n C_{2n}\right\},$$

ubi uniones, biniones, terniones etc. elementorum:

$$\eta_1,\ \eta_2,\ \dots\ \eta_{2n},$$

sine repetitione respective denotantur per:

$$C_1,\ C_2,\ C_3,\ \dots$$

Deinde ex aequationibus identicis (34.), formulis (29.) et (32.) adhibitis has earundem functionum formas derivamus:

$$47. \quad \begin{cases} \varrho(z) = (-1)^{n-1} \dfrac{F(z)}{\psi(1)-\psi(-1)} \sum_1^n {}^{\varkappa} \left(\dfrac{\psi(\beta_\varkappa)}{\beta_\varkappa F'(b_\varkappa)} \cdot \dfrac{(m_{2n+2}-b_\varkappa)^{n-1}}{z-b_\varkappa} \right), \\ \varphi(z) = F(z) \left\{ 1 + \dfrac{(-1)^n}{\psi(1)+\psi(-1)} \sum_1^n {}^{\varkappa} \left(\dfrac{\psi(\beta_\varkappa)}{F'(b_\varkappa)} \cdot \dfrac{(m_{2n+2}-b_\varkappa)^n}{z-b_\varkappa} \right) \right\}. \end{cases}$$

Quibus collatis patet quantitates $a, a_1, a_2, \ldots a_{n-2}$ dari ut radices aequationis $(n-1)$ti gradus hac forma indutae:

$$\psi\left\{\sqrt{\frac{m_{2n+1}-z}{m_{2n+2}-z}}\right\} - \psi\left\{-\sqrt{\frac{m_{2n+1}-z}{m_{2n+2}-z}}\right\} = 0,$$

sive etiam hac:

$$\sum_1^n {}^{\varkappa} \frac{\psi(\beta_\varkappa)}{\beta_\varkappa F'(b_\varkappa)} \cdot \frac{(m_{2n+2}-b_\varkappa)^{n-1}}{z-b_\varkappa} = 0;$$

nec non quantitates $x, x_2, \ldots x_{2n-2}$ ut radices aequationis $n$ti gradus, quae hac forma:

$$\psi\left\{\sqrt{\frac{m_{2n+1}-z}{m_{2n+2}-z}}\right\} + \psi\left\{-\sqrt{\frac{m_{2n+1}-z}{m_{2n+2}-z}}\right\} = 0,$$

sive hac:

$$\psi(1)+\psi(-1)+(-1)^n \sum_1^n {}^{\varkappa} \frac{\psi(\beta_\varkappa)}{F'(b_\varkappa)} \cdot \frac{(m_{2n+2}-b_\varkappa)^n}{z-b_\varkappa} = 0,$$

gaudet.

7.

Iam transeamus ad naturam radicum:

$$a, \; a_1, \; \ldots \; a_{n-2},$$
$$x, \; x_2, \; \ldots \; x_{2n-2},$$

propius investigandam. Primum ex $2n$ aequationibus formae (6.) statim concluditur, et ipsum c et functionum $\varrho(z)$ et $\varphi(z)$ coëfficientes omnes reales esse, quippe quod etiam formulis (45.) et (46.) comprobatur. Hanc ob causam nulla radicum $a, a_1, \ldots a_{n-2}$, imaginaria esse potest, nisi alteram secum fert sibi coniugatam; eademque natura gaudebunt radices: $x, x_2, \ldots x_{2n-2}$.

Formula (26.) vero docet quantitatem c adeo esse positivam, id quod etiam directe ex aequationibus formae (6.) concluditur. Quam ob causam, nec ulla radicum:

$$x, \; x_2, \; \ldots \; x_{2n-2},$$

in ullo intervallorum:

$$-\alpha \ldots m_1, \; m_2 \ldots m_3, \; m_4 \ldots m_5, \; \ldots \; m_{2n} \ldots m_{2n+1}, \; m_{2n+2} \ldots \alpha,$$

continetur, quippe in quorum intervallorum aliquo versante z, aequationis (4.) prior terminus positivo valore gaudet, hancque ob causam evanescere nequit; nec ulla quantitatum:

$$a, \; a_1, \; a_2, \; \ldots \; a_{n-2},$$

in ullo reliquorum intervallorum:

$$m_1 \ldots . m_2, \quad m_3 \ldots . m_4, \quad \ldots . \quad m_{2n+1} \ldots . m_{2n+2},$$

iacebit; in eiusmodi enim loco versante z, prior aequationis (4.) pars negativo, posterior positivo valore indueretur, id quod absurdum est.

Radicalia deinde:

$$\eta_1, \ \eta_2, \ \ldots . \ \eta_{2n},$$

ob formulam (26.), conditioni:

$$(1+\eta_1)(1+\eta_2)\ldots.(1+\eta_{2n}) = (1-\eta_1)(1-\eta_2)\ldots.(1-\eta_{2n}),$$

satisfaciant necesse est, unde sequitur, tantum 2^{2n-1} systemata diversa signorum horum radicalium assumi posse, totidemque inde prodire et quantitatis c et functionum $\varrho(z)$, $\varphi(z)$ expressiones diversas, de quibus in articulo praecedenti sermo fuit. — Formulae denique (27.), ... (32.), quia fractio $\left(\frac{\eta_\varkappa}{\eta_\lambda}\right)^2$, si habetur $\varkappa < \lambda$, unitatem haud aequat, docent expressiones binas:

$$49. \quad \begin{cases} \varrho(m_{2n+2}) & \text{atque} \quad \eta_1+\eta_2+\ldots.+\eta_{2n}, \\ (-1)^{n-1}\varrho(m_{2n+1}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2n}\left\{\frac{1}{\eta_1}+\frac{1}{\eta_2}+\ldots.+\frac{1}{\eta_{2n}}\right\}, \\ (-1)^{n-1}\varrho(m_{2n}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2n-1}, \\ (-1)^{n-1}\varrho(m_{2n-1}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2n-1}, \\ \ldots & \ldots \\ (-1)^{n-1}\varrho(m_{2h}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2h-1}, \\ (-1)^{n-1}\varrho(m_{2h-1}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2h-1}, \\ \ldots & \ldots \\ (-1)^{n-1}\varrho(m_2) & - \; - \quad \eta_1, \\ (-1)^{n-1}\varrho(m_1) & - \; - \quad \eta_1, \end{cases}$$

$$50. \quad \begin{cases} \varphi(m_{2n+2}) & \text{atque} \quad 1, \\ (-1)^n\varphi(m_{2n+1}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2n}, \\ (-1)^n\varphi(m_{2n}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2n}, \\ (-1)^n\varphi(m_{2n-1}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2n-2}, \\ (-1)^n\varphi(m_{2n-2}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2n-2}, \\ \ldots & \ldots \\ (-1)^n\varphi(m_{2h-1}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2h-2}, \\ (-1)^n\varphi(m_{2h-2}) & - \; - \quad \eta_1\eta_2\ldots.\eta_{2h-2}, \\ \ldots & \ldots \\ (-1)^n\varphi(m_2) & - \; - \quad \eta_1\eta_2, \\ (-1)^n\varphi(m_1) & - \; - \quad 1, \end{cases}$$

simul positivo vel negativo valore gaudere. Inde ducimur ad hanc distributionem 2^{2n-1} casuum, quorum supra mentionem fecimus; nimirum si valores productorum singulorum

$$51. \quad \eta_1\eta_2, \quad \eta_3\eta_4, \quad \dots \quad \eta_{2n-1}\eta_{2n},$$

quomodocunque vel positivi vel negativi assumuntur, inde 2^n casus diversi prodeunt; si deinde in his singulis casibus signa valorum productorum:

$$52. \quad \eta_1\eta_3, \quad \eta_1\eta_5, \quad \dots \quad \eta_1\eta_{2n-1}$$

quomodocunque assumuntur, pro singulis 2^{n-1} suppositiones emanant.

E numero 2^n casuum tales secernere placet, in quibus radices

$$53. \quad x, \quad x_2, \quad \dots \quad x_{2n-2},$$

non modo *semper* reales sunt, sed adhuc singulae in singulis n horum $n+1$ intervallorum:

$$54. \quad m_1 \dots m_2, \quad m_3 \dots m_4, \quad \dots \quad m_{2n+1} \dots m_{2n+2},$$

continentur, qui casus numero $n+1$ e serie (50.) facile desumuntur. Producta enim seriei (51.) vel omnia negativos valores habent, in quo casu *primo* radices (53.) singulae continentur in singulis n primis intervallorum (54.); vel omnia uno excepto negativis valoribus gaudent, unde n *reliqui* casus prodeunt tales, ut, generaliter in $\varkappa$to casu, ubi productum $\eta_{2\varkappa-3}\eta_{2\varkappa-2}$ illud unum positivum est, radices (53.) singulae in n intervallis (54.), intervallo $m_{2\varkappa-3}\dots m_{2\varkappa-2}$ excepto, contineantur. Adiiciatur in casu primo quantitatum $a, a_1, a_2, \dots a_{n-2}$ numerum parem vel imparem contineri in quocunque intervallo

$$m_{2h} \dots m_{2h+1},$$

prout productum $\eta_{2h-1}\eta_{2h+1}$ negativo vel positivo valore gaudeat, idemque fieri in casu $\varkappa$to, excepto intervallo $m_{2\varkappa-2}\dots m_{2\varkappa-1}$, in quo par vel impar numerus earundem quantitatum iacebit prout productum:

$$\eta_{2\varkappa-3}\eta_{2\varkappa-1}$$

positivo vel negativo valore gaudebit.

Maiori tamen attentione digna videtur regula simplex, secundum quam iudicare licet de signis quantitatum:

$$\varrho(m_1), \quad \varrho(m_3), \quad \dots \quad \varrho(m_{2h-1}) \dots \varrho(m_{2n-1}),$$

quae cum signis ipsarum:

$$\varrho(m_2), \quad \varrho(m_4), \quad \dots \quad \varrho(m_{2h}) \dots \dots \varrho(m_{2n}),$$

convenire, e serie (49.), nec non cum signis quantitatum:

$$\varrho(x), \quad \varrho(x_2), \quad \dots \quad \varrho(x_{2h-2}) \dots \varrho(x_{2n-2}),$$

ex considerationibus huius articuli sponte patet. Quam regulam in sequentibus adhibebimus. Nimirum ex eadem serie (49.) concludere licet, signa illarum quantitatum in primo casu congruere cum signis quantitatum:

$$(-1)^{n-1}\eta_1,\ (-1)^{n-2}\eta_3,\ \ldots .\ (-1)^{n-h}\eta_{2h-1},\ \ldots .\ \eta_{2n-1},$$

in secundo casu, cum signis quantitatum:

$$(-1)^{n-1}\eta_1,\ (-1)^{n-1}\eta_3,\ \ldots .\ (-1)^{n-h+1}\eta_{2h-1},\ \ldots .\ -\eta_{2n-1},$$

in tertio casu cum signis quantitatum:

$$(-1)^{n-1}\eta_1,\ (-1)^{n-2}\eta_3,\ (-1)^{n-2}\eta_5,\ \ldots .\ (-1)^{n-h+1}\eta_{2h-1},\ \ldots .\ -\eta_{2n-1},$$

nec non generaliter in $\varkappa$to casu signa binarum quantitatum:

$$\varrho(m_{2\varkappa-2h-3}),\qquad (-1)^{n-\varkappa+h+1}\eta_{2\varkappa-2h-3}$$

inter se congruere, aeque ac signa quantitatum binarum:

$$\varrho(m_{2\varkappa+2H-1}),\qquad (-1)^{n-\varkappa-H+1}\eta_{2\varkappa+2H-1},$$

ubi littera $\varkappa$ designatur numerus quilibet integer unitate maior, numerum $n+1$ haud superans, atque per h quilibet numerorum:

$$0,\ 1,\ 2,\ \ldots .\ \varkappa-2,$$

nec non per H quilibet numerorum:

$$0,\ 1,\ 2,\ \ldots .\ (n-\varkappa)$$

denotatur.

Haec de natura radicum duplicium aequationis formae:

$$c\{(z-a)(z-a_1)\ldots .(z-a_{n-2}\}^2(z-m_{2n+2})(z-m_{2n+1})+(z-m_1)(z-m_2)\ldots .(z-m_{2n})=0$$

atque de quantitatibus c, a, a_1, ... a_{n-2} attulisse sufficiat. Qua forma brevitatis gratia denotata per:

$$(2n+2,\ 2n+1)=0,$$

similique denotatione, si in priori aequationis termino factores $(z-m_{2n+2})$ atque $(z-m_{2n+1})$ cum aliis duobus quibuslibet e numero $2n$ factorum:

$$(z-m_1),\ (z-m_2),\ \ldots .\ (z-m_{2n}),$$

commutantur, adhibita, $(2n+1)$ formas:

$$(2n+1,2n)=0,\ (2n,2n-1)=0,\ \ldots .\ (2,1)=0,\ (1,2n+2)=0,$$

quae et ipsae similibus suppositionibus ad n reales radices duplices ducunt, ope substitutionis:

$$Z=p.\frac{n-z}{m-z},$$

ubi quantitas m respective conditionibus:

$$m_{2n+2}>m\geqq m_{2n+1},\quad m_{2n+1}>m\geqq m_{2n+2},\quad \ldots .\quad m_2>m\geqq m_1,$$

satisfacit, nec non habetur:

$$p(n-m)>0,$$

ad illam formam $(2n+2,2n+1)=0$, pro argumento Z revocare licet. Ceterarum vero omnium formarum naturam indagare hic non placet, quippe quae realem problematis algebraici solutionem haud admittunt. Facile enim demonstratur, has formas, pro realibus ipsius c et coëfficientium functionis $\varrho(z)$ valoribus, radicibus binis inter se aequalibus neutiquam gaudere posse.

8.

Problematis algebraici solutio in antecedentibus exposita ut adhibeatur in theoria integralium *Abelianorum* $(n-1)$ti ordinis, generaliores quasdam de his integralibus, quas adhuc pro theorematis *Abeliani* applicationibus habere licet, propositiones hic peculiari methodo deducere velimus ex aequatione identica:

$$55.\quad C(P(z))^2(z-m_{2n+2})(z-m_{2n+1})+f(z) = (1+C)\pi(z)\Pi(z),$$

ubi ponitur:

$$56.\quad \begin{cases} (z-m_1)(z-m_2)\dots\dots(z-m_{2n}) = f(z), \\ (z-A_1)(z-A_2)\dots\dots(z-A_{n-1}) = P(z), \\ (z-y_1)(z-y_2)\dots\dots(z-y_n) = \pi(z), \\ (z-Y_1)(z-Y_2)\dots\dots(z-Y_n) = \Pi(z), \end{cases}$$

ita ut quantitates:

$$57.\quad y_1, y_2, \dots\dots y_n, \quad Y_1, Y_2, \dots\dots Y_n,$$

radices sint aequationis $2n$ti gradus:

$$58.\quad C(P(z))^2(z-m_{2n+2})(z-m_{2n+1})+f(z) = 0.$$

Deinde per signa

$$e_1, e_2, \dots\dots e_n, \quad E_1, E_2, \dots\dots E_n,$$

tales positivae vel negativae denotentur unitates ut, littera ν designante quemlibet numerorum 1, 2, ... n, generales habeantur formulae:

$$59.\quad \begin{cases} \sqrt{C}\,P(y_\nu) = \dfrac{e_\nu\sqrt{(\varDelta y_\nu)}}{(m_{2n+2}-y_\nu)(m_{2n+1}-y_\nu)}, \\ \sqrt{C}\,P(Y_\nu) = \dfrac{E_\nu\sqrt{(\varDelta Y_\nu)}}{(m_{2n+2}-Y_\nu)(m_{2n+1}-Y_\nu)}, \end{cases}$$

ubi brevitatis gratia ponitur:

$$\varDelta z = -(z-m_1)(z-m_2)\dots\dots(z-m_{2n+2}).$$

Ex aequatione (55.) tria placet derivare systemata formularum. Primum enim ibi posito: $z = A_\mu$, $z = m_{2n+2}$, $z = m_{2n+1}$, ubi littera μ quemlibet numerorum 1, 2, ... $n-1$ denotat, prodeunt formulae numero $n+1$ hae:

$$60.\quad f(A_\mu) = (1+C)\,\pi(A_\mu)\,\Pi(A_\mu),$$

$$61.\quad f(m_{2n+2}) = (1+C)\,\pi(m_{2n+2})\,\Pi(m_{2n+2}),$$

$$62.\quad f(m_{2n+1}) = (1+C)\,\pi(m_{2n+1})\,\Pi(m_{2n+1}).$$

Deinde utriusque ipsius termini logarithmos, ipso z ut variabili assumto, differentiando, positoque $z = A_\mu$, emanant hae $n-1$ aequationes:

$$63.\quad \frac{1}{A_\mu - m_1} + \frac{1}{A_\mu - m_2} + \dots + \frac{1}{A_\mu - m_{2n}} = \sum_1^n{}^\nu \left(\frac{1}{A_\mu - y_\nu} + \frac{1}{A_\mu - Y_\nu}\right).$$

Eiusdem denique aequationis et aequationum (60.), (61.), (62.) utramque partem logarithmice differentiando, quantitatibus $C, A_1, A_2, \dots A_{n-1}, y_1, y_2, \dots y_n, Y_1, Y_2, \dots Y_n$ pro variabilibus assumtis, si adhibetur aequatio (63.), nanciscimur has $n+2$ formulas differentiales:

$$64.\quad d\log\left(1 - \frac{C(Pz)^2(z-m_{2n+2})^2(z-m_{2n+1})^2}{\Delta z}\right) = \frac{dC}{1+C} - \sum_1^n{}^\nu \left(\frac{dy_\nu}{z-y_\nu} + \frac{dY_\nu}{z-Y_\nu}\right),$$

$$65.\quad \frac{dC}{1+C} = \sum_1^n{}^\nu \left(\frac{dy_\nu}{(A_\mu - y_\nu)} + \frac{dY_\nu}{A_\mu - Y_\nu}\right),$$

$$66.\quad \frac{dC}{1+C} = \sum_1^n{}^\nu \left(\frac{dy_\nu}{m_{2n+2} - y_\nu} + \frac{dY_\nu}{m_{2n+2} - Y_\nu}\right),$$

$$67.\quad \frac{dC}{1+C} = \sum_1^n{}^\nu \left(\frac{dy_\nu}{m_{2n+1} - y_\nu} + \frac{dY_\nu}{m_{2n+1} - Y_\nu}\right).$$

Inde, eliminatione ipsius $\frac{dC}{1+C}$ facta, ducimur ad has $n+1$ aequationes:

$$68.\quad \begin{cases} d\log\left(1 - \frac{C(Pz)^2(z-m_{2n+2})^2(z-m_{2n+1})^2}{\Delta z}\right) \\ = (A_1 - z)\sum_1^n{}^\nu \left\{\frac{dy_\nu}{(y_\nu - A_1)(z-y_\nu)} + \frac{dY_\nu}{(Y_\nu - A_1)(z-Y_\nu)}\right\}, \\ = (A_2 - z)\sum_1^n{}^\nu \left\{\frac{dy_\nu}{(y_\nu - A_2)(z-y_\nu)} + \frac{dY_\nu}{(Y_\nu - A_2)(z-Y_\nu)}\right\}, \\ \dots\dots\dots\dots\dots\dots \\ = (A_{n-1} - z)\sum_1^n{}^\nu \left\{\frac{dy_\nu}{(y_\nu - A_{n-1})(z-y_\nu)} + \frac{dY_\nu}{(Y_\nu - A_{n-1})(z-Y_\nu)}\right\}, \\ = (m_{2n+2} - z)\sum_1^n{}^\nu \left\{\frac{dy_\nu}{(y_\nu - m_{2n+2})(z-y_\nu)} + \frac{dY_\nu}{(Y_\nu - m_{2n+2})(z-Y_\nu)}\right\}, \\ = (m_{2n+1} - z)\sum_1^n{}^\nu \left\{\frac{dy_\nu}{(y_\nu - m_{2n+1})(z-y_\nu)} + \frac{dY_\nu}{(Y_\nu - m_{2n+1})(z-Y_\nu)}\right\}, \end{cases}$$

quibus ex ordine multiplicatis per expressiones:

$$\frac{1}{\sqrt{C}\, P'(A_1)(A_1 - m_{2n+2})(A_1 - m_{2n+1})} \cdot \frac{1}{z - A_1},$$

$$\frac{1}{\sqrt{C}\, P'(A_2)(A_2 - m_{2n+2})(A_2 - m_{2n+1})} \cdot \frac{1}{z - A_2},$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{1}{\sqrt{C}\, P'(A_{n-1})(A_{n-1} - m_{2n+2})(A_{n-1} - m_{2n+1})} \cdot \frac{1}{z - A_{n-1}},$$

$$\frac{1}{\sqrt{C}\,P(m_{2n+2})\,(m_{2n+2}-m_{2n+1})}\cdot\frac{1}{z-m_{2n+2}},$$
$$\frac{1}{\sqrt{C}\,P(m_{2n+1})\,(m_{2n+1}-m_{2n+2})}\cdot\frac{1}{z-m_{2n+1}},$$

quorum factorum summa secundum theorema notissimum transit in expressionem:

$$\frac{1}{\sqrt{C}\,P(z)\,(z-m_{2n+2})(z-m_{2n+1})},$$

additioneque instituta, prodit aequatio differentialis:

$$\frac{-d.\log\left\{1-\frac{C(P(z))^2(z-m_{2n+2})^2(z-m_{2n+1})^2}{\Delta z}\right\}}{\sqrt{C}\,P(z)\,(z-m_{2n+2})(z-m_{2n+1})} =$$

$$\sum_1^n{}^{\nu}\left\{\frac{dy_\nu}{\sqrt{C}(z-y_\nu)P(y_\nu)(y_\nu-m_{2n+2})(y_\nu-m_{2n+1})}+\frac{dY}{\sqrt{C}(z-Y_\nu)P(Y_\nu)(Y_\nu-m_{2n+2})(Y_\nu-m_{2n+1})}\right\},$$

nec non post facilem reductionem, si in singulis secundae partis terminis e singulis aequationibus (59.) valores ipsius $\sqrt{C}$ substituuntur, respective integratione facta inde a valoribus

$$C^0,\quad A_1^0,\ A_2^0,\ \dots\ A_{n-1}^0,\quad y_1^0,\ y_2^0,\ \dots\ y_n^0,\quad Y_1^0,\ Y_2^0,\ \dots\ Y_n^0,$$

qui valores initiales quantitatum variabilium

$$C,\quad A_2,\ A_2,\ \dots\ A_{n-1},\quad y_1,\ y_2,\ \dots\ y_n,\quad Y_1,\ Y_2,\ \dots\ Y_n$$

sunt, usque ad hos valores finales, haec elegans formula obtinetur:

$$69.\quad \sum_1^n{}^{\nu}\left\{\int_{y_\nu^0}^{y_\nu}\frac{e_\nu\,dy}{(z-y)\sqrt{(\Delta y)}}+\int_{Y_\nu^0}^{Y_\nu}\frac{E_\nu\,dy}{(z-y)\sqrt{(\Delta y)}}\right\}$$
$$=-\frac{2}{\sqrt{(-\Delta z)}}\left\{\operatorname{arc\,tang}\frac{\sqrt{C}\,P(z)\,(z-m_{2n+2})(z-m_{2n+1})}{\sqrt{(-\Delta z)}}\right.$$
$$\left.-\operatorname{arc\,tang}\frac{\sqrt{C^0}\,P^0(z)(z-m_{2n+2})(z-m_{2n+1})}{\sqrt{(-\Delta z)}}\right\},$$

ubi ponitur:

$$P^0(z)=(z-A_1^0)(z-A_2^0)\dots(z-A_{n-1}^0).$$

In hac formula continentur multae aliae quae per evolutionem secundum descendentes ipsius z potestates inde derivantur, nimirum haec:

$$70.\quad \sum_1^n{}^{\nu}\left\{\int_{y_\nu^0}^{y_\nu}\frac{e_\nu\,y^\mu\,dy}{\sqrt{(\Delta y)}}+\int_{Y_\nu^0}^{Y_\nu}\frac{E_\nu\,Y^\mu\,dY}{\sqrt{(\Delta Y)}}\right\}=0,$$

ubi littera μ quilibet numerorum 0, 1, 2, ... $n-1$ denotatur, atque haec pro qualibet quantitate α valens, ubi denotatio pro coëfficiente evolutionis usitata adhibetur, nec non $\Phi(z)$ designat functionem integram ipsius z quamlibet:

$$71.\quad \sum_1^n{}^{\nu}\left\{\int_{y_\nu^0}^{y_\nu}\frac{e_\nu\,\Phi(y)\,dy}{(y-\alpha)\sqrt{(\varDelta y)}}+\int_{Y_\nu^0}^{Y_\nu}\frac{E_\nu\,\Phi(y)\,dy}{(y-\alpha)\sqrt{(\varDelta y)}}\right\}=$$

$$-2\left[\frac{\Phi(z)}{(z-\alpha)\sqrt{(-\varDelta z)}}\left\{\text{arc tang}\frac{\sqrt{C(P(z))(z-m_{2n+2})(z-m_{2n+1})}}{\sqrt{(-\varDelta z)}}-\text{arc tang}\frac{\sqrt{C^0(P^0(z))(z-m_{2n+2})(z-m_{2n+1})}}{\sqrt{(-\varDelta z)}}\right\}\right]_{z^{-1}}$$
$$+2\frac{\Phi(\alpha)}{\sqrt{(-\varDelta\alpha)}}\left\{\text{arc tang}\frac{\sqrt{C(P(\alpha))(\alpha-m_{2n+2})(\alpha-m_{2n+1})}}{\sqrt{(-\varDelta\alpha)}}-\text{arc tang}\frac{\sqrt{C^0(P^0(\alpha))(\alpha-m_{2n+2})(\alpha-m_{2n+1})}}{\sqrt{(-\varDelta\alpha)}}\right\}.$$

Aequationis (55.) forma talis est, ut n relationes algebraicae inter $2n$ ipsius radices valeant, unde patet, n valores Y_1, Y_2, ... Y_n algebraice eosque ut radices aequationis $n$ti gradus determinari ex quantitatibus $y_1, y_2, \dots y_n$. Quam determinationem, una cum computatione signorum E_1, E_2, ... E_n, atque quantitatum in formulis integralibus involutarum C et $P(z)$, hoc modo instituere placet.

E formulis (59.) sequitur haec:

$$72.\quad \sqrt{C\,P(z)}=\pi(z)\sum_1^n{}^{\nu}\left(\frac{e_\nu\sqrt{(\varDelta y_\nu)}}{(m_{2n+2}-y_\nu)(m_{2n+1}-y_\nu)}\cdot\frac{1}{\pi'(y_\nu)}\cdot\frac{1}{z-y_\nu}\right);$$

unde cum ex aequatione (55.) posito $z=m_h$, derivetur haec:

$$73.\quad C(P(m_h))^2(m_{2n+2}-m_h)(m_{2n+1}-m_h)=(C+1)\pi(m_h).\Pi(m_h),$$

deducuntur hae relationes:

$$74.\quad \begin{cases}\Pi(m_{2n+2}) & \dfrac{f(m_{2n+2})}{\pi(m_{2n+2})}\\ :\Pi(m_{2n+1}) & :\dfrac{f(m_{2n+1})}{\pi(m_{2n+1})}\\ :\Pi(m_h) \quad = & :(m_{2n+2}-m_h)(m_{2n+1}-m_h)\pi(m_h)\left\{\sum_1^n{}^{\nu}\dfrac{e_\nu\sqrt{(\varDelta y_\nu)}}{(m_{2n+2}-y_\nu)(m_{2n+1}-y_\nu)\pi'(y_\nu)}\cdot\dfrac{1}{m_h-y_\nu}\right\}^2\\ :1 & :1+\left\{\sum_1^n{}^{\nu}\dfrac{e_\nu\sqrt{(\varDelta y_\nu)}}{(m_{2n+2}-y_\nu)(m_{2n+1}-y_\nu)\pi' y_\nu}\right\}^2\end{cases}$$

atque, quantitatibus b_1, b_2, ... b_n ut supra denotationeque (35.) introductis, haec aequationis, radicibus Y_1, Y_2, ... Y_n gaudentis, forma:

$$75.\quad 1+\left\{\sum_1^n{}^{\nu}\frac{e_\nu\sqrt{(\varDelta y_\nu)}}{(m_{2n+2}-y_\nu)(m_{2n+1}-y_\nu)\pi' y_\nu}\right\}^2+\sum_1^n{}^{\varkappa}\sum_1^n{}^{\nu}\left(\frac{\pi(b_\varkappa)}{F' b_\varkappa}\cdot\frac{(m_{2n+2}-b_\varkappa)(m_{2n+1}-b_\varkappa)}{(m_{2n+2}-y_\nu)(m_{2n+1}-y_\nu)}\cdot\frac{e_\nu\sqrt{(\varDelta y_\nu)}}{\pi' y_\nu(b_\varkappa-y_\nu)}\cdot\frac{1}{z-b_\varkappa}\right)=0.$$

Signa E_1, E_2, ... E_n deinde computantur ope formulae ex aequationibus (59.) et (72.) derivatae:

$$76.\quad E_\varkappa=(m_{2n+2}-Y_\varkappa)(m_{2n+1}-Y_\varkappa)\pi(Y_\nu)\sum_1^n{}^{\nu}\frac{e_\nu\sqrt{(\varDelta y_\nu)}}{(m_{2n+2}-y_\nu)(m_{2n+1}-y_\nu)\pi'(y_\nu)}\cdot\frac{1}{Y_\varkappa-y_\nu}.$$

Iam adhuc aliam formam functionis $\sqrt{C.Pz}$ proponere placet, ex omnibus radicibus aequationis (55.) compositam atque in formula (71.) substituendam.

Nimirum valore ipsius $1+C$ ope formulae (61.) determinato, atque in aequatione (73.) substituto, emanat haec

$$77. \quad \sqrt{C}\,P(m_h) = \pm\frac{1}{\sqrt{(m_{2n+2}-m_h)(m_{2n+1}-m_h)}}\sqrt{\left\{\frac{\pi(m_h).\Pi(m_h)\,f(m_{2n+2})}{\pi(m_{2n+2})\,\Pi(m_{2n+2})}\right\}},$$

unde loco ipsius m_h quantitatibus $b_1, b_2, \ldots\ b_n$, atque loco signorum ± 1 respective signis: $e^{(1)}, e^{(2)}, \ldots\ e^{(n)}$ introductis, haec deducitur formula:

$$78. \quad \sqrt{C}\,P(z) =$$

$$F(z)\sum_1^n{}^{\nu}\left(\frac{e^{(\nu)}}{F'(b_\nu)}\cdot\frac{1}{\sqrt{(m_{2n+2}-b_\nu)(m_{2n+1}-b_\nu)}}\sqrt{\left(\frac{\pi(b_\nu)\,\Pi(b_\nu)\,f(m_{2n+2})}{\pi(m_{2n+2})\,\Pi(m_{2n+2})}\right)}\frac{1}{z-b_\nu}\right).$$

9.

Casum ubi n quantitates datae:

$$y_1,\ y_2,\ \ldots\ y_n,$$

reales sunt et continentur respective in n intervallis:

$$79. \quad m_1\ldots m_2,\ m_3\ldots m_4,\ \ldots\ m_{2n-1}\ldots m_{2n},$$

pro fundamentali hic habere placet, quippe in quo quantitatem C positivo valore gaudere, nec non quantitates:

$$Y_1,\ Y_2,\ \ldots\ Y_n,$$

reales esse atque in iisdem, singulas in singulis, intervallis contineri, ex aequatione (55.) extemplo concluditur.

Huius enim aequationis pars prior, si quantitas C negativa esset, loco ipsius z valore radicis, quae in quolibet intervallorum (79.) continetur, substituto, ut aggregatum duorum terminorum negativorum evanescere non posset, eademque parte priori pro $z=m_{2\nu-1}$ et pro $z=m_{2\nu}$ positivis valoribus gaudente, radix y_ν in intervallo $m_{2\nu-1}\ldots m_{2\nu}$ iacens, alteram in eodem intervallo secum ferat necesse est.

Deinde patet, quamcunque e numero quantitatum:

$$A_1,\ A_2,\ \ldots\ A_{n-1},$$

quae in quolibet intervallorum (79.) contineatur, etiam inter radices duas in eodem intervallo iacentes contineri. Sit enim hoc intervallum $m_{2\nu-1}\ldots m_{2\nu}$, atque radicum y_ν, Y_ν minor v_ν, maior Υ_ν. Iam si quaecunque quantitatum A in intervallo $m_{2\nu-1}\ldots v_\nu$ vel in intervallo $\Upsilon_\nu\ldots m_{2\nu}$ contineretur, etiam radix alia aequationis (55.) in eodem intervallo versaretur, et hanc ob rem haec ut tertia in intervallo $m_{2\nu-1}\ldots m_{2\nu}$ iaceret, id quod fieri nequit. Inde adhuc sequitur, binas quantitates:

$$P(m_{2\nu-1}) \quad \text{et} \quad P(v_\nu)$$

generaliter simul positivas vel negativas esse, eademque natura binas quantitates

$$P(m_{2\nu}) \quad \text{et} \quad P(\Upsilon_\nu)$$

gaudere.

Quae cum ita sint, habetur haec propositio:

Si radices aequationis:

$$C(z-A_1)^2(z-A_2)^2\dots(z-A_{n-1})^2(z-m_{2n+2})(z-m_{2n+1})+(z-m_1)(z-m_2)\dots(z-m_{2n}) = 0$$

ex ordine scriptae:

$$v_1,\ \Upsilon_1,\ v_2,\ \Upsilon_2,\ \dots\ v_n,\ \Upsilon_n,$$

binae in singulis intervallis:

$$m_1\dots m_2,\ m_3\dots m_4,\ \dots\ m_{2n-1}\dots m_{2n},$$

continentur, radices aequationis similis formae:

$$C^0(z-A_1^0)^2(z-A_2^0)^2\dots(z-A_{n-1}^0)^2(z-m_{2n+2})(z-m_{2n+1})+(z-m_1)(z-m_2)\dots(z-m_{2n}) = 0,$$

ex ordine scriptae:

$$v_1^0,\ \Upsilon_1^0,\ v_2^0,\ \Upsilon_2^0,\ \dots\ v_n^0,\ \Upsilon_n^0$$

tales functiones quantitatum C^0, A_1^0, A_2^0, ... A_n^0 sunt, ut quantitatibus A_1^0, A_2^0, ... A_{n-1}^0 in certis quibusdam intervallis respective usque ad valores:

$$A_1,\ A_2,\ \dots\ A_{n-1}$$

continue progredientibus, nec non ipso C^0 simul a nihilo usque ad valorem C apte crescente, ipsae continue pergant:

v_1^0 ab m_1 usque ad v_1,
Υ_1^0 ab m_2 usque ad Υ_1,
v_2^0 ab m_3 usque ad v_2,
Υ_2^0 ab m_4 usque ad Υ_2,
. . . .
Υ_n^0 ab m_{2n} usque ad Υ_n,

nec generaliter signa expressionum formae:

$$(v_\nu^0-A_1^0)(v_\nu^0-A_2^0)\dots(v_\nu^0-A_{n-1}^0),$$
$$(\Upsilon_\nu^0-A_1^0)(\Upsilon_\nu^0-A_2^0)\dots(\Upsilon_\nu^0-A_{n-1}^0),$$

interea commutantur, nisi quantitatum A aliqua per aliquem valorum:

$$m_1,\ m_2,\ \dots\ m_{2n},$$

transmigrat.

Hac propositione, cuius demonstratione facili, quippe quam in casu speciali $n=2$ antea exposuimus, hic supersedemus, docemur quantitates:

$$m_1,\ m_2,\ \dots\ m_{2n}$$

in aequationibus integralibus (70.) et (71.) pro valoribus variabilium initialibus

$$y_1^0, \quad Y_1^0, \quad \dots \quad Y_n^0$$

assumi posse, ideoque expressionem:

$$80. \qquad \overset{n}{\underset{1}{\Sigma}}{}^{\nu} \left\{ \int_{m_{2\nu-1}}^{v_\nu} \frac{\varepsilon_\nu\, y^\mu\, dy}{\sqrt{(\varDelta y)}} + \int_{m_{2\nu}}^{\Upsilon_\nu} \frac{\mathcal{E}_\nu y^\mu\, dy}{\sqrt{(\varDelta y)}} \right\}$$

(ubi signa e_ν et E_ν ad v_ν et Υ_ν respective pertinentia denotantur per ε_ν et $\mathcal{E}_\nu$) evanescere pro:

$$\mu = 0, \quad \mu = 1, \quad \mu = 2, \quad \dots \quad \mu = n-1,$$

atque pro $\mu = n$ transire in expressionem:

$$81. \qquad -2 \operatorname{arc\,tang} \overset{n}{\underset{1}{\Sigma}}{}^{\nu} \frac{e_\nu \sqrt{(\varDelta y_\nu)}}{(m_{2n+2}-y_\nu)(m_{2n+1}-y_\nu)} \cdot \frac{1}{\pi'(y_\nu)},$$

nec non integralium aggregatum hoc:

$$82. \qquad \overset{n}{\underset{1}{\Sigma}}{}^{\nu} \left\{ \int_{m_{2\nu-1}}^{v_\nu} \frac{\varepsilon_\nu\, \varPhi(y)\, dy}{(y-\alpha)\sqrt{(\varDelta y)}} + \int_{m_{2\nu}}^{\Upsilon_\nu} \frac{\mathcal{E}_\nu\, \varPhi(y)\, dy}{(y-\alpha)\sqrt{(\varDelta y)}} \right\}$$

congruere cum expressione, ope formulae (72.) composita:

$$83. \quad -2 \left[\frac{\varPhi(z)}{(z-\alpha)\sqrt{(-\varDelta z)}} \operatorname{arc\,tang} \left\{ \frac{\pi(z)\,(z-m_{2n+2})(z-m_{2n+1})}{\sqrt{(-\varDelta z)}} \overset{n}{\underset{1}{\Sigma}}{}^{\nu} \left(\frac{e_\nu \sqrt{(\varDelta y_\nu)}}{(m_{2n+2}-y_\nu)(m_{2n+1}-y_\nu)} \cdot \frac{1}{\pi'(y_\nu)} \cdot \frac{1}{z-y_\nu} \right) \right\} \right]_{z^{-1}}$$
$$+ 2 \frac{\varPhi(\alpha)}{\sqrt{(-\varDelta \alpha)}} \operatorname{arc\,tang} \left\{ \frac{\pi(\alpha)(\alpha-m_{2n+2})(\alpha-m_{2n+1})}{\sqrt{(-\varDelta \alpha)}} \overset{n}{\underset{1}{\Sigma}}{}^{\nu} \left(\frac{e_\nu \sqrt{(\varDelta y_\nu)}}{(m_{2n+2}-y_\nu)(m_{2n+1}-y_\nu)} \cdot \frac{1}{\pi'(y_\nu)} \cdot \frac{1}{\alpha-y_\nu} \right) \right\}.$$

Si contra formulam (78.) in aequatione (71.) substituere velimus, loco quantitatum, e numero $2n$ quantitatum:

$$m_1, \quad m_2, \quad \dots \quad m_{2n},$$

arbitrarie electarum, $b_1, b_2, \dots b_n$, positis

$$m_1, \quad m_3, \quad \dots \quad m_{2n-1},$$

simul signa:

$$e_1, \quad e_2, \quad \dots \quad e_n,$$

ex antecedentibus congruere debent cum signis:

$$\varepsilon_1, \quad \varepsilon_2, \quad \dots \quad \varepsilon_n;$$

unde sequitur, expressionem (81.) adhuc commutari posse cum hac:

$$-2 \operatorname{arc\,tang} \left\{ \overset{n}{\underset{1}{\Sigma}}{}^{\nu} \frac{\varepsilon_\nu}{F'(m_{2\nu-1})} \cdot \frac{1}{\sqrt{\{(m_{2n+2}-m_{2\nu-1})(m_{2n+1}-m_{2\nu-1})\}}} \sqrt{\left(\frac{\pi(m_{2\nu-1})\, \varPi(m_{2\nu-1})\, f(m_{2n+2})}{\pi(m_{2n+2})\, \varPi(m_{2n+2})} \right)} \right\},$$

atque posito:

$$\sqrt{C}\, P(z) =$$
$$F(z) \overset{n}{\underset{1}{\Sigma}}{}^{\nu} \left\{ \frac{\varepsilon_\nu}{F'(m_{2\nu-1})} \cdot \frac{1}{\sqrt{\{(m_{2n+2}-m_{2\nu-1})(m_{2n+1}-m_{2\nu-1})\}}} \sqrt{\left(\frac{\pi(m_{2\nu-1})\, \varPi(m_{2\nu-1})\, f(m_{2n+2})}{\pi(m_{2n+2})\, \varPi(m_{2n+2})} \cdot \frac{1}{z-m_{2\nu-1}} \right)} \right\},$$

ipsam (83.) cum hac:

$$-2\left[\frac{\Phi(z)}{(z-\alpha)\sqrt{(-\Delta z)}}\operatorname{arc\,tang}\frac{\sqrt{C}\,P(z)\,(z-m_{2n+2})\,(z-m_{2n+1})}{\sqrt{(-\Delta z)}}\right]_{z^{-1}}$$
$$+2\,\frac{\Phi(\alpha)}{\sqrt{(-\Delta\alpha)}}\operatorname{arc\,tang}\frac{\sqrt{C}\,P(\alpha)\,(\alpha-m_{2n+2})\,(\alpha-m_{2n+1})}{\sqrt{(-\Delta\alpha)}}.$$

10.

Disquisitiones denique antecedentes de integralibus *Abelianis* $(n-1)$ti ordinis adhibere restat ad *functiones Abelianas* eiusdem ordinis. Ubi ut casum principalem, ad quem ceteri ob earundem functionum periodicitatem revocantur, ponamus hunc:

$$\varepsilon_1=\mathcal{E}_1,\quad \varepsilon_2=\mathcal{E}_2,\quad \dots\dots\quad \varepsilon_n=\mathcal{E}_n,$$

quippe qui cum casu primo problematis specialis in articulo 6. exposito convenit. Cum vero ibi signa quantitatum:

$$P(m_{2n-1}),\quad P(m_{2n-3}),\quad \dots\dots\quad P(m_3),\quad P(m_1),$$

congruentium cum ipsis:

$$\varrho(m_{2n-1}),\quad \varrho(m_{2n-3}),\quad \dots\dots\quad \varrho(m_3),\quad \varrho(m_1),$$

eadem fuerint ac ipsorum:

$$\eta_{2n-1},\quad -\eta_{2n-3},\quad \dots\dots\quad (-1)^{n-2}\eta_3,\quad (-1)^{n-1}\eta_1,$$

sequitur signa quantitatum:

$$\eta_{2n},\ \eta_{2n-1},\ \eta_{2n-2},\ \eta_{2n-3},\ \dots\dots\quad \eta_{2h},\quad \eta_{2h-1},\quad \dots\dots\quad \eta_2,\quad \eta_1,$$

aequalia esse ipsis:

$$-\varepsilon_n,\ +\varepsilon_n,\ +\varepsilon_{n-1},\ -\varepsilon_{n-1},\ \dots\dots\ (-1)^{n-h-1}\varepsilon_h,\ (-1)^{n-h}\varepsilon_h,\ \dots\dots\ (-1)^{n-2}\varepsilon_1,\ (-1)^{n-1}\varepsilon_1.$$

Quae cum ita se habeant, ex articulis (9.), (8.), (7.), (6.) prodeunt haec de *functionibus Abelianis* theoremata:

Theorema V.

„Introducantur brevitatis gratia hae denotationes:

$$\Delta z=-(z-m_1)(z-m_2)\dots\dots(z-m_{2n+2}),$$
$$f(z)=\ (z-m_1)(z-m_2)\dots\dots(z-m_{2n}),$$
$$\pi(z)=\ (z-v_1)(z-v_2)\dots\dots(z-v_n),$$

„atque definiantur argumenta $u,\ u',\ u'',\ \dots\dots\ u^{(n-1)}$ his aequationibus:

$$\sum_1^n{}^{\nu}\left(\int_{m_{2\nu-1}}^{v_\nu}\frac{\varepsilon_\nu\,a\,dy}{\sqrt{(\Delta y)}}\right)=2u,$$
$$\sum_1^n{}^{\nu}\left(\int_{m_{2\nu-1}}^{v_\nu}\frac{\varepsilon_\nu\,a'y\,dy}{\sqrt{(\Delta y)}}\right)=2u',$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\sum_1^n{}^{\nu}\left(\int_{m_{2\nu-1}}^{v_\nu}\frac{\varepsilon_\nu\,a^{(n-1)}y^{n-1}dy}{\sqrt{(\Delta y)}}\right)=2u^{(n-1)},$$

„ubi a, a', ... $a^{(n-1)}$ quantitates quaelibet sunt. Iam horum integralium „limitibus:

$$v_1, \quad v_2, \quad \dots \quad v_n;$$

„tanquam functionibus argumentorum:

$$u, \quad u', \quad \dots \quad u^{(n-1)}$$

„consideratis, ponatur generaliter:

$$\pi(m_h) = \lambda_h(u, u', \dots u^{(n-1)}),$$

„nec non:

$$\sum_1^n{}^{\nu} \left(\int_{m_{2\nu-1}}^{v_\nu} \frac{\varepsilon_\nu \, \Phi(y) \, dy}{(y-\alpha)\sqrt{\Delta(y)}} \right) = 2G(u, u', \dots u^{(n-1)}).$$

„Quibus statutis, si argumenta u, u', ... $u^{(n-1)}$ pro:

$$v_1 = m_2, \quad v_2 = m_4, \quad \dots \quad v_n = m_{2n},$$

„respective valores M, M', ... M^{n-1} induunt, habentur formulae hae:

$$\begin{aligned}
&\lambda_{2n+2}(u, u', \dots u^{(n-1)}) \,.\, \lambda_{2n+2}(M-u, M'-u', \dots M^{(n-1)}-u^{(n-1)}) \\
&: \lambda_{2n+1}(u, u', \dots u^{(n-1)}) \,.\, \lambda_{2n+1}(M-u, M'-u', \dots M^{(n-1)}-u^{(n-1)}), \\
&: \quad \lambda_h(u, u', \dots u^{(n-1)}) \,.\, \quad \lambda_h(M-u, M'-u', \dots M^{(n-1)}-u^{(n-1)}), \\
&: 1 \\
= \;& f(m_{2n+2}) \\
&: f(m_{2n+1}) \\
&: (m_{2n+2}-m_h)(m_{2n+1}-m_h)\pi(m_h) \left\{ \sum_1^n{}^{\nu} \frac{\varepsilon_\nu \sqrt{\Delta(v_\nu)}}{(m_{2n+2}-v_\nu)(m_{2n+1}-v_\nu)\pi'(v_\nu)} \cdot \frac{1}{m_h - v_\nu} \right\}^2 \\
&: 1 + \left\{ \sum_1^n{}^{\nu} \frac{\varepsilon_\nu \sqrt{\Delta(v_\nu)}}{(m_{2n+2}-v_\nu)(m_{2n+1}-v_\nu)\pi'(v_\nu)} \right\}^2,
\end{aligned}$$

„ubi h quilibet est numerorum 1, 2, ... $2n$; atque

$$G(u, u', \dots u^{(n-1)}) + G(M-u, M'-u', \dots M^{(n-1)}-u^{(n-1)}) - G(M, M', \dots M^{(n-1)})$$

$$= -\left[\frac{\Phi(z)}{(z-\alpha)\sqrt{-\Delta(z)}} \operatorname{arc\,tang} \frac{\sqrt{C}\,P(z)(z-m_{2n+2})(z-m_{2n+1})}{\sqrt{-\Delta(z)}} \right]_{z^{-1}} + \frac{\Phi(\alpha)}{\sqrt{-\Delta(\alpha)}} \operatorname{arc\,tang} \frac{\sqrt{C}\,P(\alpha)(\alpha-m_{2n+2})(\alpha-m_{2n+1})}{\sqrt{-\Delta(\alpha)}},$$

„ubi, posito:

$$F(z) = (z-m_1)(z-m_3) \dots (z-m_{2n-1})$$

„expressio $\sqrt{C}.P(z)$ determinatur formula:

$$\sqrt{C}.\,P(z) = F(z) \sqrt{\left(\frac{f(m_{2n+2})}{\lambda_{2n+2}(u, u', \dots u^{(n-1)})\,\lambda_{2n+2}(M-u, M'-u', \dots M^{(n-1)}-u^{(n-1)})} \right)}$$

$$\times \sum_1^n{}^{\nu} \frac{\varepsilon_\nu \sqrt{[\lambda_{2\nu-1}(u, u', \dots u^{(n-1)})\,\lambda_{2\nu-1}(M-u, M'-u', \dots M^{(n-1)}-u^{(n-1)})]}}{F'(m_{2\nu-1})\sqrt{[(m_{2n+2}-m_{2\nu-1})(m_{2n+1}-m_{2\nu-1})]}\,(z-m_{2\nu-1})}."$$

Theorema VI.

„Iisdem denotationibus, ac in theoremate I., adhibitis, atque posito:

$$\psi(z)=\left(z-\varepsilon_n\sqrt{\frac{m_{2n+1}-m_{2n}}{m_{2n+2}-m_{2n}}}\right).\left(z+\varepsilon_{n-1}\sqrt{\frac{m_{2n+1}-m_{2n-2}}{m_{2n+2}-m_{2n-2}}}\right)\dots\left(z+(-1)^n\,\varepsilon_1\sqrt{\frac{m_{2n+1}-m_2}{m_{2n+2}-m_2}}\right)$$
$$\times\left(z+\varepsilon_n\sqrt{\frac{m_{2n+1}-m_{2n-1}}{m_{2n+2}-m_{2n-1}}}\right).\left(z-\varepsilon_{n-1}\sqrt{\frac{m_{2n+1}-m_{2n-3}}{m_{2n+2}-m_{2n-3}}}\right)\dots\left(z+(-1)^{n-1}\varepsilon_1\sqrt{\frac{m_{2n+1}-m_1}{m_{2n+2}-m_1}}\right),$$

„ubi signa $\varepsilon_1,\ \varepsilon_2,\ \dots\ \varepsilon_n$ conditioni:

$$\psi(1)>\psi(-1)$$

satisfaciunt, habentur formulae memorabiles:

$$\begin{array}{ll} \lambda_{2n+2}(\tfrac{1}{2}M,\tfrac{1}{2}M',\dots\tfrac{1}{2}M^{(n-1)}) & 2(m_{2n+2}-m_{2n+1})^n \\ :\lambda_{2n+1}(\tfrac{1}{2}M,\tfrac{1}{2}M',\dots\tfrac{1}{2}M^{(n-1)}) & :\sqrt{\left(4(m_{2n+2}-m_{2n+1})^{2n}\dfrac{f(m_{2n+1})}{f(m_{2n+2})}\right)} \\ :\lambda_{2\nu}\ (\tfrac{1}{2}M,\tfrac{1}{2}M',\dots\tfrac{1}{2}M^{(n-1)}) \quad = & :(m_{2n+2}-m_{2n+1})^n\,\psi\left((-1)^{n-\nu-1}\varepsilon_\nu\sqrt{\dfrac{m_{2n+1}-m_{2\nu}}{m_{2n+2}-m_{2\nu}}}\right) \\ :\lambda_{2\nu-1}(\tfrac{1}{2}M,\tfrac{1}{2}M',\dots\tfrac{1}{2}M^{n-1}) & :(m_{2n+2}-m_{2n+1})^n\,\psi\left((-1)^{n-\nu}\ \varepsilon_\nu\sqrt{\dfrac{m_{2n+1}-m_{2\nu-1}}{m_{2n+2}-m_{2\nu-1}}}\right) \\ :\ 1 & :\psi(1)+\psi(-1) \end{array}$$

$$2G(\tfrac{1}{2}M,\tfrac{1}{2}M',\dots\tfrac{1}{2}M^{(n-1)})-G(M,M',\dots M^{n-1})$$
$$=-\left[\frac{\Phi(z)}{(z-\alpha)\sqrt{-\Delta(z)}}\,\text{arc tang}\,\chi(z)\right]+\frac{\Phi(\alpha)}{\sqrt{-\Delta(\alpha)}}\,\text{arc tang}\,\chi(\alpha),$$

„ubi ponitur:

$$\chi(z)=\frac{z-m_{2n+1}}{\sqrt{-\Delta(z)}}\left(\frac{z-m_{2n+2}}{m_{2n+2}-m_{2n+1}}\right)^n\sqrt{f(m_{2n+2})}.\frac{\psi\left(\sqrt{\dfrac{z-m_{2n+1}}{z-m_{2n+2}}}\right)-\psi\left(-\sqrt{\dfrac{z-m_{2n+1}}{z-m_{2n+1}}}\right)}{2\sqrt{\dfrac{z-m_{2n+1}}{z-m_{2n+2}}}}.$$

Si in utroque theoremate ponitur $a=a'=\sqrt{m_{2n+2}}$, atque loco functionis $\Phi(z)$ vel expressio: $\sqrt{m_{2n+2}}.(z-\alpha)\Phi_1(z)$ vel $\sqrt{m_{2n+2}}.\Phi_2(z)$, ubi $\Phi_1(z)$ et $\Phi_2(z)$ functiones integras denotant, quarum posterioris gradus numerum n haud superat, dum prioris gradus numerum $2n-1$ aequat, pro valore ipsius m_{2n+2} infinite magno, prodeunt theoremata duo de tribus *functionum Abelianarum* $(n-1)$ti ordinis generibus principalibus.

Theorema VII.

„Posito:

$$(y-m_1)(y-m_2)\dots(y-m_{2n+1})=D(y),$$

$$\overset{n}{\underset{1}{\Sigma^\nu}}\left(\int_{m_{2\nu-1}}^{v_\nu}\frac{\varepsilon_\nu\,dy}{\sqrt{D(y)}}\right) = 2u,$$

$$\overset{n}{\underset{1}{\Sigma^\nu}}\left(\int_{m_{2\nu-1}}^{v_\nu}\frac{\varepsilon_\nu\,y\,dy}{\sqrt{D(y)}}\right) = 2u',$$

. . . .

$$\overset{n}{\underset{1}{\Sigma^\nu}}\left(\int_{m_{2\nu-1}}^{v_\nu}\frac{\varepsilon_\nu\,y^{n-1}\,dy}{\sqrt{D(y)}}\right) = 2u^{(n-1)},$$

$$\overset{n}{\underset{1}{\Sigma^\nu}}\left(\int_{m_{2\nu-1}}^{v_\nu}\frac{\varepsilon_\nu\,\Phi_1(y)\,dy}{\sqrt{D(y)}}\right) = 2E(u, u', \ldots . u^{(n-1)}),$$

$$\overset{n}{\underset{1}{\Sigma^\nu}}\left(\int_{m_{2\nu-1}}^{v_\nu}\frac{\varepsilon_\nu\,\Phi_2(y)\,dy}{(y-\alpha)\sqrt{D(y)}}\right) = 2G(u, u', \ldots . u^{(n-1)}),$$

„integralium limites $v_1, v_2, \ldots . v_n$ argumentorum: $u, u', \ldots . u^{(n-1)}$ tales functiones „sunt, ut brevitatis gratia posito:

$$\lambda_\varkappa(u, u', \ldots . u^{(n-1)}) = (m_\varkappa - v_1)(m_\varkappa - v_2) \ldots . (m_\varkappa - v_n),$$

„ubi $\varkappa$ denotat quemlibet numerorum $1, 2, \ldots . 2n-1$, habeantur hae formulae:

$$\lambda_{2n+1}(u, u', \ldots . u^{(n-1)})\,\lambda_{2n+1}(M-u, M'-u', \ldots . M^{(n-1)}-u^{(n-1)})$$
$$= (m_{2n+1}-m_1)(m_{2n+1}-m_2) \ldots . (m_{2n+1}-m_{2n}),$$
$$\lambda_h(u, u', \ldots . u^{(n-1)}).\lambda_h(M-u, M'-u', \ldots . M^{(n-1)}-u^{(n-1)})$$
$$= (m_{2n+1}-m_h)\overset{n}{\underset{1}{\Sigma^\nu}}\left(\frac{\varepsilon_\nu\sqrt{\Delta(v_\nu)}}{m_{2n+1}-v_\nu}\cdot\frac{\pi(m_h)}{\pi'(v_\nu)(m_h-v_\nu)}\right),$$

„quantitatibus $M, M', \ldots . M^{(n-1)}$ eos argumentorum valores designantibus, „quibus ipsa pro:

$$v_1 = m_2, \quad v_2 = m_4, \quad \ldots . \quad v_n = m_{2n},$$

„gaudent; nec non hae aequationes memorabiles:

$$E(u, u', \ldots . u^{(n-1)}) + E(M-u, M'-u', \ldots . M^{(n-1)}-u^{(n-1)}) - E(M, M', \ldots . M^{(n-1)})$$
$$= -\left[\frac{\Phi_1(z)}{\sqrt{-D(z)}}\left(\chi(z) - \frac{\chi(z)^3}{3} + \frac{\chi(z)^5}{5} + \ldots . + (-1)^{n-1}\frac{\chi(z)^{2n-1}}{2n-1}\right]_{z^{-1}},$$
$$G(u, u', \ldots . u^{(n-1)}) + G(M-u, M'-u', \ldots . M^{(n-1)}-u^{(n-1)}) - G(M, M', \ldots . M^{(n-1)})$$
$$= +\frac{\Phi_2(\alpha)}{\sqrt{-D(\alpha)}}\,\text{arc tang}\,\chi(\alpha),$$

„ubi functio $\chi(z)$ determinatur formula:

$$\chi(z) =$$
$$-\frac{(z-m_{2n+2})F(z)}{\sqrt{-D(z)}}\overset{n}{\underset{1}{\Sigma^\nu}}\left(\frac{\varepsilon_\nu}{(z-m_{2\nu-1})F'(m_{2\nu-1})}\sqrt{\frac{\lambda_{2\nu-1}(u, u', \ldots . u^{(n-1)})\,\lambda_{2\nu-1}(M-u, M'-u', \ldots . M^{(n-1)}-u^{(n-1)})}{m_{2n+1}-m_{2\nu-1}}}\right)."$$

Theorema VIII.

„Iisdem denotationibus adhibitis nec non brevitatis gratia introducta functione:

$$\psi(z) = (z - \varepsilon_n \sqrt{(m_{2n+1} - m_{2n})})\ (z + \varepsilon_{n-1}\sqrt{(m_{2n+1} - m_{2n-2})}) \dots (z + (-1)^n\ \varepsilon_1 \sqrt{(m_{2n+1} - m_2)})$$
$$\times (z + \varepsilon_n \sqrt{(m_{2n+1} - m_{2n-1})})(z - \varepsilon_{n-1}\sqrt{(m_{2n+1} - m_{2n-3})}) \dots (z + (-1)^{n-1}\varepsilon_1 \sqrt{(m_{2n+1} - m_1)}),$$

„ubi signa $\varepsilon_1,\ \varepsilon_2,\ \dots\ \varepsilon_n$ conditioni

$$\psi(1) > \psi(-1)$$

„satisfaciunt, habentur aequationes notatu dignae:

$$\lambda_{2n+1}(\tfrac{1}{2}M, \tfrac{1}{2}M', \dots \tfrac{1}{2}M^{(n-1)}) = \sqrt{f(m_{2n+1})},$$
$$\lambda_{2\nu}(\tfrac{1}{2}M, \tfrac{1}{2}M', \dots \tfrac{1}{2}M^{(n-1)}) = \tfrac{1}{2}\psi((-1)^{n-\nu-1}\varepsilon_\nu \sqrt{(m_{2n+1} - m_{2\nu})}),$$
$$\lambda_{2\nu-1}(\tfrac{1}{2}M, \tfrac{1}{2}M', \dots \tfrac{1}{2}M^{(n-1)}) = \tfrac{1}{2}\psi((-1)^{n-\nu}\varepsilon_\nu \sqrt{(m_{2n+1} - m_{2\nu-1})}),$$
$$2E(\tfrac{1}{2}M, \tfrac{1}{2}M', \dots \tfrac{1}{2}M^{(n-1)}) - E(M, M', \dots M^{(n-1)})$$
$$= -\left[\frac{\Phi_1(z)}{\sqrt{-D(z)}}\left(\chi(z) - \frac{\chi(z)^3}{3} + \dots + (-1)^{n-1}\frac{\chi(z)^{2n-1}}{2n-1}\right)\right]_{z^{-1}},$$

$$2G(\tfrac{1}{2}M, \tfrac{1}{2}M', \dots \tfrac{1}{2}M^{(n-1)}) - G(M, M', \dots M^{(n-1)}) = +\frac{\Phi_2 \alpha}{\sqrt{-D(\alpha)}} \text{arc tang}\, \chi(\alpha),$$

„ubi functio $\chi(z)$ determinatur formula:

$$\chi(z) = (-1)^n \frac{z - m_{2n+1}}{\sqrt{-D(z)}} \cdot \frac{\psi(\sqrt{-1}\sqrt{(z - m_{2n+1})}) - \psi(-\sqrt{-1}\sqrt{(z - m_{2n+1})})}{2\sqrt{-1}\sqrt{(z - m_{2n+1})}}.$$