THE MEANING OF PLÜCKER'S EQUATIONS FOR A REAL CURVE.

J. L. Coolidge (Cambridge Mass. U. S. A.).

Adunanza del 28 novembre 1915.

INTRODUCTION.

Suppose that we have given an algebraic plane curve

$$a_x^n = 0$$

which is *real* in the sense that it contains at least one real continuous branch. We associate therewith certain numbers such as the order and class, which we call PLÜCKER's numbers. The definitions which we give for these numbers are geometric in form, but the numbers themselves depend upon the degrees of certain real algebraic equations. The result is that what these numbers really give us is, either the degrees of the equations which arise when we try to do thus and so, or else the numbers of points and lines which fulfill assigned conditions, when some or all of these points and lines are imaginary, and so non-existent in the real domain in which we are specially interested. For instance, what real geometric meaning may we properly attach to the number n itself in the above equation? It may represent the maximum number of real intersections with a real line, or it may represent some number far above this maximum. The curve

$$x^{100} + y^{100} = 1$$

can not intersect any real line in more than *two* real points. What real geometric fact do we learn about this curve by noticing that the degree of the equation is 100? It is the object of the present paper to find a real geometric meaning for the order and class of a curve, as well as for one other number, invariant under a linear transformation. PLÜCKER's numbers are all rationally expressible in terms of these three.

§ 1.

Polar Systems.

Suppose that we have a real curve with the equation

$$a_x^n = 0.$$

We place the single restriction that the left hand side of this equation, if reducible, shall have no multiple factor. The first polar of a point (y) is

$$a_y a_x^{n-1} = 0.$$

This equation will, again, represent a real curve if (y) be a point in one of a finite number of connected two-dimensional regions, which include all non-singular points of the curve, and, in fact, all points of the plane not singular points of the curve when n is an even number. Moreover, this polar equation will not usually have a multiple factor, for if

$$a_\eta a_x^{n-1} \equiv \Phi^m(x_1 x_2 x_3)\Psi(x_1 x_2 x_3 \eta_1 \eta_2 \eta_3)$$

then

$$a_x^n \equiv \Phi^m \Psi$$

which is contrary to hypothesis. We remember also that the first polars of four collinear points are curves of a pencil, and have cross ratios equal to the corresponding cross ratios of the points, and that the polar of every non-singular point of the curve is tangent to the curve thereat. Lastly, we note that we can form $n-1$ successive polars. This suggests the idea of defining the order of a curve by means of the number of successive polar systems needful to determine it.

Let a correspondence be set up as follows.

1) Each line corresponds to a single point, and each point to a single line.

2) Four collinear points correspond to four concurrent lines, and corresponding cross ratios are equal.

3) An infinite number of points lying on a curve, not a straight line, lie upon the corresponding lines of the system, and at each such point the corresponding line is tangent to the new curve.

A correspondence of this sort shall be called a *polar system of the first order* [1]) The curve which contains all points which lie on their corresponding lines shall be called a *curve of the second order*. The line corresponding to any point shall be called its *first polar*.

Analytically it is immediately shown that our polar system is expressed in the form

$$\rho u_i' = \sum_{j=1}^{j=3} a_{ij} x_j, \qquad a_{ij} = a_{ji}, \qquad |a_{ij}| \neq 0$$

where the expression $\sum_{i,j=1}^{i,j=3} a_{ij} x_i x_j$ is a real indefinite quadratic form. In other symbols, if the equation of the curve be

$$a_x^2 = 0$$

the line corresponding to (y) is

$$a_y a_x = 0.$$

---

[1]) This definition is, of course, too strong, in the sense that an unnecessary amount has been assumed. Conf. G. Darboux, *Sur le théorème fondamental de la géométrie projective* [Mathematische Annalen, Bd. XVII (1880), pp. 55-61]. The definition here given is that which can be extended with the least verbal change to the case of the general polar system.

Suppose that we have such infinite system of curves of the second order, and that the first polars of every point with regard to then are either identical, or form a pencil. Such a system shall be called a *pencil of curves of the second order*. If three curves be

$$a_x^2 = 0, \qquad b_x^2 = 0, \qquad c_x^2 = 0$$

a necessary and sufficient condition that the first polars of every point with regard to them be either identical or cuncurrent is that the Jacobian should vanish identically, i. e. that the equations should be linearly dependant. If, thus, we write the pencil in the form

$$\lambda a_x^2 + \mu b_x^2 = 0$$

we see that the first polars of any point with regard to four curves of the system will, when not identical, have cross ratios independant of the point chosen. These we may define as the *cross ratios* of the four curves.

We next assume that we have succeeded in defining a curve of order $n-1$ by means of successive polar systems, have noted that the order of the curve is the same as the degree of its equation, and have defined a pencil of such curves and the cross ratios of four members of the pencil. We set up a fresh correspondence as follows.

1) Every member of an infinite system, not a pencil, of curves of order $n-1$ corresponds to a distinct point of one of a finite number of connected two-dimensional regions, and every point of such a region corresponds to a curve of the system.

2) Four collinear points always correspond to four curves of a pencil and vice versa, and corresponding cross ratios of points and curves are equal.

3) An infinite number of points lying upon a curve, not of order $n-1$ or less, lie upon the corresponding curves of the system, and at each such point the curve of the system and the new curve are tangent.

A system of this sort shall be defined as a *polar system of order* $n-1$ and the new curve shall be called a *curve of order* n.

Analytically, since collinear points correspond to curves of a pencil and corresponding cross ratios are equal, we see that our curves must form a linear net, and if we set up the coordinate triangle and unit point in one of our connected regions, the curve corresponding to the point $|\eta|$ will have an equation

$$\sum_{i=1}^{i=3} \eta_i f_i = 0.$$

The points which lie upon the corresponding curves of order $n-1$ will have coordinates which satisfy the equation

$$F \equiv \sum_{i=1}^{i=3} x_i f_i = 0.$$

There will be no loss of generality in assuming that no one of the polynomials f_i is a factor of F. If then, (x) be a point which satisfies our condition 3) we must have

$$F = 0, \qquad \frac{\partial F}{\partial x_i} \equiv \left[f_i + \sum_{k=1}^{k=3} x_k \frac{\partial f_k}{\partial x_i}\right] = \lambda \sum_{k=1}^{k=3} x_k \frac{\partial f_k}{\partial x_i}.$$

We see then, that for an infinite number of points of

$$F = 0$$

we have

$$\frac{\partial F}{\partial x_i} = \frac{\lambda}{\lambda - 1} f_i$$

and, hence

$$f_i \frac{\partial F}{\partial x_j} - f_j \frac{\partial F}{\partial x_i} = 0.$$

It is not possible that for an infinite number of points we should have

$$\frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial x_j} = F = 0$$

for in that case F would have a multiple factor, and our points in question would lie on a curve of order less than n. Hence

$$f_i \frac{\partial F}{\partial x_j} - f_j \frac{\partial F}{\partial x_i} \equiv F \Phi_k.$$

We next assume, for the moment, that we have what is clearly the *general case*, namely, that the three equations

$$f_1 = f_2 = f_3 = 0$$

have no common solution, and that each two have $(n - 1)^2$ distinct solutions. We may also assume that the coordinate triangle has been so chosen that no solution of

$$f_i = f_j = 0$$

is also a solution of

$$x_k = 0.$$

It appears, then, from the law of formation of F, that the equations

$$f_i = f_j = F = 0$$

have no common solution. This leads to the conclusion that the $(n - 1)^2$ common solutions of

$$f_i = f_j = 0$$

are also solutions of

$$\Phi_k = 0.$$

This shows, however, that Φ_k must be identically 0, as otherwise its degree $(n - 2)$ would be too low to admit of all these solutions. Hence, in our general case

$$\frac{\partial F}{\partial x_i} \equiv \frac{\lambda}{\lambda - 1} f_i.$$

To cover the special cases, let us introduce a parameter t into our equations whose vanishing gives the special case desired. Then the algebraic identity

$$\frac{\partial F}{\partial x_i} \equiv \frac{\lambda(t)}{\lambda(t) - 1} f_i$$

which holds for an infinite number of values, will hold also when $t = 0$ and our proof is complete. We thus see that if F be written

$$a_x^n = 0$$

the curve corresponding to (η) is

$$a_\eta a_x^{n-1} = 0$$

and our polar system is exactly the first polar system we require.

§ 2.

Definitions.

Definition 1). — *The order of a curve is a number greater by unity than the number of successive polar systems necessary to determine the curve.*

We see that when the polynomial which is equated to zero in order to give the equation of the curve is irreducible, its degree is the order of the curve.

Definition 2). — *The class of a curve is the order of its polar reciprocal in any curve of the second order.*

Definition 3). — *The* ν *characteristic of a curve is the order of the locus of the intersection of the line polars of a point with regard to the curve, and to an arbitrary curve of the second order, when the point traces an arbitrary first polar with regard to the given curve.*

Definition 4). — *The deficiency of a curve of order* n *and class* m *is the number* p *defined by the equation*

$$p = m + \frac{(n-1)(n-2)}{2} - \nu.$$

The last definition is justified as follows. If a point P trace the first polar of a point O, its line polar continuously passes through O which has a multiplicity, real or imaginary, $n-1$ for the curve of order ν. The number ν is, thus, greater by $n-1$ than the number of variable intersections of a line through O with the new curve, which latter is the number of variable intersections of two first polars with regard to our given curve. If, thus, our curve hawe δ double points and $\varkappa$ cusps

$$\nu = n - 1 + (n-1)^2 - \delta - 2\varkappa,$$

$$= n(n-1) - \delta - 2\varkappa,$$

$$m = n(n-1) - 2\delta - 3\varkappa,$$

$$p = \frac{(n-1)(n-2)}{2} - \delta - \varkappa,$$

$$= m + \frac{(n-1)(n-2)}{2} - \nu.$$

Let us point out, finally, that, since

$$\frac{m}{2} - (n - 1) = p - \frac{\varkappa}{2}$$

we may replace definitions 3) and 4) by.

Definition 5). — The deficiency of a curve is the maximum value of the difference between one half of the class and one less than the order, of any curve birationally equivalent to the given one.

The difficulty with such a definition is that, a priori, there is no reason why the maximum value of such a difference should not become infinite, and it is also hard to show that we may always find a real birational transformation which will make $\varkappa = 0$.

Cambridge (Mass. U. S. A.), October 1915.

J. L. Coolidge.