A Theory of Magnetic Action upon Light.

By

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1. The object of this communication is to remove an objection to a theory of magnetic action upon light, which was first suggested by Prof. Rowland*) of America in 1881, and afterwards fully developed by myself**) in 1890. The defect of this theory is that it makes the tangential component of the electromotive force *discontinuous* at the surface of separation of two different media; and I propose to show that the theory can be modified in such a manner as to get rid of this objection. The notation employed by Maxwell and other British mathematicians will be used through out.

It will be desirable to give a brief account of the principal Anglican theories on this subject.

2. At the date of the publication of Maxwell's *Electricity and Magnetism*, Faraday's discovery of the rotatory polarization produced by a magnetic field was the only known phenomenon of this character. To account for it, Maxwell***) introduced into the kinetic energy of the medium an additional term, which was supposed to arise from the displacement of certain hypothetical vortices. This theory suffices to explain Faraday's experiments, but no attempt was made to connect it with the author's previous electromagnetic theory of light.

In 1879, Prof Fitz Gerald of Dublin†) improved Maxwell's theory by establishing a connection between it and the electromagnetic theory of isotropic and doubly-refracting media. Fitz Gerald introduced a new vector A, whose time variation is the magnetic force; hence the curl of this vector is equal to the electric displacement multiplied by

*) *Phil. Mag.* April 1881, p. 254.

**) *Phil Trans.* 1891, p. 371; Basset's. *Physical Optics*, Chapter XX.

***) *Electricity and Magnetism*, vol II. Chap. XXI.

†) *Phil. Trans.* 1880. p. 691.

4π. Accordingly if ξ, η, ζ be the components of A, the electrostatic energy per unity of volume is

$$(1)\quad \frac{2\pi}{K}(f^2+g^2+h^2)=\frac{1}{8\pi K}\left\{\left(\frac{d\zeta}{dy}-\frac{d\eta}{\delta z}\right)^2+\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)^2+\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)^2\right\}$$

whilst the portion of the electrokinetic energy due to the motion of the medium is

$$(2)\qquad \frac{\mu}{8\pi}(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2).$$

To account for magnetic action, Fitz Gerald, following Maxwell, introduced into the kinetic energy the additional term

$$(3)\qquad 4\pi C\left(\dot{f}\,\frac{d\xi}{d\omega}+\dot{g}\,\frac{d\eta}{d\omega}+\dot{h}\,\frac{d\zeta}{d\omega}\right)$$

which is supposed to arise from the displacement of the abovementioned hypothetical vortices. In (3), f, g, h are the components of electric displacement; $\alpha_0, \beta_0, \gamma_0$ of the magnetic forces *due to external causes; C* is a constant which depends upon the nature of the magnetized medium, and which varies for different media; and*)

$$(4)\qquad d/d\omega=\alpha_0\, d/dx+\beta_0\, d/dy+\gamma_0\, d/dz.$$

By means of these expressions for the energy of the medium, Fitz Gerald proceeded to deduce the equations of motion and the boundary conditions by means of the Principle of Least Action: but on working out the theory, a difficulty presented itself from the fact that the boundary conditions were too numerous.

In 1893, Mr. Larmor of Cambridge**) proposed to remedy this defect in the following manner. He observed that ξ, η, ζ are not independent, but are connected together by the equation

$$(5)\qquad d\xi/dx+d\eta/dy+d\zeta/dz=0;$$

it is therefore necessary to introduce into the variational equation the above expression multiplied by an undetermined quantity λ. The effect of this additional term is to furnish four equations of motion and four boundary conditions connecting the four unknown quantities $\xi, \eta, \zeta, \lambda$. There is consequently no superfluous equation as was the case in Fitz Gerald's theory; but there are just sufficient equations, and no more, for determining all the unknown quantities.

If this theory were satisfactory, it would be unnecessary to propose another; but as a matter of fact Mr. Larmor's theory contains two serious defects. In the first place, it involves the introduction into the general equations of the electromagnetic field of a new quantity λ, for which there is no justification on any electrical or optical ground

*) Fitz Gerald and Larmor write $d/d\theta$ for $d/d\omega$.

**) *British Association Report*, 1893, p. 346.

whatever. In the second place, the theory makes the tangential component of the electromotive force discontinuous at an interface.

3. We must now consider there objections in detail.

The equations of motion obtained by Larmor (see, *Brit. Assoc. Rep.* 1893, p. 348) are

$$\frac{da}{dt} = \frac{4\pi}{K}\left(\frac{dg}{dz} - \frac{dh}{dy}\right) - 32\pi^2 C \frac{d\dot{f}}{d\omega} - 4\pi \frac{d\lambda}{dx}. \tag{6}$$

With the exception of the term $d\lambda/dx$, these equations are of the same form as equations (10) of my paper in the *Phil. Trans.* *) 1891. Moreover if we eliminate λ, we obtain three equations of electric displacement which are exactly the same as equations (12) of that paper; hence, *so far as the propagation of light is concerned*, the two theories an identical. There is, however, an essential difference between them, owing to the introduction of the quantity λ which in Mr. Larmor's theory cannot be put equal to zero.

4. The introduction of the additional term (3) into the electro-kinetic energy must necessarily produce some modification in Maxwell's general equations of the electromagnetic field. We must therefore enquire, which of the different sets of equations are modified? What is the nature of the modifications? Can any evidence, experimental or otherwise, be adduced in support of these modifications? But to all these interesting and important questions Mr. Larmor maintains an impenetrable silence, and not a single hint is any where given with regard to their solution.

For a non-conducting medium, Maxwell's equations are

$$P = -dF/dt - d\psi/dx, \tag{7}$$

$$P = 4\pi f/K, \tag{8}$$

$$a = dH/dy - dG/dz, \tag{9}$$

$$4\pi\mu\dot{f} = dc/dy - db/dz. \tag{10}$$

Equation (10) connects the electric displacement with the magnetic induction; and as this equation is expressly assumed in the theory, it cannot be modified.

Equation (9) connects the vector potential with the magnetic induction. Now the use of the vector potential is a mere mathematical artifice introduced by Maxwell, who found it convenient to employ this quantity in certain portions of the work. The vector itself is indeterminate, for if the magnetic force is given, and F, G, H are any *particular* values which satisfy (9), these equations will also be satisfied by $F + d\varphi/dx$, $G + d\varphi/dy$, $H + d\varphi/dz$, where φ is an arbitrary function. The indeterminateness of the vector potential is

*) See also, *Physical Optics* p. 394, equation (6).

taken into account in (7) by the introduction of the function ψ; and in the theory of light all difficulty arising from this cause can be got rid of by eliminating it. Equation (9) cannot therefore be modified.

Equation (8) gives the relation between electromotive force and electric displacement; and as Mr. Larmor assumes this relation in his expression for the electrostatic energy, equation (8) cannot be modified.

Equation (7) is consequently the only one which is capable of modification. To ascertain the necessary modification, let Φ be any solution of Laplace's equation $\nabla^2\Phi = 0$, and let us write in the place of (7)

$$(11) \qquad P = -\frac{dF}{dt} - p_3\dot{g} + p_2\dot{h} + z\frac{d\Phi}{dy} - y\frac{d\Phi}{dz} - \frac{d\psi}{dx}$$

with two similar equations, where $p_1 = 32\pi^2 C\alpha_0$ etc.

Substitute the value of P from (8) in (11), differentiate the first of (11) will respect to y and the second with respect to x and subtract; then recollecting that f, g, h satisfy (5), we shall obtain

$$(12) \qquad \frac{dc}{dt} = \frac{4\pi}{K}\left(\frac{df}{dy} - \frac{dg}{dx}\right) - \frac{d\dot{h}}{d\omega} + \frac{d}{dx}\left(\Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right).$$

Now according to Mr. Larmor's theory, λ is a potential function; we may therefore put

$$(13) \qquad -4\pi\lambda = \Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}$$

in which case (12) will be identical with (6).

We have therefore shown that Mr. Larmor's theory requires that Maxwell's equation (7) should be modified in the manner expressed by (11). The second and third terms are equivalent to the introduction of Hall's effect, which was the procedure originally suggested by Prof. Rowland and afterwards more fully developed by myself; but for the introduction of the two terms in Φ, there is no justification whatever: they are not required in optics nor in electromagnetism.

5. Having disposed of the equations of motion, we must now consider the boundary conditions.

Mr. Larmor takes the axis of z at right angles to the surface of separation, and on page 349 of the *British Association Report* for 1893, he obtains the condition that*)

$$\frac{4\pi g}{K} + 4\pi C\frac{d\beta}{d\omega} - 16\pi^2 C\gamma_0\frac{df}{dt}$$

should be continuous at an interface. By (8) ts requihires that

$$Q + 4\pi C\frac{d\beta}{d\omega} - 16\pi^2 C\gamma_0\frac{df}{dt}$$

*) Larmor writes $d/d\theta$ for $d/d\omega$.

should be continuous. In other words, *the tangential component of the electromotive force must be discontinuous at an interface.*

It thus appears that, so for from being any improvement on its predecessors, Mr. Larmor's theory is not only open to the original objection of making the electromotive force discontinuous at an interfaces, but at the same time necessitates the introduction of a new quantity for which there is no justification whatever.

Whether or not this theory is capable of accounting for the experiments of Kerr and Kundt on the reflection of light from magnetized substances is a question which it is unnecessary to discuss; since I shall now explain how the theory of Rowland and myself, which has already been shown to be capable of giving an explanation of this phenomenon*), can be modified to as to remedy the objection to which it was originally subject.

6. The theory of Rowland and myself consisted in introducing Hall's effect into Equations (7), which were accordingly replaced by

$$(14) \qquad P = -\dot{F} - p_3\dot{g} + p_2\dot{h} - d\psi/dx$$

where $p_1 = C\alpha_0$ etc.; α_0, β_0, γ_0 being the components of the magnetic force produced by external causes, and C is Hall's constant. The boundary conditions are carefully discussed in my paper**), from which it will be found that we are obliged to suppose that the tangential component of the electromotive force is discontinuous at an interface. This discontinuity constitutes a serious defect in the theory, which I shall now proceed to remedy.

In the absence of any experimental evidence to the contrary (of which there appears to be none), we may equally well assume that the effect of the external magnetic field may be represented by introducing the additional terms into (8) instead of (7). We shall therefore replace (8) by

$$(15) \qquad P = 4\pi f/K + p_3\dot{g} - p_2\dot{h}$$

keeping (7), (9) and (10) unchanged. This is the first hypothesis.

According of Maxwell's theory, when electromotive force acts upon a dielectric it produces electric displacement; and the work done

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*) *Provisional Theory of Kerr's Experiments. Proc. Camb. Phil. Soc.* vol. VIII p. 68. In this paper the results obtained in my paper in the Phil. Trans. 1891 are transformed by the method of Cauchy and Eisenlohr, by assuming that the refractive index for a metal is a complex quantity of the form $R\varepsilon^{\iota\alpha}$. The values of R and α are obtained by means of the experimental values of the principal incidence and azimuth given by Jamin and Conroy, and the results agree very well with experiment.

**) *Phil. Trans.* 1891, p. 381; and *Physical Optics* p. 400.

is equal to half the product of the force into the electric displacement produced. Whence the electrostatic energy of the medium is equal to

$$\tag{16} \tfrac{1}{2}(Pf + Qg + Rh).$$

We shall now suppose that this theorem is true when the relation between electromotive force and electric displacement is given by (15) instead of (8); whence the electrostatic energy per unit of volume will be given by (16), where the values of P, Q, R are given by (15); whilst the electrokinetic energy will be given by (2). This is the second hypothesis.

Since

$$\tag{17} 4\pi f = \frac{d\zeta}{dy} - \frac{d\eta}{dz} \text{ \&c.}$$

all the quantities can be expressed in terms of ξ, η, ζ, and the equations of motion and the boundary conditions can be obtained by the Principle of Least Action

$$\tag{18} \iiiint \delta(T - W)\, dx\, dy\, dz\, dt = 0$$

where T and W are the electrokinetic and electrostatic energies respectively.

Substituting in terms of ξ, η, ζ the values of T from (2) and W from (16), (15) and (17), and working out the integral in (18) by the ordinary methods of the Calculus of Variations*), we shall find that the *non-magnetic* portion involving $\delta\xi$ is

$$\tag{19} \iiiint \left\{ -\frac{\mu}{4\pi}\frac{d^2\xi}{dt^2} + \frac{1}{K}\left(\frac{dg}{dz} - \frac{dh}{dy}\right)\right\} \delta\xi\, dx\, dy\, dz\, dt \\ - \frac{1}{K}\iiint (ng - mh)\, \delta\xi\, dS\, dt$$

where l, m, n are the direction cosines of the normal to dS drawn outwards.

The magnetic part of δW is

$$\tag{20} \tfrac{1}{2}\iiiint \{(p_3\dot{g} - p_2\dot{h})\,\delta f - (p_3 g - p_2 h)\,\delta\dot{f} + \cdots\}\, dx\, dy\, dz\, dt \\ = \iiiint \{(p_3\dot{g} - p_2\dot{h})\,\delta f + \cdots\}\, dx\, dy\, dz\, dt$$

the terms at the limits of the time being omitted. The right hand side of (20), so far as it involves $\delta\xi$, can be shown by an integration by parts to be equal to

*) In strictness, we ought to have introduced an indeterminante multiplier λ, as in § 2, but it will be found that $\lambda = 0$.

$$(21)\qquad \frac{1}{4\pi}\iiiint \frac{d\dot{f}}{d\varpi}\,\delta\xi\,dx\,dy\,dz\,dt$$
$$+\frac{1}{4\pi}\iiint \{n(p_1\dot{h}-p_3\dot{f})-m(p_2\dot{f}-p_1\dot{g})\}\,\delta\xi\,dS\,dt.$$

Subtracting this expression from (19) and equating the result to zero, we obtain the equation

$$(22)\qquad \mu\frac{d^2\xi}{dt^2}=\frac{4\pi}{K}\left(\frac{dg}{dz}-\frac{dh}{dy}\right)-\frac{d\dot{f}}{d\varpi}$$

and the condition that

$$(23)\quad \left\{n\left(\frac{4\pi g}{K}+p_1\dot{h}-p_3\dot{f}\right)-m\left(\frac{4\pi h}{K}+p_2\dot{f}-p_1\dot{g}\right)\right\}\delta\xi+\cdots$$

should be continuous at an interface.

Since $\mu\,d\xi/dt=a$, equation (22) is identical with equation (10) of my paper in the *Phil. Trans.* 1891. The modified theory accordingly furnishes exactly the same equations of motion as the original theory.

By virtue of (15) the condition (23), when written out in full, is equivalent to the condition that

$$(24)\qquad (nQ-mR)\,\delta\xi+(lR-nP)\,\delta\eta+(mP-lQ)\,\delta\zeta.$$

should be continuous at an interface. If therefore we take the axis of x to be perpendicular to the interface, $l=1$, $m=n=0$ and (24) requires that

$$R\delta\eta-Q\delta\zeta.$$

should be continuous.

Now Newton's third Law of Motion requires that the electromotive and the magnetic forces should be continuous at an interface; the latter condition requires that $\delta\eta$ and $\delta\zeta$ should be continuous, and this requires that Q and R should be continuous, which is the symbolical expression for the first condition. Hence the two assumptions, on which the modified theory has been based, lead to a consistent scheme of equations which do not violate any of the fundamental principles of Dynamics, as Mr. Larmor's theory does. The theory also gives a fairly good explanation of the experiments of Kerr and Kundt upon magnetic reflection.

7. With regard to the hypothesis on which the modified theory depends, we may point out that there are some grounds for thinking that the portion of the energy due to the external magnetic field is static instead of kinetic. For when magnetic force acts upon a magnetic substance, such as iron, a state of magnetic stress is produced within the substance, which must necessarily modify the

passage of electromagnetic waves. The kinetic energy, being due to the motion of the medium, will be unchanged by magnetic action; but the forces by which electromagnetic motions are resisted, will necessarily be affected by the presence of the external magnetic field; and the additional portion of the forces, due to this cause, will give rise to a term in the energy which will be static rather than kinetic.

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