The Elementary Theory of Cauchy's Principal Values. *By* G. H. Hardy.
Received March 12th, 1901. Read March 14th, 1901.

1. Definite integrals are of two kinds—*finite* and *infinite.** In finite integrals the range of integration is finite, and the subject of integration finite throughout it. A finite integral, simple or multiple, is defined as a *single limit*; thus, for instance, the simple integral

$$\int_a^A f(x)\,dx \qquad (1)$$

is the limit when n tends to infinity of a certain finite sum

$$\sum_{r=1}^{n} f(\xi_{r-1,r})(x_r - x_{r-1}),$$

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = A, \quad x_{r-1} \leqq \xi_{r-1,r} \leqq x_r.$$

When a or A is infinite, or $f(x)$ has infinities lying within (a, A), the integral (1) can be defined, if at all, only as a double limit. Thus, if, to take the simplest case, $f(x)$ has a single infinity ξ in (a, A), it must be defined as the limit of

$$\left(\int_a^{\xi-\epsilon} + \int_{\xi+\epsilon'}^{A}\right) f(x)\,dx,$$

when the positive quantities ϵ, ϵ' tend, independently or otherwise, to zero; that is to say, as the limit of the sum of two single limits, *i.e.*, a double limit. The only case which is considered in any detail in the books is that in which this limit is determinate when ϵ, ϵ' tend independently to zero; that is, when it is the same for all possible ways in which they can do so. We shall say then that the integral (1) is *unconditionally* convergent. Unconditional convergence may be of two kinds—*absolute* or *relative*; but this is a distinction with which we need not concern ourselves at present.

* There are, so far as I know, no English words of general use in this connexion equivalent to the German *eigentlich*, *uneigentlich*. "Finite" and "infinite" do, I think, really express the distinction in a way the German words do not. It has indeed been suggested to me as a possible objection that "infinite integral" ought to mean "integral whose value is infinite, divergent integral." But nobody supposes that an "infinite series" is necessarily divergent, and I hardly see why confusion should be more likely to arise in one case than in the other.

Conditionally Convergent Integrals.

2. We shall now suppose that this is not the case. We have then to consider the possibility that a definite limit may result if the quantities ϵ, ϵ', while they tend to zero, continue to satisfy one or more relations. If such relations can be found, we shall say that the function under the integral sign is *conditionally integrable*; that the integral is *conditionally convergent*; and that the limit which corresponds to any such particular set of relations is a *particular value* of the integral.

These definitions can only be useful when the subject of integration changes its sign within the range; the integral of a function of constant sign is either unconditionally (and absolutely) convergent or determinately divergent. And when a function is only conditionally integrable different sets of relations will generally lead to different results.

3. Consider, for example, the simple case of a function $f(x)$ which is finite and integrable throughout the range (a, A) except at one point X; and is positive throughout a finite interval $(X-\xi, X)$, negative throughout a finite interval $(X, X+\xi')$. Suppose, moreover, that

$$\lim_{\epsilon=0} \int_a^{X-\epsilon} = +\infty, \quad \lim_{\epsilon'=0} \int_{X+\epsilon'}^{A} = -\infty.$$

Then *any quantity m whatever is a particular value of*

$$\int_a^A f\,dx.$$

For let $\eta_1, \eta_2, \ldots$ and $\eta_1', \eta_2', \ldots$ be any two sequences of descending positive quantities whose limits are zero. Let

$$\int_a^{\xi-\eta_i} = H_i, \quad \int_{X+\eta_i'}^{A} = -H_i'.$$

Let $m_1, m_2, \ldots$ be any sequence of quantities whose limit is m. We can determine M_1, M_1' so that $M_1 > H_1$, $M_1' > H_1'$, and $M_1 - M_1' = m_1$. Then we can determine $M_2 > M_1$, $M_2' > M_1'$ so that $M_2 > H_2$, $M_2' > H_2'$, and $M_2 - M_2' = m_2$; and so on.

We can then determine ϵ_i, ϵ_i' by the equations

$$\int_a^{X-\epsilon_i} = M_i, \quad \int_{X+\epsilon_i'}^{A} = -M_i';$$

so that
$$\int_a^{X-\epsilon_i} + \int_{X+\epsilon_i'}^{A} = m_i.$$

Also $\epsilon_i < \eta_i$, $\epsilon_i' < \eta_i'$; so that $\lim \epsilon_i = 0$, $\lim \epsilon_i' = 0$. Hence, if ϵ tend to 0 through the sequence of values $\epsilon_1, \epsilon_2, \ldots$, and ϵ' through the sequence $\epsilon_1', \epsilon_2', \ldots$,
$$\lim_{\epsilon,\, \epsilon'=0} \left(\int_a^{X-\epsilon} + \int_{X+\epsilon'}^{A} \right) = m.$$

Thus m is a particular value of $\int_a^A$. For instance
$$\lim_{\epsilon=\kappa\epsilon'=0} \left(\int_a^{X-\epsilon} + \int_{X+\epsilon'}^{A} \right) \frac{dx}{x-X} = \log \left(\kappa \frac{A-X}{X-a} \right),$$
which may be made equal to any quantity we please by choice of κ.

4. There is only one form of "particular value" with which we shall be concerned in the following pages.

The Principal Value of an Integral.

Suppose that $f(x)$ possesses a convergent integral over any part of (a, A) which does not include any of a finite number n of points X_i, distinct from a or A, and that
$$\left(\int_a^{X_1-\epsilon_1} + \int_{X_1+\epsilon_1}^{X_2-\epsilon_2} \cdots + \int_{X_n+\epsilon_n}^{A} \right) f(x)\, dx$$
tends to a finite limit when the quantities $\epsilon_1, \ldots, \epsilon_n$ tend independently to 0. Then this limit will be called *the principal value of the integral* $\int_a^A$, and will be denoted by
$$P \int_a^A f(x)\, dx.$$

5. *Historical and Critical Note.*—The set of relations which serves to define the principal value is, in fact,
$$\epsilon_i = \epsilon_i' \quad (i = 1, 2, \ldots, n).$$

The principal value was first defined by Cauchy. But Cauchy's ideas on the subject of infinite integrals had not the degree of precision required by modern analysts. So far as I am aware, he does not recognize the distinction between unconditionally and conditionally convergent infinite integrals at all. In some of his earlier memoirs, indeed, he does not distinguish principal values from ordinary finite integrals. And he does not seem to have observed that, if the subject of integration becomes infinite like $(x-X)^\mu$, the only case in which the definition of the principal value is useful is that of $\mu = 1$.

There is so much that seems arbitrary, at any rate from the point of view of the theory of functions of a real variable, about the conditions

$$\epsilon_i = \epsilon_i'$$

by which the principal value is defined, that its theory has been practically neglected. Thus Riemann, who was the first to give a precise form to the definition of the infinite integral, expressly excludes it from consideration. And in the best theoretical treatises (as, *e.g.*, Stolz, Jordan, Harnack) it is generally dismissed with a remark; sometimes its very legitimacy appears to be called in question. For instance:

"Cauchy hat . . . Hauptwerthe (*valeurs principales*) in Betracht gezogen, auch wenn . . . keinen Sinn hat. *Es ist jedoch uns unseren Ueberlegungen klar, dass man besser thut, dieser Einfuhrung nicht zu folgen*" (Kronecker, *Vorlesungen*, p. 211).

"Dass der Begriff der *valeur principale* einen Integrales, den Cauchy aufstellt, nicht statthaft sei, braucht nicht erörtert zu werden" (Schläfli, *Acta Math.*, Vol. VII., p. 187).

It is, at any rate, quite clear that the principal value is not what the last writer asserts it to be: "was er [Cauchy] so nennt, ist *eine Summe von Integralen, die einander nichts angehn.*" For instance, if $f(x)$ be a function of the complex variable, analytic near the origin,

$$P\int_{-1}^{1} \frac{f(x)}{x}\,dx = \lim_{\epsilon=0} \left(\int_{\epsilon}^{1} + \int_{-1}^{-\epsilon}\right)$$

is determinate. But the principal value is neither the sum of $\int_{\epsilon}^{1}$, $\int_{-1}^{-\epsilon}$ nor the sum of $\int_{0}^{1}$, $\int_{-1}^{0}$, which are not convergent.

Again, in the *Encyc. d. Math. Wiss.* (I. ii. 1, p. 38) it is asserted that the ordinary formula of transformation cannot be applied to principal values. We shall see later that in such cases as ordinarily occur this statement is untrue. The fact is that the interest of the principal value depends upon its frequent occurrence in connexion with the ordinary, elementary functions of analysis, such as

$$\frac{1}{x}, \quad \log x, \quad \operatorname{cosec} x.$$

These are of course extremely *special* functions. But we must distinguish, with Borel ("Mémoire sur les Séries divergentes," *Ann. de l'E. N.*, XVI.), between the *theoretically general* and the *practically general*. The simplest special kind of infinity of a function across which its integral is only conditionally convergent is a *simple pole*; and in general analysis this is, of course, the most important kind of all. Indeed, when we look at the matter from the standpoint of the theory of functions of a complex variable, and consider the methods used in contour integration, in which poles are often excluded from the range of integration by semicircles having the poles as centres, the particular significance of the at first sight arbitrary conditions $\epsilon_i = \epsilon_i'$ becomes quite plain.

It is worth remarking that formulæ involving principal values are very often simpler than the corresponding formulæ which involve ordinary integrals. Thus, for instance,

$$\int_0^\infty \frac{dx}{a^2+x^2} = \frac{\pi}{2a}, \quad P\int_0^\infty \frac{dx}{a^2-x^2} = 0;$$

$$\int_0^\pi \frac{dx}{\cosh \alpha - \cos x} = \frac{\pi}{\sinh \alpha}, \quad P\int_0^\pi \frac{dx}{\cos \alpha - \cos x} = 0.$$

The consequence of this is that the easiest way of evaluating an ordinary integral is often by means of its connexion with a principal value. And by the use of principal values the range of some of the fundamental double limit problems of the integral calculus can be considerably extended. I hope to illustrate these points systematically in a series of further papers. For the present I may refer to a paper in the *Quart. Jour. of Math.* (June and September, 1900).

Elementary Properties.

6. (i.) $$P\int_a^A f(x)\,dx = -P\int_A^a f(x)\,dx.$$

(ii.) $$P\int_a^A \sum_1^n f_i(x)\,dx = \sum_1^n P\int_a^A f_i(x)\,dx.$$

(iii.) $$P\int_a^A \kappa f(x)\,dx = \kappa P\int_a^A f(x)\,dx.$$

(iv.) *If* $P\int_a^A f(x)\,dx$ *is determinate, so is* $P\int_a^c f(x)\,dx$ $(a < c < A)$, *except possibly for a finite number of values of* c, *and*

$$P\int_a^c + P\int_c^A = P\int_a^A.$$

(v.) $P\int_a^A f(x)\,dx$ *is a continuous function of* A *for all values of* A *for which it is defined. It has a derivate equal to* $f(A)$ *for all values of* A *for which* $f(A)$ *is continuous.*

We need not delay to prove the above almost obvious theorems, which result immediately from the definition and the corresponding properties of ordinary integrals. We will only remark that they are equally true of any particular value of an integral $\int_a^A$, when the range only includes a finite number of points across which the integral is not unconditionally convergent.

(vi.) *If $f(x)$ be continuous in general in (a, A), and $F(x)$ be continuous except at a finite number of points X_i, distinct from a or A, and there become infinite or discontinuous in such a way that*

$$\lim_{\epsilon=0} \{f(X_i-\epsilon)-f(X_i+\epsilon)\} = 0,$$

and if $F(x)$ have a derivate equal to $f(x)$ at all points at which the latter is continuous, then, for all values of x in (a, A) other than X_i,

$$F(x)-F(a) = P\int_a^x f(x)\,dx.$$

For, if, *e.g.*, $X_i < x < X_{i+1}$,

$$\int_{X_i+\epsilon_i}^{x} f(x)\,dx = F(x)-F(X_i+\epsilon_i),$$

$$\int_{X_{i-1}+\epsilon_{i-1}}^{X_i-\epsilon_i} f(x)\,dx = F(X_i-\epsilon_i)-F(X_{i-1}+\epsilon_{i-1}),$$

$$\dots \quad \dots \quad \dots$$

and on adding and proceeding to the limit the theorem follows.

7. Suppose, in particular, that $\psi(x)\,\phi(x)$ is a product of two functions which satisfies the conditions imposed upon $F(x)$, while $\psi(x)\,\phi'(x)$, $\psi'(x)\,\phi(x)$ satisfy those imposed upon $f(x)$. Then

$$P\int_a^A \{\psi(x)\,\phi'(x)+\psi'(x)\,\phi(x)\}\,dx = \Big[\phi(x)\,\psi(x)\Big]_a^A,$$

the formula for *integration by parts*.

Let, *e.g.*, $a=0$, $A=\infty$, $\psi(x)=x$, $\phi(x)=\log\left(1-\frac{p}{x^2}\right)^2$, where $p>0$. Then

$$\int_0^\infty \log\left(1-\frac{p}{x^2}\right)^2 dx = 4P\int_0^\infty \frac{p\,dx}{p-x^2} = 0.$$

We only defined $P\int_a^A$ when A was finite. But, if there be only a finite number of points X_i, all $< H$, $P\int_a^\infty$ is simply $P\int_a^H + \int_H^\infty$, if this be determinate.

Convergence of the Principal Value over an Isolated Infinity.*

7. The principal value $P\int_a^A$ has so far only been defined in the case in which (a, A) includes but a finite number of points X_i across which the integral is not unconditionally convergent. Wider definitions will be given shortly. But first we must consider more in detail the possible characters of these points X_i. There is only one case with which we need seriously concern ourselves. As we have already pointed out, the principal value is a special notion which derives its interest from its frequent occurrence in connexion with certain familiar functions. It would therefore be futile to attempt to state theorems connected with it with the utmost generality of which they are capable. That would be to try to generalize what is essentially a special case. Our object will be rather to prove a few theorems general enough to give an account of such cases as we shall meet.

Infinities X^r.

8. We define the functions

$$lx,\ l^2x,\ \ldots$$

by the equations $\qquad lx = \log|x|,\quad l^2x = llx,\ \ldots.$

We shall say that a function $f(x)$ has an infinity X^r at a point $x = X$, if a finite interval $(X-\xi, X+\xi)$ can be found within which $f(x)$ can be expressed in the form

$$\psi_\nu(x-X)\,\Theta(x),$$

where (i.) $\Theta(x)$ is a function which possesses a continuous derivate throughout $(X-\xi, X+\xi)$, and

$$\text{(ii.)}\quad \psi_\nu(u) \equiv |u|^{-r}\,|lu|^{r_1}\,|l^2u|^{r_2}\ldots|l^\nu u|^{r_\nu}.$$

* It is, no doubt, verbally inaccurate to say that "the principal value is convergent"; the principal value is a limit to which something else converges. However the expression saves a good many rather awkward periphrases. And it is quite usual to say that an ordinary infinite integral converges, although this is open to the same criticism. Strictly speaking, to say that

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

is convergent is only a short way of saying that it is a limit to which something else converges.

It is to be understood that some or all of the symbols of the absolute value on the right may be omitted, provided no difficulty as to reality arises. Thus $\psi_\nu(u)$ might be

$$u^{\frac{1}{2}}, \quad |u|^{-\frac{1}{2}}, \quad u^{\frac{1}{2}}\,|lu|^{-1}, \quad \dots .$$

It is also to be observed that, if $r < 0$, or if $r = r_1 \dots = r_{i-1} = 0$, $r_i < 0$, the point is really not an infinity, but a zero. But no confusion will arise from this.

9. To avoid misunderstanding later we add the following remarks on the subject of these logarithmic factors. All of

$$lu, \quad l^2u, \quad l^3u, \quad \dots$$

become infinite for $u = 0, \infty$. But also

$$\left.\begin{aligned} |l^2u| &= \infty, \quad u = \pm 1 \\ &= 0, \quad u = \pm e, \quad \pm e^{-1} \end{aligned}\right\},$$

$$\left.\begin{aligned} |l^3u| &= \infty, \quad u = \pm 1, \quad \pm e, \quad \pm e^{-1} \\ &= 0, \quad u = \pm e^{e}, \quad \pm e^{-e}, \quad \pm e^{e^{-1}}, \quad \pm e^{-e^{-1}} \end{aligned}\right\},$$

and so on. We are, however, interested in these logarithmic products only in connexion with the behaviour of a function *in the immediate neighbourhood of* $u = 0$. So we shall suppose all these other possible infinities excluded from consideration, either by a sufficient restriction of the range of integration, or by a suitable choice of exponents $r, r_1, \dots, r_\nu$.

Since, if $\phi(u)$ be real,

$$\frac{d}{du}\log|\phi(u)| = \frac{|\phi(u)|}{\phi(u)}\,\frac{\phi'(u)}{|\phi(u)|} = \frac{\phi'(u)}{\phi(u)},$$

$$\frac{d}{du}\,l^\nu u = \frac{1}{u\,lu\,l^2u \dots l^{\nu-1}u};$$

just as

$$\frac{d}{du}\log^\nu u = \frac{1}{u \log u \log^2 u \dots \log^{\nu-1} u}.$$

As soon as u is small enough $\quad l^\nu u = \log^\nu u$;

but, by using the functions l^ν, a good deal of possible ambiguity as to sign and reality is avoided.

It is well known that *the integral* $\int f(x)\,dx$ *is absolutely convergent across* X, *if* $r < 1$, *or if* $r = 1$, $r_1 < -1$, *or if* $r = 1$, $r_1 = r_2 \dots r_{i-1} = -1$, $r_i < -1$. When $r = 1$, we shall write Ω_ν for ψ_ν; so that

$$\Omega_\nu(u) = \frac{1}{u}\prod_1^\nu |l^i u|^{r_i}.$$

Some of the symbols of the absolute value may possibly be omitted.

In what follows we shall suppose, for simplicity, as we evidently may, that the range is so restricted that all of the functions $l^i u$, $i = 1, \ldots, \nu$ are positive.

THEOREM.—*If* $$f(x) = \Omega_\nu(x-X)\,\Theta(x),$$

where $\Theta(x)$ *is a function which has a continuous derivate near* $x = X$, *and*

$$\Omega_\nu(u) = \frac{1}{u}\prod_1^\nu \{l^i u\}^{r_i};$$

and ξ *be a sufficiently small finite quantity, then*

$$P\int_{X-\xi}^{X+\xi} f(x)\,dx$$

will be convergent.

For $$\left(\int_{X+\epsilon}^{X+\xi} + \int_{X-\xi}^{X-\epsilon}\right) f(x)\,dx$$

$$= \int_\epsilon^\xi \{f(X+u)+f(X-u)\}\,du$$

$$= \int_\epsilon^\xi \Omega_\nu(u)\,\{\Theta(X+u)-\Theta(X-u)\}\,du$$

$$= 2\int_\epsilon^\xi \prod_1^\nu \{l^i u\}^{r_i}\,\Theta'(X+\theta u)\,du, \quad (-1 \leqq \theta \leqq 1);$$

and the limit of this for $\epsilon = 0$ is plainly finite and determinate, since

$$\lim_{u=0} u^\gamma \prod_1^\nu \{l^i u\}^{r_i} = 0,$$

for any positive value of γ. Hence the theorem follows.

It should be observed (i.) that, if the exponents r_i satisfy certain conditions, not only $P\int_{X-\xi}^{X+\xi}$, but $\int_{X-\xi}^{X+\xi}$ also, is convergent; (ii.) that $\int (x-X)\,f(x)\,dx$ is evidently convergent across $x = X$; and (iii.) that $\int^x f(x)\,dx$ becomes at most logarithmically infinite for $x = X$, in the cases in which it does not converge up to $x = X$.

That is to say, as $\phi(x) = \int^x f(x)\,dx$ becomes infinite, its modulus remains less than the value of a certain logarithmic product. This may be proved as follows:—

In the first place

$$\phi(X-\epsilon) = \int_\epsilon f(X-u)\,du$$

$$= -\int_\epsilon \frac{1}{u}\prod_1^\nu \{l^i u\}^{r_i}\,\Theta(X-u)\,du.$$

We may suppose the upper limit (say ξ) so small that Θ is of constant sign (say > 0), and

$$l^i u > 1,\ i = 1,\ \ldots,\ \nu \quad (0 < u \leqq \xi). \qquad (a)$$

Also we may suppose $r_2, r_3, \ldots, r_\nu$ all > 0, and $r_1+1 > r_2+r_3\ldots+r_\nu$. For, if these conditions are not satisfied by the indices r_i, we can substitute for f a function f' whose indices r_i' satisfy the conditions

$$r_2', r_3', \ldots, r_\nu' > 0, \quad r_i' \geqq r_i, \quad r_1'+1 > r_2'+r_3'\ldots+r_\nu';$$

and then, if we can prove our conclusion for f', it will follow *a fortiori* for f, in virtue of (a). Then

$$\phi(X-\epsilon) = -\frac{1}{r_1+1}\int_\epsilon^\xi \frac{d}{du}\{lu\}^{r_1+1}\prod_2^\nu \{l^i u\}^{r_i}\,\Theta(X-u)\,du$$

$$= -\frac{1}{r_1+1}\left[\{lu\}^{r_1+1}\prod_2^\nu\{l^i u\}^{r_i}\,\Theta\right]_\epsilon^\xi - \frac{1}{r_1+1}\int^\xi \{lu\}^{r_1+1}\prod_2^\nu\{l^i u\}^{r_i}\,\Theta'\,du$$

$$+\frac{1}{r_1+1}\sum_2^\nu r_i\int_\epsilon^\xi \frac{1}{u}\{lu\}^{r_1}\prod_2^i\{l^\kappa u\}^{r_\kappa-1}\prod_{i+1}^\nu\{l^\kappa u\}^{r_\kappa}\,\Theta\,du$$

$$= \Phi(\epsilon)+\frac{1}{r_1+1}\sum_2^\nu r_i\psi_i(\epsilon), \text{ say.}$$

Here $\Phi(\epsilon)$ is the sum of

$$\frac{1}{r_1+1}\{l\epsilon\}^{r_1+1}\prod_2^\nu\{l^i\epsilon\}^{r_i}\,\Theta(X-\epsilon)$$

and terms which remain finite as ϵ tends to zero. And we may therefore suppose ξ so small that

$$\Phi(\epsilon) > 0, \quad 0 < \epsilon \leqq \xi.$$

Also $\psi_i(\epsilon) > 0$, and, since the subject of integration in ψ_i is less than that in ϕ, for every value of u in question [by (a)]

$$\phi(X-\epsilon) > \psi_i(\epsilon) \quad (i = 2, 3, \ldots, \nu).$$

Hence $$0 < \left(1-\frac{\sum_2^\nu r_i}{r_1+1}\right)\phi(X-\epsilon) < \phi(X-\epsilon)-\frac{1}{r_1+1}\sum_2^\nu r_i\psi_i(\epsilon) = \Phi(\epsilon);$$

i.e., $$\phi(X-\epsilon) < \frac{r_1+1}{r_1+1-\sum_2^\nu r_i}\,\Phi(\epsilon) < \frac{1}{r_1+1-\sum_2^\nu r_i}\left[H\{l\epsilon\}^{r_1+1}\prod_2^\nu\{l^i\epsilon\}^{r_i}+K\right],$$

where H, K are some finite constants.

Thus our conclusion is established.

11. We shall not have occasion to consider in any detail principal values other than those in which the subject of integration behaves like this near each isolated

infinity across which the integral ceases to be unconditionally convergent. An example of another kind is

$$P\int_{-1}^{1} \sin\frac{1}{x}\,\frac{dx}{x^2} = \lim_{\epsilon=0}\left\{\cos\frac{1}{\epsilon}-\cos\frac{1}{\epsilon}\right\} = 0.$$

12. We shall also find the following lemma useful in the sequel.

A Lemma analogous to the First Theorem of the Mean.

LEMMA.—*If* $\Theta(x)$, $\phi(x)$ *be functions which have continuous derivates in* $(X-\xi,\ X+\xi)$, *and* $\Theta(x)$ *do not change its sign, and*

$$f(x) = \Omega_\nu(x-X)\,\Theta(x),$$

then will

$$P\int_{X-\xi}^{X+\xi} f(x)\,\phi(x)\,dx$$
$$= \phi(X)\,P\int_{X-\xi}^{X+\xi} f(x)\,dx + \phi'(X+\mu)\int_{X-\xi}^{X+\xi}(x-X)f(x)\,dx,$$

where $$-\xi \leqq \mu \leqq \xi.$$

That each of these three terms is determinate follows immediately from what precedes. Also

$$\left(\int_{X-\xi}^{X-\epsilon} + \int_{X+\epsilon}^{X+\xi}\right) f\phi\,dx$$
$$= \int_{\epsilon}^{\xi}\{f(X+u)\,\phi(X+u)+f(X-u)\,\phi(X-u)\}\,du$$
$$= \phi(X)\int_{\epsilon}^{\xi}\{f(X+u)+f(X-u)\}\,du$$
$$+\int_{\epsilon}^{\xi} f(X+u)\,\{\phi(X+u)-\phi(X)\}\,du$$
$$-\int_{\epsilon}^{\xi} f(X-u)\,\{\phi(X)-\phi(X-u)\}\,du.$$

When ϵ tends to zero the first term on the right tends to

$$\phi(X)\,P\int_{X-\xi}^{X+\xi} f(x)\,dx.$$

The second is

$$\int_{\epsilon}^{\xi} f(X+u)\,\left[u\phi'(X+\theta u)\right]du \quad (0 \leqq \theta \leqq 1)$$
$$= \phi'(X+\lambda)\int_{\epsilon}^{\xi} uf(X+u)\,du \quad (0 < \lambda \leqq \xi).$$

The first form shows that the term tends to a limit for $\epsilon = 0$. The second factor of the second form also tends to a limit; and therefore $\phi'(X+\lambda)$ tends to a limit, which must evidently be

$$\phi'(X+\kappa) \quad (0 \leqq \kappa \leqq \xi).$$

Thus the limit of the second term is

$$\phi'(X+\kappa)\int_0^{\xi} uf(X+u)\,du = \phi'(X+\kappa)\int_X^{X+\xi}(x-X)f(x)\,dx.$$

Similarly the third term tends to

$$\phi'(X-\kappa')\int_{X-\xi}^{X}(x-X)f(x)\,dx.$$

Since $(x-X)f(x)$ is of constant sign, we may replace the sum of these two limits by

$$\phi'(X+\mu)\int_{X-\xi}^{X+\xi}(x-X)f(x)\,dx,$$

and the lemma is proved.

If, in particular, $\Theta \equiv 1$,

$$P\int_{X-\xi}^{X+\xi}\Omega_\nu(x-X)\,\phi(x)\,dx = \phi'(X+\mu)\int_{X-\xi}^{X+\xi}(x-X)\,\Omega_\nu(x-X)\,dx.$$

Thus, *e.g.*, $$P\int_{X-\xi}^{X+\xi}\frac{\phi(x)\,dx}{x-X} = 2\xi\phi'(X+\mu) \quad (-\xi \leqq \mu \leqq \xi).$$

Infinite Limits.

13. We shall now consider the case in which the range is infinite. If there be but a finite number of infinities across which $\int f(x)\,dx$ is not convergent, no new point arises. It may indeed happen that

$$\lim_{H=\infty}\int_{-H}^{H} f(x)\,dx$$

is finite and determinate, although $\int_{-\infty}^{\infty}$ is not. If so, we call the former a principal value. We shall not be concerned with this, however; and indeed principal values of this kind are not particularly interesting, and may always be reduced by simple substitutions to those of the kind which has already been considered.

We suppose then that $f(x)$ has an infinity of such infinities,

$$X_1 < X_2 < X_3 \dots \quad (\lim X_i = +\infty);$$

further, that each of them is an infinity X^1 (§ 8), and that there is a positive constant H, such that

$$X_{i+1} - X_i > H \quad (i = 1, 2, \dots).$$

Then

$$P\int_a^x f(x)\,dx \tag{1}$$

is determinate for any finite value of $x > a$ and distinct from any X_i; in particular, for values of $x > a$ which satisfy the conditions

$$|x - X_i| > \delta \quad (i = 1, 2, \dots), \tag{c}$$

where δ is any small fixed positive quantity.

Then, *if, when x tends to ∞ through any system of values which satisfy condition (c), the principal value (1) tend, however small be δ, to a finite limit independent of the particular system chosen, this limit* (which must evidently be independent of δ) *will be called* the principal value of the integral $\int_a^\infty$, *and denoted by*

$$P\int_a^\infty f(x)\,dx.$$

Similarly for an infinite lower limit.

14. We shall shortly give a still more general definition, which may be used when condition (c) cannot be satisfied, or when $f(x)$ has infinitely many infinities X^1 within a finite range. But we may first illustrate the preceding definition by some examples.

15. (i.) Let
$$f(x) = \frac{\sin ax}{\sin x}\,\frac{1}{\theta^2 + x^2}.$$

Here $X_i = i\pi$. Suppose $0 < \delta < \frac{1}{2}\pi$. Then

$$P\int_0^{N\pi+\delta} = \int_0^{\frac{1}{2}\pi} + \sum_1^N P\int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} - \int_{N\pi+\delta}^{(N+\frac{1}{2})\pi}. \tag{1}$$

Also
$$P\int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} = P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (-)^i \frac{\sin a(x+i\pi)}{\sin x}\,\frac{dx}{\theta^2 + (x+i\pi)^2}$$
$$= (-)^i \cos ai\pi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin ax}{\sin x}\,\frac{dx}{\theta^2 + (x+i\pi)^2}$$
$$+ (-)^i \sin ai\pi \, P\int_{-\frac{1}{2}\pi}^{i\pi} \frac{\cos ax}{\sin x}\,\frac{dx}{\theta^2 + (x+i\pi)^2}.$$

We can determine a finite constant L, such that

$$\left|\frac{\sin ax}{\sin x}\right| < L, \quad -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi;$$

and then the modulus of the first term is

$$< \frac{L}{\theta^2 + \{(i-\frac{1}{2})\pi\}^2},$$

the general term of a convergent series. Again, the conditions of the lemma of § 12 are satisfied by the principal value in the second term, if we put

$$f(x) = \frac{1}{\sin x}, \quad \phi(x) = \frac{\cos ax}{\theta^2 + (x+i\pi)^2}.$$

Also
$$P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{dx}{\sin x} = 0.$$

Hence

$$P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos ax}{\sin x}\,\frac{dx}{\theta^2+(x+i\pi)^2} = \phi'(\mu)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{x}{\sin x}\,dx \quad (-\tfrac{1}{2}\pi \leqq \mu \leqq \tfrac{1}{2}\pi)$$

$$= \left[-\frac{a\sin a\mu}{\theta^2+(\mu+i\pi)^2} - \frac{2\cos a\mu\,(\mu+i\pi)}{\{\theta^2+(\mu+i\pi)^2\}^2}\right] K,$$

i.e.,
$$< \left[\frac{a}{\theta^2+\{(i-\frac{1}{2})\pi\}^2} + \frac{2(i+\frac{1}{2})\pi}{\{\theta^2+[(i-\frac{1}{2})\pi]^2\}^2}\right] K,$$

in absolute value, K being some finite constant. And this again is the general term of a convergent series.

Hence $P\int_0^{N\pi+\delta}$ tends to a finite limit for $N=\infty$. For the last term of (1) evidently tends to zero. And so does $\int_{N\pi+\delta}^{x}$ $[N\pi+\delta < x < (N+1)\pi-\delta]$. Hence, according to our definition,

$$P\int_0^\infty \frac{\sin ax}{\sin x}\,\frac{dx}{\theta^2+x^2}$$

is convergent. As a matter of fact its value, if $a<1$, $\theta>0$, is

$$\frac{\pi}{2\theta}\,\frac{\sinh a\theta}{\sinh\theta}.$$

16. (ii.) We shall now establish the convergence of a general class of principal values which includes the preceding example as a particular case. In the general case, however, we are obliged to use an argument which is not quite so simple.

A General Convergence Theorem.

THEOREM.—*If $\psi(x)$ be a function which possesses a continuous derivate $\psi'(x)$ for all positive values of x, and $\psi(x)$, $\psi'(x)$ tend* steadily *to zero* *for* $x = \infty$, *then*

$$P\int^{\infty} \frac{\sin ax}{\sin x}\psi(x)\,dx, \quad P\int_0^{\infty} \frac{\cos ax}{\cos x}\psi(x)\,dx$$

*will be convergent, provided a be not an integer.**

Take, for instance, the former of the two. It will be clear, after the discussion of the preceding section, that it is sufficient for our purpose to prove that we can choose N so large that

$$\left| \sum_{N+1}^{N'} P\int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} \right| < \sigma,$$

for all values of $N' > N$; σ being an arbitrarily small positive quantity. Now

$$\sum_{N+1}^{N'} P\int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} = \sum_{N+1}^{N'} (-)^i P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin a(x+i\pi)}{\sin x}\psi(x+i\pi)\,dx$$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin ax}{\sin x}\sum_{N+1}^{N'} (-)^i \cos ai\pi\,\psi(x+i\pi)\,dx$$

$$+ P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos ax}{\sin x}\sum_{N+1}^{N'} (-)^i \sin ai\pi\,\psi(x+i\pi)\,dx. \quad (1)$$

Now the series

$$\sum^{\infty} (-)^i \cos ai\pi\,\psi(x+i\pi), \quad \sum^{\infty} (-)^i \sin ai\pi\,\psi(x+i\pi)$$

are *uniformly convergent* for values of x in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, if a is not an odd integer.

For, by a well known lemma due to Abel,

$$\sum_n^{n'} u_i v_i = \sum_n^{n'-1} (u_i - u_{i+1}) V_i + u_{n'} V_{n'},$$

if $$V_i = v_n + v_{n+1} \dots + v_i.$$

Let, for instance, $v_i = (-)^i \cos ai\pi$, $u_i = \psi(x+i\pi)$.

Then V_i oscillates between finite limits as i increases. And, owing to the conditions imposed upon $\psi(x)$,

$$\sum (u_i - u_{i+1})$$

---

* If a is an odd integer, the integrals are of the ordinary kind, and are convergent or divergent according as

$$\int_0^{\infty} \psi(x)\,dx$$

is convergent or divergent. If a is an even integer, the first of them is an ordinary integral, and convergent; the second is a principal value, and convergent.

converges *absolutely* and *uniformly*, and u_i tends uniformly to zero. Hence $\Sigma u_i v_i$ converges uniformly. A similar proof applies if

$$v_i = (-)^i \sin ai\pi.$$

It follows at once that we can make the modulus of the first term of (1) assignedly small by choice of N. And the second, by the lemma of § 12, used as in the preceding section, is

$$\left[\cos ax \sum_{N+1}^{N'} (-)^i \sin ai\pi \, \psi(x+i\pi)\right]'_{x=\mu} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{x}{\sin x}\, dx,$$

where $$-\tfrac{1}{2}\pi \leqq \mu \leqq \tfrac{1}{2}\pi.$$

The quantity in square brackets is

$$-a \sin a\mu \sum_{N+1}^{N'} (-)^i \sin ai\pi \, \psi(\mu+i\pi) + \cos a\mu \sum_{N+1}^{N'} (-)^i \sin ai\pi \, \psi'(\mu+i\pi).$$

Now $$\sum^{\infty} (-)^i \sin ai\pi \, \psi'(a+i\pi)$$

is also uniformly convergent, by the same argument as before. Hence the modulus of the second term of (1) can also be made assignedly small by choice of N. And so the theorem follows.

A similar conclusion holds for

$$P\int_0^\infty \frac{\sin ax}{\cos x}\psi(x)\, dx, \quad P\int_c^\infty \frac{\cos ax}{\sin x}\psi(x)\, dx \quad (0 < c),$$

$$P\int_0^\infty \sin ax \tan x \, \psi(x)\, dx, \ \ldots,$$

$$P\int_0^\infty \frac{\sin ax}{\sin x - \sin a}\psi(x)\, dx,$$

$$P\int_0 \frac{\cos ax}{\cos x - \cos a}\psi(x)\, dx \quad (0 < a < \pi),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots.$$

Thus $\psi(x)$ may be, for example,

$$x^{-\mu} \quad (0 < \mu < 1),$$

$$e^{-\lambda x}, \quad \frac{1}{x+\theta}, \quad \frac{x}{x^2+\theta^2}, \quad x^\mu e^{-\lambda x} \quad (\mu > -1), \quad \ldots.$$

It is obviously sufficient if the conditions as to the *steady decrease* of $\psi(x)$, $\psi'(x)$ are satisfied after some finite value of x.

Principal values of these types are interesting in many ways. Cauchy evaluated some of them, as *e.g.*,

$$P\int_0^\infty \frac{\sin ax}{\sin bx}\frac{dx}{1+x^2},\quad P\int_0^\infty \frac{\cos ax}{\cos bx}\frac{dx}{1+x^2};$$

but he never defined precisely the sense in which they are convergent; nor, so far as I am aware, has any later writer done so.

A more general Theorem.

17. (iii.) THEOREM.—*If $\psi(x)$ satisfy the conditions of the preceding section, and $\phi(u)$ be a function which has a continuous derivate for all values of u, $0 \leqq u \leqq 1$, then*

$$P\int^\infty \frac{\cos ax}{\cos x}\phi(\cos^2 x)\,\psi(x)\,dx,\quad P\int^\infty \frac{\sin ax}{\sin x}\phi(\sin^2 x)\,\psi(x)\,dx$$

will be convergent.

Arguing as before, we obtain, instead of equation (1) of § 16,

$$\sum_{N+1}^{N'} P\int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin ax}{\sin x}\phi(\sin^2 x)\sum_{N+1}^{N'}(-)^i \cos ai\pi\,\psi(x+i\pi)\,dx$$

$$+P\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos ax}{\sin x}\phi(\sin^2 x)\sum_{N+1}^{N'}(-)^i \sin ai\pi\,\psi(x+i\pi)\,dx;$$

and, as before, the first line can be made assignedly small by choice of N, and the second is (using the lemma of § 12 once more)

$$\left[\cos ax\,\phi(\sin^2 x)\sum_{N+1}^{N'}(-)^i \sin ai\pi\,\psi(x+i\pi)\right]'_{x=\mu}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{x}{\sin x}\,dx.$$

The conclusion follows as before.

It would not be difficult to generalize these theorems further; but what we have proved will be sufficient for our present purpose. We may mention, among formulæ of other types, the two

$$\left.\begin{aligned} P\int_0^\infty \frac{dx}{\cos x - px\sin x} &= 0 \\ P\int_0^\infty \frac{1}{\cos x - px\sin x}\frac{dx}{a^2+x^2} &= \frac{\pi}{2a(\cosh a + pa\sinh a)} \end{aligned}\right\}\quad (p>0).$$

These follow easily from Cauchy's theorem. We have only to observe that the roots of

$$\cot x = px \quad (p>0)$$

are all real.

Transformation of Principal Values.

18. We shall now consider the question of the transformation of a principal value by the substitution of a new variable. We shall begin by considering the case of a principal value defined as in § 4. When we attempt to apply the same process to the case in which the limits are infinite, and the principal value defined as in § 13, we shall find that the definitions already given are inadequate. We shall thus be led to a more general definition.

19. Let us suppose that $P\int_a^A f(x)\,dx$ is convergent, the range of integration including one, and only one, infinity X^1, viz., $x = X$; and that, if ξ be any positive quantity, however small,

$$\int_a^{X-\xi}, \quad \int_{X+\xi}^{A},$$

can be transformed in the ordinary way by the substitution

$$x = \phi(y), \quad y = \phi^{-1}(x) = \psi(x);$$

finally, that ϕ and its first *two* derivates are continuous in the immediate neighbourhood of $x = X$, and that $\psi'(y)$ is not zero when $x = X$. *Then* $P\int_a^A$ *can be transformed by the ordinary rule*, that is to say,

$$P\int_a^A f(x)\,dx = P\int_b^B f\left[\phi(y)\right]\phi'(y)\,dy.$$

In the proof of this theorem we shall need the following lemma.

20. Lemma.—*If* $\qquad f(u) = \Omega_\nu(u)\,\Theta(u),$

where $\Omega_\nu(u)$, $\Theta(u)$ *are functions of the type considered in* §§ 8, 9, 10, *and* κ, κ' *tend to zero in such a way that*

$$\lim \frac{\kappa' - \kappa}{\kappa^{1+\mu}} = 0 \quad (\mu > 0);$$

then $$\lim P\int_{-\kappa}^{\kappa'} f(u)\,du = \lim P\int_{-\kappa}^{\kappa} = 0;$$

and so, if ξ *be small enough,*

$$\lim\left(\int_{-\xi}^{-\kappa} + \int_{\kappa'}^{\xi}\right) = P\int_{-\xi}^{\xi}.$$

For $$\int_\kappa^{\kappa'} f(u)\,du = \int_\kappa^{\kappa'} \Omega_\nu(u)\,\Theta(u)\,du.$$

Now we may suppose κ, κ' so small that $\Omega_\nu(u)$ does not change its sign in (κ, κ'), and increases as u decreases, and $\kappa' > \kappa$. Then this is

$$\Theta(\lambda)\int_\kappa^{\kappa'} \Omega_\nu(u)\,du \quad (\kappa < \lambda < \kappa')$$
$$< \Theta(\lambda)(\kappa'-\kappa)\,\Omega_\nu(\kappa),$$

which tends to zero with κ, κ'.

21. Let $$b = \psi(a), \quad B = \psi(A),$$

and suppose, *e.g.*, $b < B$. Then

$$P\int_a^A f\,dx = \lim_{\xi=0}\left(\int_a^{X-\xi} + \int_{X+\xi}^A\right) f(dx)$$
$$= \lim_{\xi=0}\left(\int_b^{\psi(X-\xi)} + \int_{\psi(X+\xi)}^B\right) f[\phi(y)]\,\phi'(y)\,dy. \qquad (1)$$

Since $\phi'(y)$ is not zero for $x = X$,

$$b < \psi(X-\xi) < \psi(X+\xi) < B.$$

Now $$f[\phi(y)] = \Omega_\nu\{\phi(y)-X\}\,\Theta[\phi(y)].$$

and $\phi(y)-X = \phi(y)-\phi(Y)$, say,

$$= (y-Y)\,\phi'(Y)+\tfrac{1}{2}(y-Y)^2\,\phi''\{Y+\theta(y-Y)\},$$

where $$0 \leqq \theta \leqq 1.$$

Hence $$\frac{1}{x-X} = \frac{1}{(y-Y)\,\phi'(Y)}$$

+ terms which remain continuous at $x = X$.

Also $$l(x-X) = l(y-Y)+l\phi'(Y)+l\left[1+\tfrac{1}{2}\frac{y-Y}{\phi'(Y)}\phi''\right]$$
$$= l(y-Y)+l\phi'(Y)+(y-Y)\,\Theta_1(y),$$

where Θ_1 is continuous.

We shall suppose for simplicity that

$$\Omega_\nu u \equiv \frac{1}{u}\,|\,lu\,|^{\rho_1}\,l^2 u\,;$$

when there are more factors the argument is more complicated, but in principle the same. Also we shall suppose that c is the greatest integer in r_1,

$$r_1 = s + c \quad (0 \leqq s < 1).$$

Then

$$|\, l(x - X)\,|^{r_1}$$

$$= |\, l(y - Y)\,|^{r_1} \left|\, 1 + \frac{l\phi' Y}{l(y-Y)} + \frac{(y-Y)\,\Theta_1}{l(y-Y)} \right|^{r_1}$$

$$= |\, l(y-Y)\,|^{r_1} + \sum_{i=1}^{c+1} a_i \,|\, l(y-Y)\,|^{r_1 - i}$$

$$+ |\, l(y-Y)\,|^{s-2} p(y) + (y-Y)\,|\, l(y-Y)\,|^{r_1 - 1} q(y),$$

where a_i is a constant, and $p(y)$, $q(y)$ are continuous functions.

Again,

$$l^2(x-X) = l^2(y-Y) + l\left\{1 + \frac{l\phi'(Y)}{|\, l(y-Y)\,|} + \frac{(y-Y)\,\Theta_1}{|\, l(y-Y)\,|}\right\}$$

$$= l^2(y-Y) + \sum_{\kappa=1}^{c+1} \gamma_\kappa \,|\, l(y-Y)\,|^{-\kappa} + |\, l(y-Y)\,|^{-c-2} \lambda(y)$$

$$+ (y-Y)\,\mu(y),$$

where γ_κ is a constant, and $\lambda(y)$, $\mu(y)$ are continuous functions.

On forming the product Ω_ν we see that it is the sum of—

(i.) A finite number of terms each of which possesses a convergent integral across $y = Y$. It is to be remembered that

$$(y-Y)^{-1}\,|\, l(y-Y)\,|^{\rho}\, l^2(y-Y)\,\phi(y)$$

is a term of this kind, if $\phi(y)$ is continuous and $\rho < -1$; so that these terms include, *e.g.*,

$$(y-Y)^{-1}\,|\, l(y-Y)\,|^{s-2} \lambda(y)$$

and

$$(y-Y)^{-1}\,|\, l(y-Y)\,|^{s-2}\, l^2(y-Y)\, p(y).$$

(ii.) A finite number of terms of the form

$$A\Omega_{\nu'}(y-Y),$$

where A is a constant. Hence

$$P\int_b^B f[\phi(y)]\,\phi'(y)\,dy$$

is convergent. Also, if

$$\psi(X-\xi) = Y-\eta, \quad \psi(X+\xi) = Y+\eta',$$

$$X-\xi = \phi(Y)-\eta\phi'(Y)+\tfrac{1}{2}\eta^2\phi''(Y-\theta\eta),$$

$$X+\xi = \phi(Y)+\eta'\phi'(Y)+\tfrac{1}{2}\eta'^2\phi''(Y+\theta'\eta'),$$

$$(0 \leqq \theta, \ \theta' \leqq 1).$$

Hence

$$\frac{\eta'-\eta}{\eta^3}$$

remains finite as η, η' tend to zero with ξ; and therefore, by equation (1) and the lemma of § 20,

$$P\int_a^A f(x)\,dx = P\int_b^B f[\phi(y)]\,\phi'(y)\,dy.$$

22. Thus, for instance, if $x = y^2$, and $H \neq n\pi$,

$$P\int_0^H \frac{\sin ax}{\sin x}\,\frac{dx}{x^\mu} = 2P\int_0^{\surd H} \frac{\sin ay^2}{\sin y^2}\,y^{1-2\mu}\,dy \quad (0<\mu<1).$$

If H tend to ∞ through a series of values included in the intervals

$$\{n\pi+\delta, \ (n+1)\pi-\delta\} \quad (n = 1, 2, \ldots),$$

each side of the equation tends to a finite limit, and it is natural to write

$$P\int_0^\infty \frac{\sin ax}{\sin x}\,\frac{dx}{x_\mu} = 2P\int^\infty \frac{\sin ay^2}{\sin y^2}\,y^{1-2\mu}\,dy. \tag{1}$$

But the right hand cannot be defined as in § 13, since the intervals

$$\{\surd n\pi, \ \surd(n+1)\pi\} \quad (n = 1, 2, \ldots)$$

diminish indefinitely as n increases. This suggests that our former definition may be extended.

23. The following general definition includes as particular cases those which we have been considering, and justifies equation (1) of § 22.

A General Definition.

Let $f(x)$ be a function which possesses a convergent integral over any part of an interval (a, A), where A may be ∞, which does not include any one of an infinite series of isolated points X_i,

$$(a < \ldots < X_i < X_{i+1}, \ldots, \lim_{i=\infty} X_i = A);$$

while

$$P\int f(x)\,dx$$

is convergent across any point X_i. Let the points X_i be included in a series of open* intervals

$$\xi_{i,\delta},$$

no two of which have any point in common. And suppose that the interval $\xi_{i,\delta}$ depends on a parameter δ, and that as δ tends to zero each of its extremities tends steadily to X_i. Let the remainder of (a, A) be denoted by R_δ.

If $x < A$ be any point of R_δ,

$$P\int_a^x f(x)\,dx \tag{1}$$

is convergent.

Then, *if, when x tends to A through any series of values lying entirely in R_δ,* (1) *tends, however small be δ, to a finite limit independent of the particular series chosen, this limit*—which must evidently be independent of δ—*will be called the principal value of the integral* $\int_a^A$, and will be denoted by

$$P\int_a^A f(x)\,dx.$$

Further generalizations are at once suggested. It is clear, for instance, that we may extend our definition to meet the case in which the infinities X_i form any enumerable set, and that similar definitions are possible for "conditionally convergent" integrals other than principal values. But, for the reasons stated in § 7, I shall not enter into this.

24. I shall conclude this paper with an illustration of the theorem of §§ 19–21, and the definition of § 23. I shall determine

* An *open* interval is an interval which does not include its extremities.

whether

$$P\int_0^\infty \frac{1}{\sin\left(ax+\dfrac{b}{x}\right)}\frac{dx}{x},$$

where $\quad a,\, b>0, \quad ab<\frac{1}{4}\pi^2,$

is determinate or not.

The infinities of the subject of integration are

$$x=0, \quad ax=\tfrac{1}{2}n\pi \pm \sqrt{\{(\tfrac{1}{2}n\pi)^2-ab\}} \quad (n=1,2,\ldots);$$

and, except for 0, are infinities X^1. As n increases

$$\tfrac{1}{2}n\pi+\sqrt{\{(\tfrac{1}{2}n\pi)^2-ab\}}$$

tends to ∞,

$$\tfrac{1}{2}n\pi-\sqrt{\{(\tfrac{1}{2}n\pi)^2-ab\}}$$

to zero.

Consider the transformation

$$y=ax+\frac{b}{x}, \quad x=\frac{1}{2a}\{y\pm\sqrt{(y^2-4ab)}\}.$$

As η increases from $2\sqrt{(ab)}$ to ∞ the upper value of x increases steadily from $\sqrt{\left(\dfrac{b}{a}\right)}$ to ∞, and the lower value decreases steadily from $\sqrt{\left(\dfrac{b}{a}\right)}$ to 0. Also

$$x'=\frac{1}{2a}\left\{1\pm\frac{y}{\sqrt{(y^2-4ab)}}\right\},$$

$$x''=\mp\frac{2b}{(y^2-4ab)^{\frac{3}{2}}};$$

and these are continuous for all values of $y>2\sqrt{(ab)}$. Finally,

$$\frac{1}{x}\frac{dx}{dy}=\frac{1}{\sqrt{(y^2-4ab)}}, \quad y>2\sqrt{(ab)}.$$

Hence, so long as A is distinct from any of the infinities,

$$P\int_{\sqrt{(b/a)}+\epsilon}^{A}\frac{1}{\sin\left(ax+\dfrac{b}{x}\right)}\frac{dx}{x}=P\int_{2\sqrt{(ab)}+\eta}^{aA+b/A}\frac{1}{\sin y}\frac{dy}{\sqrt{(y^2-4ab)}},$$

however large be A, and however small be $\epsilon>0$. Here η is >0, and tends to zero with ϵ.

Let δ be an arbitrarily small positive quantity. We can make A tend to ∞ in such a way that $aA+\frac{b}{A}$ tends to ∞ through a series of values entirely included in the intervals

$$\{n\pi+\delta,\ (n+1)\pi-\delta\} \quad (n=1, 2, \ldots),$$

and then the limit of the right hand is

$$P\int_{2\sqrt{(ab)}+\eta}^{\infty} \frac{1}{\sin y} \frac{dy}{\sqrt{(y^2-4ab)}};$$

and it is easy to see that the left becomes $P\int_{\sqrt{(b/a)}+\epsilon}^{\infty}$ according to the definition of § 18.

The limit of the right hand for $\epsilon = 0$ is determinate, and so

$$P\int_{\sqrt{(b/a)}}^{A} \frac{1}{\sin\left(ax+\frac{b}{x}\right)} \frac{dx}{x} = P\int_{2\sqrt{(ab)}}^{\infty} \frac{1}{\sin y} \frac{dy}{\sqrt{(y^2-4ab)}}.$$

Similarly

$$P\int_{0}^{\sqrt{(b/a)}} \frac{1}{\sin\left(ax+\frac{b}{x}\right)} \frac{dx}{x} = P\int_{2\sqrt{(ab)}}^{\infty} \frac{1}{\sin y} \frac{dy}{\sqrt{(y^2-4ab)}}.$$

Finally,

$$P\int_{0}^{\infty} \frac{1}{\sin\left(ax+\frac{b}{x}\right)} \frac{dx}{x} = 2P\int_{2\sqrt{(ab)}}^{\infty} \frac{1}{\sin y} \frac{dy}{\sqrt{(y^2-4ab)}}.$$

This is easily verified by the help of Cauchy's theorem. In fact each integral $=\frac{\pi}{2\sqrt{(ab)}}$. And more generally, if

$$(n+1)\pi > u = 2\sqrt{(ab)} > n\pi,$$

each of the integrals $\quad = \pi \sum_{-n}^{n} \frac{1}{\sqrt{\{u^2-(i\pi)^2\}}}.$

Similarly, if $a, b > 0$,

$$P\int_{0}^{\infty} \frac{1}{\sin\left(ax-\frac{b}{x}\right)} \frac{dx}{x} = P\int_{-}^{\infty} \frac{1}{\sin y} \frac{dy}{\sqrt{(y^2+4ab)}} = 0.$$

We may perhaps mention the following formulæ of the same kind:—

$$P\int_0^\infty \frac{\sin\left(ax+\frac{b}{x}\right)}{\sin\left(cx+\frac{d}{x}\right)}\frac{dx}{\theta^2+x^2} \quad (\theta>0,\ |c|>|a|,\ |d|>|b|)$$

$$=\frac{\pi}{2\theta}\frac{\sinh\left(a\theta-\frac{b}{\theta}\right)}{\sinh\left(c\theta-\frac{d}{\theta}\right)} \quad (cd<0)$$

$$=\frac{\pi}{2\theta}\frac{\sinh\left(a\theta-\frac{b}{\theta}\right)}{\sinh\left(c\theta-\frac{d}{\theta}\right)}+\frac{\pi\sinh\left(a\sqrt{\frac{d}{c}}-b\sqrt{\frac{c}{d}}\right)}{d-c\theta^2}$$

$$\left(0<cd<\frac{\pi^2}{4}\right). \quad (1)$$

$$P\int_0^\infty \frac{\sin\left(ax+\frac{b}{x}\right)}{\sin\left(cx+\frac{d}{x}\right)}\frac{dx}{x^2-\theta^2} \quad \left[\theta>0,\ |c|>|a|,\ |d|>|b|,\ \sin\left(c\theta+\frac{d}{\theta}\right)\neq 0\right]$$

$$=0 \quad (cd<0)$$

$$=\frac{\pi\sinh\left(a\sqrt{\frac{d}{c}}-b\sqrt{\frac{c}{d}}\right)}{d+c\theta^2} \quad \left(0<cd<\frac{\pi^2}{4}\right). \quad (2)$$

These and many other similar formulæ, which may all be proved by means of Cauchy's theorem, afford more examples of the use of the definitions of this paper.