The ordinary symbols of integration might thus be dispensed with, while the successive steps of the integration would be exhibited in their natural order and without a break.

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*On Quadric Transformations.** *By* Mr. W. Spottiswoode, P.R.S.

A quantic is said to be transformed when the variables originally contained in it are replaced by others, and the transformation is called linear, or quadratic, or of a higher dègree, according as the equations connecting the two sets of variables are of the first, or second, or of higher degrees. The transformations usually considered are linear, whereby, for example, a rational homogeneous function U of x, y, is transformed into another of the same degree in ξ, η, either by a substitution of the form

$$\xi : \eta = lx + my : l'x + m'y,$$

or by a substitution of the form

$$x : y = \alpha\xi + \beta\eta : \alpha'\xi + \beta'\eta.$$

The theory of linear transformations has, as is well known, been investigated by Boole, by Sylvester, by Cayley, and by many others; and notably by the last-mentioned mathematician in his classical "Memoirs on Quantics" (*Phil. Trans.*, 1856, *et seqq.*).

Tchirnhausen's transformation is an instance of one of a higher degree; and the substitution is of the first form, viz., it is as follows:

$$\xi : \eta = V : V',$$

where V, V' are homogeneous functions of x, y, of a given degree n. And, if U be the given function of x, y, which is to be the subject of transformation, the process is effected by eliminating x, y from the equations

$$U = 0, \quad \eta V - \xi V = 0;$$

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* Some papers by the late Mr. W. Spottiswoode, P.R.S., which had apparently been written with a view to their being brought before this Society, were, at the instance of the Council, submitted to Prof. Cayley for him to report upon the advisability of their being printed in the *Proceedings*. In accordance with Prof. Cayley's report, the following portions, which have been kindly edited by him, are printed here.

and the result, say $W = 0$, will clearly be of the same degree in ξ, η as was U in x, y. This done, we may calculate the derivatives (invariants and covariants) of W, and examine the relations between the derivatives in question and those of U, V, V', taken separately or in combination.

But, precisely as in linear transformations there are two forms, one in which ξ, η are expressed in terms of x, y, and the other in which x, y are expressed in terms of ξ, η (which forms may be termed complementary to one another); so also in higher transformations there are two forms, one expressed, as above, by Tchirnhausen's formula, and the other by the formula

$$x : y = \Upsilon : \Upsilon',$$

where Υ, Υ' are homogeneous functions of ξ, η, of a given degree. And, even although there be not at present any applications of the latter method whose utility is immediately apparent, it may still be a legitimate problem to calculate the derivatives of W, and to examine their relation to those of U, Υ, Υ'. This is the subject proposed in the present communication, wherein, however, I confine myself to quadratic transformations, and in fact to an elementary instance of such.

The question, in the case of a Tchirnhausen transformation, was noticed by Prof. Cayley in a paper entitled "An Example of the Higher Transformation of a Binary Form" (*Mathematische Annalen*, vol. IV., p. 359), and it was more fully investigated, especially with reference to quadratic transformations, by Gordan, in a paper, "Ueber die Invarianten binären Formen bei höheren Transformationen" (*Crelle*, vol. LXVII., p. 164). In connexion with the same subject, the following Memoirs may also be mentioned:—Hermite, "Sur quelques théorèmes d'algèbre, et la résolution de l'équation du quatrième degré" (*Comptes Rendus*, vol. XLVI., p. 961, 1858); also, Cayley, "On Tchirnhausen's Transformation" (*Crelle*, vol. LVIII., pp. 259–262, 263–269, 1858, and *Phil. Trans.*, 1862, pp. 566–578).

The case which we first propose to examine is the simplest, viz., the quadratic transformation of the quadric. Let the quadratic be

$$U = (a, b, c)(x, y)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and the equation of transformation, or the transformant,

$$x : y = (\alpha, \beta, \gamma)(\xi, \eta)^2 : (\alpha', \beta', \gamma')(\xi, \eta)^2, \; = V : V' \quad \ldots\ldots(2).$$

But, before proceeding to the actual transformation, it will be convenient to enumerate in detail the quadrics in x, y; as well as those in ξ, η; together with the several derivatives which will occur in the sequel.

Adopting the following notation,

$$D = ac - b^2,$$

$$\Delta = \alpha\gamma - \beta^2,$$

$$2\Delta' = \alpha\gamma' + \alpha'\gamma - 2\beta\beta',$$

$$\Delta'' = \alpha'\gamma' - \beta'^2,$$

$$\square = 4(\Delta\Delta'' - \Delta'^2),$$

$$K = a\Delta + 2b\Delta' + c\Delta'',$$

then D is the discriminant of U,

Δ " " V,

Δ'' " " V',

$2\Delta'$ is the intermediate of Δ, Δ'',

$\square$ is the resultant of V, V',

K is the intermediate of D, $\square$;

$\square$ is also the discriminant of a quantic U', to be hereafter particularised.

Further, putting $\begin{Vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{Vmatrix} = A, B, C,$

we have the relations

$$\alpha A + \beta B + \gamma C = 0,$$

$$\alpha' A + \beta' B + \gamma' C = 0,$$

and $$4AC - B^2 = 4(\Delta\Delta'' - \Delta'^2) = \square\,;$$

also, putting $\begin{Vmatrix} c, & -2b, & a \\ \Delta, & 2\Delta', & \Delta'' \end{Vmatrix} = P, Q, R,$

we have the relations $$cP - 2bQ + aR = 0,$$

$$\Delta P + 2\Delta' Q + \Delta'' R = 0,$$

and $$PR - Q^2 = D\square - K^2.$$

This being premised, if we write the transformant in the following form,

$$(\alpha y - \alpha' x,\ \beta y - \beta' x,\ \gamma y - \gamma' x)(\xi, \eta)^2 = 0,$$

and then form its discriminant U' with respect to ξ, η, we shall find

$$U' = \Delta y^2 - 2\Delta' yx + \Delta'' x^2,$$

which is in fact the function, whose discriminant is $\frac{1}{4}\square$, mentioned above.

Following the course pursued by Cayley in his "Fifth Memoir on Quantics," and considering first the quantics in x, y, we have the two quadrics U, U'; their discriminants D, $\frac{1}{4}\square$; the connective of D, $\square$, viz., K; the resultant of the two quadrics, S; their Jacobian Θ; an intermediate Υ; and, lastly, the discriminant of the intermediate. Among these, the values of U, U', D, $\square$, K have been given above. Also it will be found that

$$S = 4(D\square - K^2),$$

$$\Theta = \tfrac{1}{2}(P, Q, R)(x, y)^2;$$

and, if the intermediate be

$$\Upsilon = (\lambda a + \mu\Delta'', \lambda b - \mu\Delta', \lambda c + \mu\Delta)(x, y)^2,$$

then its discriminant will be

$$(D, K, \tfrac{1}{4}\square \between (\lambda, \mu)^2.$$

If the two quadrics are harmonically related; that is, if their four roots are harmonically related; then

$$K = 0.$$

If they have a common factor, then

$$D\square - K^2 = PR - Q^2;$$

and the latter form expresses also the condition that the Jacobian shall be a perfect square, as was shown in the case of any two quadrics harmonically related, by Cayley in the Memoir above quoted.

Turning now to the quantics in ξ, η, we have the three quadrics, viz.,

$$V = (\alpha, \beta, \gamma)(\xi, \eta)^2,$$

$$V' = (\alpha', \beta', \gamma')(\xi, \eta)^2,$$

$$V'' = (\alpha y - \alpha' x, \beta y - \beta' x, \gamma y - \gamma' x)(\xi, \eta)^2,$$

and, since the invariant of these, viz.,

$$\begin{vmatrix} \alpha, & \alpha', & \alpha y - \alpha' x \\ \beta, & \beta', & \beta y - \beta' x \\ \gamma, & \gamma', & \gamma y - \gamma' x \end{vmatrix} = 0,$$

vanishes identically, it follows that there is a syzygetic relation between V, V', V''; and therefore each quadric may be considered as an intermediate of the other two. The three quadrics are then said to be in involution; that is, the six roots of the three quadrics are in involution.

If to these three quadrics we add the Jacobian of V, V, viz.,

$$\Omega = (C, \; -B, \; A \between \xi, \eta)^2,$$

we shall have four quadrics. Further, if the roots of these be represented by p, p'; q, q'; r, r'; s, s'; then the sets p, q, r, s; p', q', r', s' will be homographic, if

$$\begin{vmatrix} 1, & p, & p', & pp' \\ 1, & q, & q', & qq' \\ 1, & r, & r', & rr' \\ 1, & s, & s', & ss' \end{vmatrix} = 0.$$

which condition may be written also in the following form,

$$\begin{vmatrix} p-p', & 1, & p+p', & pp' \\ q-q', & 1, & q+q', & qq' \\ r-r', & 1, & r+r', & rr' \\ s-s', & 1, & s+s', & ss' \end{vmatrix} = 0.$$

But, multiplying the four lines of this determinant by $a, a', ay-a'x, C$, respectively, and substituting from the expressions for V, V', V'', Ω, the condition takes the form:

$$\begin{vmatrix} \sqrt{-\Delta}, & a, & \beta, & \gamma \\ \sqrt{-\Delta''}, & a', & \beta', & \gamma' \\ \sqrt{-U'}, & ay-a'x, & \beta y-\beta' x, & \gamma y-\gamma' x \\ \sqrt{-\Box}, & C, & -B, & A \end{vmatrix} = 0.$$

Then, writing for (line 3), $-y$ (line 1) $+x$ (line 2) + (line 3), the determinant may be reduced to the following form:

$$(-y\sqrt{-\Delta}+x\sqrt{-\Delta''}+\sqrt{-U'}) \begin{vmatrix} a, & \beta, & \gamma \\ a', & \beta', & \gamma \\ C, & -B, & A \end{vmatrix} = 0,$$

or $$(-y\sqrt{-\Delta}+x'\sqrt{-\Delta''}+\sqrt{-U'})\,\Box = 0.$$

If the first factor vanishes, we may transpose the last term, and then square both sides of the equation. We shall then obtain the following expression:

$$y^2\Delta - 2xy\sqrt{\Delta\Delta''} + x^2\Delta'' = y^2\Delta - 2xy\Delta' + x^2\Delta'',$$

whence also $$\Delta\Delta'' - \Delta'^2 = \tfrac{1}{4}\,\square = 0.$$

Therefore, the condition that the roots of the four quadrics may form a homographic system is $\square = 0$; that is, the quadrics V, V' must have a common factor.

To calculate the anharmonic ratio of V, V', V'', Ω. From the quadratic equations for p, q, r, s respectively, we have

$$\begin{aligned} p &= (-\beta + \sqrt{-\Delta}) &&: a, \\ q &= (-\beta' + \sqrt{-\Delta''}) &&: a', \\ r &= (-\beta y + \beta' x + \sqrt{-U'}) &&: (ay - a'x), \\ s &= \tfrac{1}{2}(B + \sqrt{-\square}) &&: C, \end{aligned}$$

and whence

$$\begin{aligned} q - r &= \{-\beta'(ay - a'x) + a'(\beta y - \beta' x) + (ay - a'x)\sqrt{-\Delta''} - a'\sqrt{-U'}\} \\ &\qquad\qquad : a'(ay - a'x) \\ &= \{-Cy + (ay - a'x)\sqrt{-\Delta''} - a'\sqrt{-U'}\} : a'(ay - a'x), \\ r - p &= \{-a(\beta y - \beta' x) + \beta(ay - a'x) - (ay - a'x)\sqrt{-\Delta} + a\sqrt{-U'}\} \\ &\qquad\qquad : a(ay - a'x) \\ &= \{\ \ Cx - (ay - a'x)\sqrt{-\Delta} + a\sqrt{-U'}\} : a(ay - a'x), \\ p - q &= \{C + a'\sqrt{-\Delta} - a\sqrt{-\Delta''}\} : aa', \\ p - s &= \{-2C\beta + 2C\sqrt{-\Delta} - aB - a\sqrt{-\square}\} : 2aC \\ &= \{2a\Delta' - 2a'\Delta + 2C\sqrt{-\Delta} - a\sqrt{-\square}\} : 2aC, \\ q - s &= \{-2C\beta' + 2C\sqrt{-\Delta''} - a'B - a'\sqrt{-\square}\} : 2a'C \\ &= \{2a\Delta'' - 2a'\Delta' + 2C\sqrt{-\Delta''} - a'\sqrt{-\square}\} : 2a'C, \\ r - s &= \{-2C(\beta y - \beta' x) + 2C\sqrt{-U'} - B(ay - a'x) - (ay - a'x)\sqrt{-\square}\} \\ &\qquad\qquad : 2C(ay - a'x) \\ &= \{2(a'\Delta' - a\Delta'')y + 2(a\Delta' - a'\Delta)x + 2C\sqrt{-U} - (ay - a'x)\sqrt{-\square}\} \\ &\qquad\qquad : 2C(ay - a'x); \end{aligned}$$

and the anharmonic ratio will be expressed by any of the following ratios:

$$(q - r)(p - s) : (r - p)(q - s) : (p - q)(r - s).$$

Each of these products has for its denominator the quantity

$$2aa'(ay-a'x),$$

which may therefore be discarded. Taking the first two of the above products, we shall have as the expression for the anharmonic ratio the following:

$$\{-Cy+(ay-a'x)\sqrt{-\Delta''}-a'\sqrt{-U'}\}$$
$$\times\{2a\Delta'-2a'\Delta+2C\sqrt{-\Delta}-a\sqrt{-\Box}\}$$
$$:\{Cx-(ay-a'x)\sqrt{-\Delta}+a\sqrt{-U'}\}$$
$$\times\{2a\Delta''-2a'\Delta'+2C\sqrt{-\Delta''}-a'\sqrt{-\Box}\}.$$

If the system be homographic, then, as was shown above, $\Box=0$, and,

$$\pm C \quad = a'\sqrt{-\Delta}-a\sqrt{-\Delta''},$$
$$\pm\sqrt{-U'}=y\sqrt{-\Delta}-x\sqrt{-\Delta''};$$

and, if we take C with the negative sign and $\sqrt{-U'}$ with the positive, we shall find that each of the factors of the anharmonic ratio will vanish, as they should, because the four quadrics have a common factor, *i.e.*, $p=q=r=s$.

If we give C the positive sign, retaining also the positive sign for $\sqrt{-U'}$, we shall find that

$$-Cy+(ay-a'x)\sqrt{-\Delta''}-a'\sqrt{-U'}=2y(a\sqrt{-\Delta''}-a'\sqrt{-\Delta}),$$
$$-Cx+(ay-a'x)\sqrt{-\Delta}\ -a\sqrt{-U'}=2x(a\sqrt{-\Delta''}-a'\sqrt{-\Delta}),$$
$$2(a\Delta'-a'\Delta+C\sqrt{-\Delta})-a\sqrt{-\Box}=4(a\Delta'-a'\Delta),$$
$$2(a\Delta''-a'\Delta'+C\sqrt{-\Delta''})-a'\sqrt{-\Box}=4(a\Delta''-a'\Delta').$$

Hence the anharmonic ratio of the roots not common to the four quadrics will be

$$=-y(a\Delta'-a'\Delta):x(a\Delta''-a'\Delta')$$
$$=y(\alpha',\beta',\gamma')(\beta,-\alpha)^2:x(\alpha,\beta,\gamma)(\beta',-\alpha')^2.$$

We now proceed to the transformed quantic itself. If we substitute in

$$U=(a,b,c)(x,y)^2,$$

by means of the transformant

$$x:y=(\alpha,\beta,\gamma)(\xi,\eta)^2:(\alpha',\beta',\gamma')(\xi,\eta)^2=V:V',$$

we shall obtain the result

$$W=(a,b,c)\{(\alpha,\beta,\gamma)(\xi,\eta)^2,\ (\alpha',\beta',\gamma')(\xi,\eta)^2\}^2,$$

and, if this be developed according to powers of ξ, η, we shall find

$$
\begin{aligned}
W = \;& (a, b, c)\,(a, a')^2 && \xi^4 \\
&+4\;(a, b, c)\,(a, a')\,(\beta, \beta') && \xi^3\eta \\
&+2\{(a, b, c)\,(a, a')\,(\gamma, \gamma')+2\,(a, b, c)\,(\beta, \beta')^2\} && \xi^2\eta^2 \\
&+4\;(a, b, c)\,(\beta, \beta')\,(\gamma, \gamma') && \xi\,\eta^3 \\
&+\;(a, b, c)\,(\gamma, \gamma')^2 && \eta^4.
\end{aligned}
$$

It is proposed to calculate the invariants and covariants of W, and to express them in terms of those of U, V, V'.

Writing $$W = (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e})\,(\xi, \eta)^4,$$

and putting

$$(a, b, c)\,(\beta, \beta')^2 = B, \quad (a, b)\,(\beta, \beta') = B_1, \quad (b, c)\,(\beta, \beta') = B_1',$$

$$a\Delta+2b\Delta'+c\Delta'' = K,$$

we have

$$
\begin{aligned}
\mathrm{a} &= (a, b, c)\,(a, a')^2, \\
\mathrm{b} &= B_1 a + B_1' a', \\
\mathrm{c} &= B+\tfrac{1}{3}K, \\
\mathrm{d} &= B_1\gamma + B_1'\gamma', \\
\mathrm{e} &= (a, b, c)\,(\gamma\gamma')^2.
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathrm{ae} &= a^2(\Delta+\beta^2)^2+4b^2(\Delta+\beta^2)(\Delta''+\beta'^2)+c^2(\Delta''+\beta'^2)^2 \\
&\quad +4ab\,(\Delta+\beta^2)(\Delta'+\beta\beta')+4bc\,(\Delta''+\beta'^2)(\Delta'+\beta\beta') \\
&\qquad +2ac\{(\Delta'+\beta\beta')^2-2\,(\Delta+\beta^2)\,(\Delta''+\beta'^2)\} \\
&= \{a\,(\Delta+\beta^2)+2\beta\,(\Delta'+\beta\beta')+c\,(\Delta''+\beta'^2)\}^2 \\
&\qquad -4\,(ac-b^2)\,\{(\Delta+\beta^2)(\Delta''+\beta'^2)-(\Delta'+\beta\beta')^2\} \\
&= (K+B)^2-4D\,(\Delta\Delta''-\Delta'^2+\Delta\beta'^2-2\Delta'\beta\beta'+\Delta''\beta^2); \\
-4\mathrm{bd} &= -4\,\{B_1^2\,(\Delta+\beta^2)+2B_1B_1'\,(\Delta'+\beta\beta')+B_1'^2\,(\Delta''+\beta'^2)\} \\
&= -4\{\Delta B_1^2+2\Delta' B_1B_1'+\Delta''B_1'^2+B^2\}; \\
3\mathrm{c}^2 &= 3B^2+2BK+\tfrac{1}{3}K^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathrm{ae}-4\mathrm{bd}+3\mathrm{c}^2 = \tfrac{4}{3}K^2+4BK-4D\,(\Delta\Delta''-\Delta'^2+\Delta\beta'^2-2\Delta'\beta\beta'+\Delta''\beta^2) \\
-4\,(\Delta B_1^2+2\Delta' B_1B_1'+\Delta''B_1'^2).
\end{aligned}
$$

But $\Delta B_1^2+2\Delta' B_1 B_1'+\Delta'' B_1'^2 = \Delta\,(a^2\beta^2+2ab\,\beta\beta'+b^2\beta'^2)$

$$+2\Delta'\,(ab\beta^2+(ac+b^2)\,\beta\beta'+bc\beta'^2)+\Delta''\,(b^2\beta^2+2bc\beta\beta'+c^2\beta'^2)$$

$$=(a\Delta+2b\Delta'+c\Delta'')(a\beta^2+2b\beta\beta'+c\beta'^2)-(ac-b^2)(\Delta''\beta^2-2\Delta'\beta\beta'+\Delta\beta'^2)$$

$$=BK-D\,(\Delta\beta'^2-2\Delta'\beta\beta'+\Delta''\beta^2).$$

Hence $\mathrm{ae}-4\mathrm{bd}+3\mathrm{c}^2=\tfrac{4}{3}K^2-D\Box.$

We next come to the quarticovariant, which may be thus expressed:

$$(\mathrm{ac}-\mathrm{b}^2),\ 2\,(\mathrm{ad}-\mathrm{bc}),\ \mathrm{ae}+2\mathrm{bd}-3\mathrm{c}^2,\ 2\,(\mathrm{be}-\mathrm{cd}),\ (\mathrm{ce}-\mathrm{d}^2\rangle\,(\xi,\eta)^4.$$

Now

$$\mathrm{ac}-\mathrm{b}^2=(a\alpha^2+2b\alpha\alpha'+c\alpha'^2)\,(B+\tfrac{1}{3}K)-(B_1^2\alpha^2+2B_1B_1'\alpha\alpha'+B_1'^2\alpha'^2)$$

$$=D\,(\alpha\beta'-\alpha'\beta)^2+\tfrac{1}{3}K\mathrm{a}.$$

Next ad $-$ bc; this is

$$=(a\alpha^2+2b\alpha\alpha'+c\alpha'^2)(B_1\gamma+B_1'\gamma')-(B_1\alpha+B_1'\alpha')\,(B+\tfrac{1}{3}K)$$

$$=aB_1\,(\Delta+\beta^2)\,\alpha+aB_1'\,(2\Delta'+2\beta\beta'-\alpha'\gamma)\,\alpha$$

$$+2bB_1\,(\Delta+\beta^2)\,\alpha'+2bB_1'\,(\Delta''+\beta'^2)\,\alpha$$

$$+\ cB_1\,(2\Delta'+2\beta\beta'-\alpha\gamma')\,\alpha'+cB_1'\,(\Delta''+\beta'^2)\,\alpha'$$

$$-\ \ B_1\,(B+\tfrac{1}{3}K)\,\alpha-B_1'\,(B+\tfrac{1}{3}K)\,\alpha'$$

$$=aB_1\,(\Delta+\beta^2)\,\alpha+2aB_1'\,(\Delta'+\beta\beta')\,\alpha-aB_1'\,(\Delta+\beta^2)\,\alpha'$$

$$+2bB_1\,(\Delta+\beta^2)\,\alpha'+2bB_1'\,(\Delta''+\beta'^2)\,\alpha$$

$$+2cB_1\,(\Delta'+\beta\beta')\,\alpha'+cB_1'\,(\Delta''+\beta'^2)\,\alpha'-cB_1\,(\Delta''+\beta'^2)\,\alpha$$

$$-B_1\,(B+\tfrac{1}{3}K)\,\alpha-B_1'\,(B+\tfrac{1}{3}K)\,\alpha'$$

$$=\{aB_1\,(\Delta+\beta^2)+2aB_1'\,(\Delta'+\beta\beta')+2bB_1'\,(\Delta''+\beta'^2)$$

$$-cB_1\,(\Delta''+\beta'^2)-B_1\,(B+\tfrac{1}{3}K)\}\,\alpha$$

$$+\{-aB_1'\,(\Delta+\beta^2)+2bB_1\,(\Delta+\beta^2)+2cB_1\,(\Delta'+\beta\beta')$$

$$+cB_1'\,(\Delta''+\beta'^2)-B_1'\,(B+\tfrac{1}{3}K)\}\,\alpha'.$$

But $2bB_1'-cB_1=bB_1'-D\beta,$

$$2bB_1-aB_1'=bB_1-D\beta'.$$

Hence $aB_1\beta^2+2aB_1'\beta\beta'+2bB_1'\beta'^2-cB_1\beta'^2-B_1B$

$$=a\,(B_1\beta+B_1'\beta')\,\beta+B_1B_1'\beta'-D\beta\beta'^2-B_1B$$

$$=a\beta B+b\beta'B-B_1B=0.$$

Hence the expression in question

$$= \{aB_1\Delta + 2aB_1'\Delta' + bB_1'\Delta'' - D\beta\Delta'' - \tfrac{1}{3}KB_1\}\,\alpha$$
$$+ \{cB_1'\Delta'' + 2cB_1\Delta' + bB_1\Delta - D\beta'\Delta - \tfrac{1}{3}KB_1'\}\,\alpha'$$
$$= \{(a^2\Delta + 2ab\Delta' + b^2\Delta'' - D\Delta'' - \tfrac{1}{3}Ka)\,\beta$$
$$+ (ab\Delta + 2ac\Delta' + bc\Delta'' - \tfrac{1}{3}Kb)\,\beta'\}\,\alpha$$
$$+ \{(bc\Delta'' + 2ac\Delta' + ab\Delta - \tfrac{1}{3}Kb)\,\beta$$
$$+ (c^2\Delta'' + 2bc\Delta' + b^2\Delta - D\Delta - \tfrac{1}{3}Kc)\,\beta'\}\,\alpha'$$
$$= 2\,\{(\tfrac{1}{3}aK - D\Delta'')\,\beta + (\tfrac{1}{3}bK + D\Delta')\,\beta'\}\,\alpha$$
$$+ 2\,\{\tfrac{1}{3}bK + D\Delta')\,\beta + (\tfrac{1}{3}cK - D\Delta)\,\beta\,\}\,\alpha'$$
$$= 2(\tfrac{1}{3}aK - D\Delta'',\ \tfrac{1}{3}bK + D\Delta',\ \tfrac{1}{3}cK - D\Delta)\,(\alpha, \alpha')\,(\beta, \beta')$$
$$= -2D\,\{\Delta\alpha'\beta' - \Delta\,(\alpha\beta' + \alpha'\beta) + \Delta''\alpha\beta\} + \tfrac{2}{3}K\mathrm{b},$$

or, finally, $\mathrm{ad} - \mathrm{bc} = -2D\,(\alpha\beta' - \alpha'\beta)\,(\gamma\alpha' - \gamma'\alpha) + \tfrac{2}{3}Kb.$

Next $\mathrm{ae} + 2\mathrm{bd} - 3\mathrm{c}^2$;

here

$$\mathrm{ae} = (B+K)^2 - D\Box - 4D\,(\Delta\beta'^2 - 2\Delta'\beta\beta' + \Delta''\beta^2),$$
$$2\mathrm{bd} = 2BK + 2B^2 - 2D\,(\Delta\beta'^2 - 2\Delta'\beta\beta' + \Delta''\beta^2),$$
$$-3\mathrm{c}^2 = -3B^2 - 2BK - \tfrac{1}{3}K^2.$$

Hence

$$\mathrm{ae} + 2\mathrm{bd} - 3\mathrm{c}^2 = \tfrac{2}{3}K^2 + 2BK - D\Box - 6D\,(\Delta\beta'^2 - 2\Delta'\beta\beta' + \Delta''\beta^2)$$
$$= -4D\,(\Delta\Delta'' - \Delta'^2 + \Delta\beta'^2 - 2\Delta'\beta\beta' + \Delta''\beta^2) + 2cK$$
$$= D\{(\gamma\alpha' - \gamma'\alpha)^2 + 2(\alpha\beta' - \alpha'\beta)(\beta\gamma' - \beta'\gamma)\} + \tfrac{1}{3}6cK.$$

The remaining terms $\mathrm{be} - \mathrm{cd}$, $\mathrm{ce} - \mathrm{d}^2$ are of course obtained by a mere interchange of letters; and, substituting the foregoing values, we find

$$\text{Quarticovariant} = D\begin{vmatrix} \eta^2, & -\eta\xi, & \xi^2 \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}^2 + \tfrac{1}{3}KW$$
$$= D\Omega^2 + \tfrac{1}{3}KW,$$

where, as above, Ω is the Jacobian of V, V'.

The next derivative which we have to calculate is the cubinvariant,

$$\mathrm{ace} - \mathrm{ad}^2 - \mathrm{b}^2\mathrm{e} + 2\mathrm{bcd} - \mathrm{c}^3.$$

This, multiplied by 3, may also be written thus:

$$\mathrm{e}\,(\mathrm{ac} - \mathrm{b}^2) - 2\mathrm{d}\,(\mathrm{ad} - \mathrm{bc}) + \mathrm{c}\,(\mathrm{ae} + 2\mathrm{bd} - 3\mathrm{c}^2) - 2\mathrm{b}\,(\mathrm{be} - \mathrm{cd}) + \mathrm{a}\,(\mathrm{ce} - \mathrm{d}^2),$$

and, substituting herein the values found above, this expression will become

$$
\begin{aligned}
= \quad & \mathrm{e}\,(ac-b^2)\,(\alpha\beta'-\alpha'\beta)^2 && +\tfrac{1}{3}\mathrm{ae}K \\
& +2\mathrm{d}\,(ac-b^2)\,(\alpha\beta'-\alpha'\beta)\,(\gamma\alpha'-\gamma'\alpha) && -\tfrac{4}{3}\mathrm{bd}K \\
& +\ \mathrm{c}\,(ac-b^2)\,\{(\gamma\alpha'-\gamma'\alpha)^2+2\,(\alpha\beta'-\alpha'\beta)\,(\beta\gamma'-\beta'\gamma)\}+2\mathrm{c}^2K \\
& +2\mathrm{b}\,(ac-b^2)\,(\beta\gamma'-\beta'\gamma)\,(\gamma\alpha'-\gamma'\alpha) && -\tfrac{4}{3}\mathrm{bd}K \\
& +\ \mathrm{a}\,(ac-b^2)\,(\beta\gamma'-\beta'\gamma)^2 && +\tfrac{1}{3}\mathrm{ae}K \\
= {} & D\,(\mathrm{a, c, e, d, c, b})\,(A,\,B,\,C)^2+\tfrac{2}{3}K\,(\tfrac{4}{3}K^2-D\square).
\end{aligned}
$$

But
$$(\mathrm{a, c, e, d, c, b})\,(A,\,B,\,C)^2$$
$$= (\mathrm{a}A+\mathrm{b}B+\mathrm{c}C)\,A+(\mathrm{b}A+\mathrm{c}B+\mathrm{d}C)\,B+(\mathrm{c}A+\mathrm{d}B+\mathrm{e}C)\,C,$$

and
$$
\begin{aligned}
\mathrm{a}A+\mathrm{b}B+\mathrm{c}C = {} & (a,\,b,\,c)\,(\alpha,\,\alpha')^2 \quad A \\
& +(a,\,b,\,c)\,(\alpha,\,\alpha')\,(\beta,\,\beta')\,B \\
& +(a,\,b,\,c)\,(\alpha,\,\alpha')\,(\gamma,\,\gamma')\,C \\
& -\tfrac{2}{3}\,\{(a,\,b,\,c)(\alpha,\,\alpha')(\gamma,\gamma')-(a,\,b,\,c)(\beta,\,\beta')^2\}\,C \\
= {} & -\tfrac{2}{3}KC\,;
\end{aligned}
$$

and, following the same process with the other terms, we should find that the whole expression

$$= -\tfrac{1}{3}K\,(4AC-B^2) = -\tfrac{1}{3}K\square.$$

Hence, finally, the cubinvariant in question

$$= \tfrac{1}{27}K\,(8K^2-3D\square). \qquad *$$

The cubicovariant may be written thus:

$$
\begin{aligned}
& \{\mathrm{a}\,(\mathrm{ad}-\mathrm{be})-2\mathrm{b}\,(\mathrm{ac}-\mathrm{b}^2)\} && \xi^6 \\
+ & \{\mathrm{a}\,(\mathrm{ae}+2\mathrm{bd}-3\mathrm{c}^2)-6\mathrm{c}\,(\mathrm{ac}-\mathrm{b}^2)\} && \xi^5\eta \\
+ & \{\mathrm{a}\,(\mathrm{be}-\mathrm{cd})-2\mathrm{d}\,(\mathrm{ac}-\mathrm{b}^2)\} && \xi^4\eta^2 \\
+ & \{-\mathrm{d}\,(\mathrm{ad}-\mathrm{be})+\mathrm{b}\,(\mathrm{be}-\mathrm{cd})\} && \xi^3\eta^3 \\
+ & \{-\mathrm{e}\,(\mathrm{ad}-\mathrm{be})+2\mathrm{b}\,(\mathrm{ce}-\mathrm{d}^2\} && \xi^2\eta^4 \\
+ & \{-\mathrm{e}\,(\mathrm{ae}+2\mathrm{bd}-3\mathrm{c}^2)+6\mathrm{c}\,(\mathrm{ce}-\mathrm{d}^2)\} && \xi\,\eta^5 \\
+ & \{-\mathrm{e}\,(\mathrm{be}-\mathrm{cd})+2\mathrm{d}\,(\mathrm{ce}-\mathrm{d}^2)\} && \eta
\end{aligned}
$$

Now, referring to former values;

$$\mathrm{a}\,(\mathrm{ad}-\mathrm{be})-2\mathrm{b}\,(\mathrm{ac}-\mathrm{b}^2)$$
$$=-\mathrm{a}DBC+\tfrac{2}{3}\mathrm{ab}K-2\mathrm{b}DC^2-\tfrac{2}{3}\mathrm{ab}K$$
$$=-DC\,(\mathrm{a}B+2\mathrm{b}C),$$

$$\mathrm{a}\,(\mathrm{ae}+2\mathrm{bd}-3\mathrm{c}^2)-6\mathrm{c}\,(\mathrm{ac}-\mathrm{b}^2)$$
$$=\mathrm{a}D\,(B^2+2AC)+2\mathrm{ac}K-6\mathrm{c}DC^2-2\mathrm{ac}K$$
$$=D\,\{-\mathrm{a}\,(4AC-B^2)+6C\,(\mathrm{a}A-\mathrm{c}C)\}$$
$$=6DC\,(\mathrm{a}A-\mathrm{c}C)-\mathrm{a}D\square,$$

$$\mathrm{a}\,(\mathrm{be}-\mathrm{cd})-2\mathrm{d}\,(\mathrm{ac}-\mathrm{b}^2)$$
$$=-\mathrm{a}DAB+\tfrac{2}{3}\mathrm{ad}K-2\mathrm{d}DC^2-\tfrac{2}{3}\mathrm{ad}K$$
$$=-D\,(\mathrm{a}AB+2\mathrm{d}C^2),$$

$$-\mathrm{d}\,(\mathrm{ad}-\mathrm{be})+\mathrm{b}\,(\mathrm{be}-\mathrm{cd})$$
$$=D\mathrm{d}BC-\tfrac{2}{3}\mathrm{bd}K-D\mathrm{b}AB+\tfrac{2}{3}\mathrm{bd}K$$
$$=DB\,(\mathrm{d}C-\mathrm{b}A),$$

$$-\mathrm{e}\,(\mathrm{ad}-\mathrm{be})+2\mathrm{b}\,(\mathrm{ce}-\mathrm{d}^2)$$
$$=D\mathrm{e}BC-\tfrac{2}{3}\mathrm{be}K+2D\,\mathrm{b}A^2+\tfrac{2}{3}\mathrm{be}K$$
$$=D\,(\mathrm{e}BC+2\mathrm{b}A^2),$$

$$-\mathrm{e}\,(\mathrm{ae}+2\mathrm{bd}-3\mathrm{c}^2)+6\mathrm{c}\,(\mathrm{ce}-\mathrm{d}^2)$$
$$=-D\mathrm{e}\,(B^2+2AC)-2\mathrm{ce}K+6\mathrm{c}DA^2+2\mathrm{ce}K$$
$$=-6DA\,(\mathrm{e}C-\mathrm{c}A)+\mathrm{e}D\square,$$

$$-\mathrm{e}\,(\mathrm{be}-\mathrm{cd})+2\mathrm{d}\,(\mathrm{ce}-\mathrm{d}^2)$$
$$=\mathrm{e}DAB-\tfrac{2}{3}\mathrm{ce}K+2\mathrm{d}DA^2+\tfrac{2}{3}\mathrm{ce}K$$
$$=DA\,(\mathrm{e}B+2\mathrm{d}A).$$

But it will further be found that

$$2\mathrm{b}C+\mathrm{a}B=(P,\,Q,\,R)\,(\alpha,\,\alpha')^2,$$
$$\mathrm{c}C-\mathrm{a}A=(P,\,Q,\,R)\,(\alpha,\,\alpha')\,(\beta,\,\beta')+\tfrac{1}{3}CK,$$
$$2\mathrm{d}C^2+\mathrm{a}AB=C\,(P,\,Q,\,R)\,(\beta,\,\beta')^2-B\,(P,\,Q,\,R)\,(\alpha,\,\alpha')\,(\beta,\,\beta'),$$
$$\mathrm{b}A-\mathrm{d}C=-(P,\,Q,\,R)\,(\beta,\,\beta')^2,$$
$$2\mathrm{b}A^2+\mathrm{e}BC=-A\,(P,\,Q,\,R)(\beta,\,\beta')^2+B\,(P,\,Q,\,R)\,(\beta,\,\beta')\,(\gamma,\,\gamma'),$$
$$\mathrm{c}A-\mathrm{e}C=-(P,\,Q,\,R)\,(\beta,\,\beta')\,(\gamma,\,\gamma')+\tfrac{1}{3}AK,$$
$$2\mathrm{d}A+\mathrm{e}B=-(P,\,Q,\,R)\,(\gamma,\,\gamma')^2.$$

Hence the cubicovariant $= -D\times$

$$\begin{array}{ll}
C(P,Q,R)(\alpha,\alpha')^2 & \xi^6 \\
+\ 6C(P,Q,R)(\alpha,\alpha')(\beta,\beta')+2C^2K+\mathrm{a}\square & \xi^5\eta \\
+\ 5C(P,Q,R)(\beta,\beta')^2-5B(P,Q,R)(\alpha,\alpha')(\beta,\beta') & \xi^4\eta^2 \\
-10B(P,Q,R)(\beta,\beta')^2 & \xi^3\eta^3 \\
+\ 5A(P,Q,R)(\beta,\beta')^2\xi^2\eta^4-5B(P,Q,R)(\beta,\beta')(\gamma,\gamma') & \xi^2\eta^4 \\
+\ 6A(P,Q,R)(\beta,\beta')(\gamma,\gamma')-2A^2K-\mathrm{e}\square & \xi\,\eta^5 \\
+\ \ A(P,Q,R)(\gamma,\gamma')^2 & \eta^6.
\end{array}$$

The expression contains terms divisible by $A\xi^2$, $B\xi\eta$, $C\eta^2$ respectively, and also the terms

$$(2C^2K+\mathrm{a}\square)\,\xi^5\eta-(2A^2K+\mathrm{e}\square)\,\xi\eta^5.$$

Now, since

$$(P,Q,R)(\alpha,\alpha')(\gamma,\gamma')-(P,Q,R)(\beta,\beta')^2=0,$$

it follows that the coefficient of $C\xi^2$ in the covariant may be written thus:

$$\begin{aligned}
&\{(P,Q,R)(\alpha,\alpha')^2\xi^2+2(P,Q,R)(\alpha,\alpha')(\beta,\beta')\xi\eta \\
&\qquad\qquad +(P,Q,R)(\alpha,\alpha')(\gamma,\gamma')\eta^2\}\,\xi^2 \\
+2&\{(P,Q,R)(\beta,\beta')(\alpha,\alpha')\xi^2+2(P,Q,R)(\beta,\beta')\xi\eta \\
&\qquad\qquad +(P,Q,R)(\beta,\beta')(\gamma,\gamma')\eta^2\}\,\xi\eta \\
+\ &\{(P,Q,R)(\gamma,\gamma')(\alpha,\alpha')\xi^2+2(P,Q,R)(\gamma,\gamma')(\beta,\beta')\xi\eta \\
&\qquad\qquad +(P,Q,R)(\gamma,\gamma')^2\eta^2\}\,\eta^2 \\
+2\ &(P,Q,R)(\alpha,\alpha')(\beta,\beta')\xi^3\eta-(P,Q,R)(\alpha,\alpha')(\gamma,\gamma')\xi^2\eta^2 \\
&\qquad -4(P,Q,R)(\beta,\beta')(\gamma,\gamma')\xi\eta^3-(P,Q,R)(\gamma,\gamma')^2\eta^4 \\
&=(P,Q,R)(V,V')^2-(P,Q,R)(\gamma,\gamma')(V,V')\eta^2 \\
&\quad +2(P,Q,R)(\alpha,\alpha')(\beta,\beta')\xi^3\eta-2(P,Q,R)(\beta,\beta')(\gamma,\gamma')\xi\eta^3.
\end{aligned}$$

Similarly the coefficient of $B\xi\eta$ is

$$\begin{aligned}
&=-(P,Q,R)(V,V')^2-(P,Q,R)(\beta,\beta')(V,V')\xi\eta \\
&+(P,Q,R)(\alpha,\alpha')^2\xi^4-2(P,Q,R)(\alpha,\alpha')(\gamma,\gamma')\xi^2\eta^2+(P,Q,R)(\gamma,\gamma')^2\eta^4,
\end{aligned}$$

and that of $A\eta^2$ is

$$\begin{aligned}
&=(P,Q,R)(V,V')^2-(P,Q,R)(\alpha,\alpha')(V,V')\xi^2 \\
&\quad +2(P,Q,R)(\gamma,\gamma')(\beta,\beta')\xi\eta^3-2(P,Q,R)(\alpha,\alpha')(\beta,\beta')\xi^3\eta.
\end{aligned}$$

Multiplying each of these results by their respective factors and adding the products together, the first terms give

$$\Omega(P,Q,R)(V,V')^2;$$

the second terms give

$$(P,\ Q,\ R)\ (V,\ V')\ (A\alpha+B\beta+C\gamma,\ A\alpha'+B\beta'+C\gamma')=0\,;$$

the third terms give

$$(P,\ Q,\ R)\ (\beta,\ \beta')\ (A\alpha+B\beta+C\gamma,\ A\alpha'+B\beta'+C\gamma')=0\,;$$

and there remain the terms

$$\{2C\,(P,\ Q,\ R)\,(\alpha,\ \alpha')\,(\beta,\ \beta')+B\,(P,\ Q,\ R)\,(\alpha,\ \alpha')^2+2C^2K+\mathfrak{a}\Box\}\,\xi^5\eta$$
$$+\{2A\,(P,\ Q,\ R)\,(\beta,\ \beta')\,(\gamma,\ \gamma')+B\,(P,\ Q,\ R)\,(\gamma,\ \gamma')^2-2A^2K-\mathfrak{e}\Box\}\,\xi\eta^5.$$

Now $2C\,(P,\ Q,\ R)\,(\alpha,\ \alpha')\,(\beta,\ \beta')+B\,(P,\ Q,\ R)\,(\alpha,\ \alpha')^2+\mathfrak{a}\Box$

$$=(P,\ Q,\ R)\,(\alpha,\ \alpha')\,(2C\beta+B\alpha,\ 2C\beta'+B\alpha')+\mathfrak{a}\Box$$
$$=2\,(P,\ Q,\ R)\,(\alpha,\ \alpha')\,(\alpha'\Delta-\alpha\Delta',\ \alpha'\Delta'-\alpha\Delta'')+\mathfrak{a}\Box$$
$$=2\ \times\ c\ .\ \ \Delta\ .\ \alpha\alpha'\Delta-\ \alpha^2\Delta'+4\mathfrak{a}\,(\Delta\Delta''-\Delta'^2)$$
$$-2b\ .\ 2\Delta'\ .\ \alpha'^2\Delta-\ \alpha^2\Delta''$$
$$+\ a\ .\ \Delta''\ .\ \alpha'^2\Delta'-\alpha\alpha'\Delta'',$$

which, on being developed, will be found to be $=-2KC^2$. Hence the terms in $\xi^5\eta$ and $\xi\eta^5$ will vanish, and the cubicovariant in question finally found to be $\quad=-D\Omega\,(P,\ Q,\ R)(V,\ V')^2$.

Recapitulating, we have, for the derivatives of W

the Quadrinvariant, $\qquad I=\frac{4}{3}K^2-D\Box,$

the Quarticovariant, $\qquad H=D\Omega+\frac{1}{3}KW,$

the Cubinvariant, $\qquad J=\frac{1}{27}K\,(8K^2-3D\Box),$

the Cubicovariant, $\qquad \Phi=-\,D\Omega\,(P,Q,R)\,(V,V')^2\,;$

whence also for the Discriminant, we have

$$I^3-27J^2=\tfrac{1}{9}D\Box\ (-32K^4+33K^2D\Box-9D^2\Box^2).$$

If $(P,\ Q,\ R)\,(V,\ V')^2$ be regarded as a quartic in $\xi,\ \eta$, then its Quadrinvariant $=\frac{4}{3}\,(P\Delta+2Q\Delta'+R\Delta'')-\Box\,(PR-Q^2)$, which, since $P\Delta+2Q\Delta'+R\Delta''=0$, is

$$=-\,\Box\ (D\Box-K^2)\,;$$

Quarticovariant $=\quad \Omega\,(D\Box-K^2),$

Cubinvariant $\quad=\quad 0;$

Cubicovariant $\quad=-\,\Omega\,(D\Box-K^2)\,(P_1,\ Q_1,\ R_1)\,(V,\ V')^2,$

where P_1, Q_1, R_1 are quantities formed with P, Q, R, in the same way as were P, Q, R with a, b, c; viz.,

$$P_1, Q_1, R_1 = \left\| \begin{matrix} R, & -2Q, & P \\ \Delta, & 2\Delta', & \Delta'' \end{matrix} \right\|;$$

and, on developing these expressions, it will be found that

$$P_1 = 2\{-\ a(\Delta\Delta''-\Delta'^2)+(a\Delta'^2+2b\Delta'\Delta''+c\Delta''^2)\},$$
$$Q_1 = 2\{-2b(\Delta\Delta''-\Delta'^2)-(a\Delta\Delta'+2b\Delta'^2+c\Delta'\Delta'')\},$$
$$R_1 = 2\{-\ c(\Delta\Delta''-\Delta'^2)+(a\Delta^2+2b\Delta\Delta'+c\Delta'^2)\};$$

but
$$a\Delta'^2+2b\Delta'\Delta''+c\Delta''^2 = K\Delta''-\tfrac{1}{4}a\square,$$
$$a\Delta\Delta'+2b\Delta'^2+c\Delta'\Delta'' = K\Delta',$$
$$a\Delta^2+2b\Delta\Delta'+c\Delta'^2 = K\Delta-\tfrac{1}{4}c\square;$$

hence
$$P_1 = 2K\Delta''-a\square,$$
$$Q_1 = -2K\Delta'-b\square,$$
$$R_1 = 2K\Delta-c\square.$$

But, on referring to the expression for W given on p. 155, and replacing a, b, c by Δ'', $-\Delta'$, Δ, respectively, it will be found that the

coefficient of
$$\xi^4 = -\ C^2,$$
$$\xi^3\eta = 2BC,$$
$$\xi^2\eta^2 = -(2AC+B^2),$$
$$\xi\eta^3 = 2AB,$$
$$\eta^4 = -\ A^2,$$

so that the whole coefficient of K is $= -(C\xi^2-B\xi\eta+A\eta^2)^2$. Hence

$$(P_1, Q_1, R_1)(V, V')^2 = -K\Omega-2\square\,(P, Q, R)(V, V')^2,$$

and the cubicovariant in question

$$= \Omega\,(D\square-K^2)\{K\Omega+2\square\,(P, Q, R)(V, V')^2\}.$$

If the quartic W has a pair of equal roots, then

$I^3-27J^2=0$, and either $\square=0$, or $(32, -33, 9)(K^2, D\square)^2=0$.

The condition $\square=0$ must, however, be excluded, since it implies that V, V' have a common factor; in which case the transformation ceases to be quadratic. For this factor may be divided out from the transformant, and the transformation then becomes linear.

If it has two pairs of equal roots, then

$$\Phi = 0, \text{ and either } \Omega = 0, \text{ or } (P, Q, R)(V, V')^2 = 0.$$

If it has three equal roots,

$$I = 0, \text{ and } J = 0, \text{ and then } K = 0, \ \square = 0,$$

viz., we have here the excluded condition.

If all the roots are equal, then

$$H = 0, \text{ and } \Omega = 0, \ K = 0.$$

As regards the quartic $(P, Q, R)(V, V')^2$, if $D\square - K^2 = 0$, *i.e.*, if the two quadrics U, U' have a common factor, then all the invariants and covariants vanish, and all the roots of the quartic in question become equal; viz.,

$$(P, Q, R)(V, V')^2 = (V\surd P + V'\surd R)^2$$
$$= \{(\alpha\surd P + \alpha'\surd R, \beta\surd P + \beta'\surd R, \gamma\surd P + \gamma'\surd R)(\xi, \eta)^2\}^2.$$

But the discriminant of the last form

$$= P\Delta + 2Q\Delta' + R\Delta'' = 0,$$

or we have $V\surd P + V'\surd R$ a perfect square, and hence the quantic

$$= \{\surd(\alpha\surd P + \alpha'\surd R)\,\xi + \surd(\gamma\surd P + \gamma'\surd R)\,\eta\}^4.$$

We now proceed to the cubic

$$U = (a, b, c, d)(x, y)^3,$$

which, if transformed by the same formula as was the quadratic, gives

$$W = (a, b, c, d)(\alpha\xi^2 + 2\beta\xi\eta + \gamma\eta^2, \alpha'\xi^2 + 2\beta'\xi\eta + \gamma'\eta^2)^3$$

and if we represent this by

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g})(\xi, \eta)^6,$$

we shall have

$$\mathfrak{a} = (a,\ldots)(\alpha, \alpha')^3,$$
$$\mathfrak{b} = (a,\ldots)(\alpha, \alpha')^2(\beta, \beta'),$$
$$\mathfrak{c} = (a,\ldots)(\alpha,\alpha')(\beta,\beta')^2 + \tfrac{1}{3}\{(a,\ldots)(\alpha,\alpha')^2(\gamma,\gamma') - (a,\ldots)(\alpha,\alpha')(\beta,\beta')^2\},$$
$$\mathfrak{d} = (a,\ldots)(\beta, \beta')^3 + \tfrac{3}{5}\{(a,\ldots)(\alpha, \alpha')(\beta, \beta')(\gamma, \gamma') - (a,\ldots)(\beta, \beta')^3\},$$
$$\mathfrak{e} = (a,\ldots)(\gamma,\gamma')(\beta,\beta')^2 + \tfrac{1}{3}\{(a,\ldots)(\alpha,\alpha')(\gamma,\gamma')^2 - (a,\ldots)(\beta,\beta')^2(\gamma,\gamma')\},$$
$$\mathfrak{f} = (a,\ldots)(\gamma, \gamma')^2(\beta, \beta'),$$
$$\mathfrak{g} = (a,\ldots)(\gamma, \gamma')^3.$$

Further, if we put

$$\mathfrak{h} = (a, b, c, d)(\beta, \beta')^3,$$
$$\mathfrak{h}_1 = (a, b, c)(\beta, \beta')^2, \quad \mathfrak{h}_1' = (b, c, d)(\beta, \beta')^2,$$
$$\mathfrak{h}_2 = (a, b)(\beta, \beta'), \quad \mathfrak{h}_2' = (b, c)(\beta, \beta'), \quad \mathfrak{h}_2'' = (c, d)(\beta, \beta'),$$
$$a\Delta + 2b\Delta' + c\Delta'' = K_1,$$
$$b\Delta + 2c\Delta' + d\Delta'' = K_1',$$
$$(ac-b^2)\Delta + (ad-bc)\Delta' + (bd-c^2)\Delta'' = K = L\Delta + M\Delta' + N\Delta'';$$

then
$$\mathfrak{a} = (a, b, c, d)(\alpha, \alpha')^3,$$
$$\mathfrak{b} = (\mathfrak{h}_2, \mathfrak{h}_2', \mathfrak{h}_2'')(\alpha, \alpha')^2,$$
$$\mathfrak{c} = (\mathfrak{h}_1 + \tfrac{1}{3}K_1, \mathfrak{h}_1' + \tfrac{1}{3}K_1')(\alpha, \alpha'),$$
$$\mathfrak{d} = \mathfrak{h} + \tfrac{3}{5}(K_1\beta + K_1'\beta'),$$
$$\mathfrak{e} = (\mathfrak{h}_1 + \tfrac{1}{3}K_1, \mathfrak{h}_1' + \tfrac{1}{3}K_1')(\gamma, \gamma'),$$
$$\mathfrak{f} = (\mathfrak{h}_2, \mathfrak{h}_2', \mathfrak{h}_2'')(\gamma, \gamma')^2,$$
$$\mathfrak{g} = (a, b, c, d)(\gamma, \gamma')^3.$$

The quadrinvariant

$$\mathfrak{ag} - 6\mathfrak{bf} + 15\mathfrak{ce} - 10\mathfrak{d}^2,$$

of this function is found to be

$$= \tfrac{8}{5}\{\Delta K_1^2 + 2\Delta' K_1 K_1' + \Delta'' K_1'^2 - 5K(\Delta\Delta'' - \Delta'^2)\}.$$

Given the Quantic

$$U = (a, b, c, f, g, h)\ (x, y, z)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1$$

and $$V = (a, \beta, \gamma, \lambda, \mu, \nu)(\xi, \eta, \zeta)^2,$$

and like values of V', V'', or, more briefly, say

$$V = (a, \ldots)(\xi, \eta, \zeta)^2,\ V' = (a', \ldots)(\xi, \eta, \zeta)^2,\ V'' = (a'', \ldots)(\xi, \eta, \zeta)^2 \ldots(2),$$

then U may be transformed by the equations

$$x : y : z = V : V' : V'' \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The result is $$W = (a, \ldots)(V, V', V'')^2 \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and, if we write for brevity

$$\left.\begin{array}{c} (a)^2 = (a, \ldots)(a, a', a'')^2 \\ (a)(\beta) = (a, \ldots)(a, a', a'')(\beta, \beta', \beta'') \\ \ldots \quad \ldots \quad \ldots \end{array}\right\} \ldots\ldots\ldots(5),$$

then

$$\left.\begin{array}{rl} W = & (a)^2\xi^4 + (\beta)^2\eta^4 + (\gamma)^2\zeta^4 \\ & +4(\beta)(\lambda)\ \eta^3\zeta + 4(\gamma)(\lambda)\ \eta\zeta^3 \\ & +4(\gamma)(\mu)\ \zeta^3\xi + 4(a)(\mu)\ \zeta\xi^3 \\ & +4(a)(\nu)\ \xi^3\eta + 4(\beta)(\nu)\ \xi\eta^3 \\ & +2[(\beta)(\gamma) + 2(\lambda)^2]\ \eta^2\zeta^2 \\ & +2[(\gamma)(a) + 2(\mu)^2]\ \zeta^2\xi^2 \\ & +2[(a)(\beta) + 2(\nu)^2]\ \xi^2\eta^2 \\ & +4[(a)(\lambda) + 2(\mu)(\nu)]\ \xi^2\eta\zeta \\ & +4[(\beta)(\mu) + 2(\nu)(\lambda)]\ \xi\eta^2\zeta \\ & +4[(\gamma)(\nu) + 2(\lambda)(\mu)]\ \xi\eta\zeta^2 \end{array}\right\} \ldots\ldots\ldots\ldots(6),$$

It is proposed to express the invariants, &c. of W in terms of those of U, V, V', V''. If we write the above value of W in the following form $$W = (\mathrm{a, b, c, f, g, h, p, p', q, q', r, r', l, m, n})(\xi, \eta, \zeta)^4,$$

we shall have

$$\mathrm{a} = (a)^2,\quad \mathrm{b} = (\beta)^2,\quad \mathrm{c} = (\gamma)^2,$$
$$3\mathrm{f} = (\beta)(\gamma) + 2(\lambda)^2,$$
$$3\mathrm{g} = (\gamma)(a) + 2(\mu)^2,$$
$$3\mathrm{h} = (a)(\beta) + 2(\nu)^2,$$
$$\mathrm{p} = (\beta)(\lambda), \qquad \mathrm{p}' = (\gamma)(\lambda),$$
$$\mathrm{q} = (\gamma)(\mu), \qquad \mathrm{q}' = (a)(\mu),$$
$$\mathrm{r} = (a)(\nu), \qquad \mathrm{r}' = (\beta)(\nu),$$

$$3\mathrm{l} = (\alpha)(\lambda)+2(\mu)(\nu),$$
$$3\mathrm{m} = (\beta)(\mu)+2(\nu)(\lambda),$$
$$3\mathrm{n} = (\gamma)(\nu)+2(\lambda)(\mu),$$

and the quantities which we have first to calculate are

$$\mathbb{A} = \mathrm{bc}+3\mathrm{f}^2-4\mathrm{pp}',$$
$$\mathbb{B} = \mathrm{ca}+3\mathrm{g}^2-4\mathrm{qq}',$$
$$\mathbb{C} = \mathrm{ab}+3\mathrm{h}^2-4\mathrm{rr}',$$
$$\mathbb{F} = \mathrm{af}+\mathrm{gh}+2\mathrm{l}^2-2\mathrm{rn}-2\mathrm{q}'\mathrm{m},$$
$$\mathbb{G} = \mathrm{bg}+\mathrm{hf}+2\mathrm{m}^2-2\mathrm{pl}-2\mathrm{r}'\mathrm{n},$$
$$\mathbb{H} = \mathrm{ch}+\mathrm{fg}+2\mathrm{n}^2-2\mathrm{qm}-2\mathrm{p}'\mathrm{l},$$
$$\mathbb{L} = 2\mathrm{fl}-\mathrm{mn}-\mathrm{gp}-\mathrm{hp}'+\mathrm{qr}',$$
$$\mathbb{M} = 2\mathrm{gm}-\mathrm{nl}-\mathrm{hq}-\mathrm{fq}'+\mathrm{rp}',$$
$$\mathbb{N} = 2\mathrm{hn}-\mathrm{lm}-\mathrm{fr}-\mathrm{gr}'+\mathrm{pq}',$$
$$\mathbb{P} = 3\mathrm{nq}'-3\mathrm{lg}-\mathrm{ap}'+\mathrm{qr}, \quad \mathbb{P}' = 3\mathrm{mr}-3\mathrm{lh}-\mathrm{ap}+\mathrm{q}'\mathrm{r}',$$
$$\mathbb{Q} = 3\mathrm{lr}'-3\mathrm{mh}-\mathrm{bq}'+\mathrm{rp}, \quad \mathbb{Q}' = 3\mathrm{np}-3\mathrm{mf}-\mathrm{bq}+\mathrm{r}'\mathrm{p}',$$
$$\mathbb{R} = 3\mathrm{mp}'-3\mathrm{nf}-\mathrm{cr}'+\mathrm{pq}, \quad \mathbb{R}' = 3\mathrm{lq}-3\mathrm{ng}-\mathrm{cr}+\mathrm{p}'\mathrm{q}'.$$

[*Mem.*—Comparing the notation of Salmon's *Higher Plane Curves* with that used here, we have

$$\mathrm{p} = b_3, \quad \mathrm{p}' = c_2, \quad P = B_3, \quad P' = C_2,$$
$$\mathrm{q} = c_1, \quad \mathrm{q}' = a_3, \quad Q = C_1, \quad Q' = A_3,$$
$$\mathrm{r} = a_2, \quad \mathrm{r}' = b_1, \quad R = A_2, \quad R' = B_1.]$$

If we now put

$$\beta\gamma-\lambda^2 = A, \quad \gamma\alpha-\mu^2 = B, \quad \alpha\beta-\nu^2 = C,$$
$$\mu\nu-\alpha\lambda = F, \quad \nu\lambda-\beta\mu = G, \quad \lambda\mu-\gamma\nu = H,$$

and

$$\beta'\gamma'-\lambda'^2 = A', \ldots \quad \beta'\gamma''+\beta''\gamma'-2\lambda'\lambda'' = A^{\backprime}, \ldots$$
$$aA+bA'+cA''+2fA^{\backprime}+2gA^{\backprime\backprime}+2hA^{\backprime\backprime\backprime} = \mathbf{A}, \ldots$$
$$\vdots$$
$$aF+bF'+cF''+2fF^{\backprime}+2gF^{\backprime\backprime}+2hF^{\backprime\backprime\backprime} = \mathbf{F},$$
$$\vdots$$
$$bc-f^2 = \mathfrak{A}, \quad ca-g^2 = \mathfrak{B}, \quad ab-h^2 = \mathfrak{C},$$
$$gh-af = \mathfrak{F}, \quad hf-bg = \mathfrak{G}, \quad fg-ch = \mathfrak{H},$$

then $\tfrac{3}{4}\cdot\mathbb{A} = \mathbf{A}^2 - 3\mathfrak{A}(A'A''-A^{\backprime 2}) - 6\mathfrak{F}(A^{\backprime\backprime}A^{\backprime\backprime\backprime}-AA^{\backprime}),\ \tfrac{3}{4}\mathbb{B}=\dots,\ \tfrac{3}{4}\mathbb{C}=\dots,$

$$\begin{aligned}
&-3\mathfrak{B}(A''A-A^{\backprime\backprime 2})-6\mathfrak{G}(A^{\backprime\backprime\backprime}A^{\backprime}-A'A^{\backprime\backprime})\\
&-3\mathfrak{C}(AA'-A^{\backprime\backprime\backprime 2})-6\mathfrak{H}(A^{\backprime}A^{\backprime\backprime}-A''A^{\backprime\backprime\backprime}),
\end{aligned}$$

$$\tfrac{3}{4}\cdot 6\mathbb{F} = \mathbf{BC}+4\mathbf{F}^2 \qquad ,\ \tfrac{3}{4}\cdot 6\mathbb{G}=\dots,\ \tfrac{3}{4}\cdot 6\mathbb{H}=\dots,$$

$$\begin{aligned}
&-3\mathfrak{A}(B'C''+B''C'-2B^{\backprime}C^{\backprime}+4F'F''-4F^{\backprime 2})\\
&-3\mathfrak{B}(B''C+BC''-2B^{\backprime\backprime}C^{\backprime\backprime}+4F''F-4F^{\backprime\backprime 2})\\
&-3\mathfrak{C}(BC'+B'C-2B^{\backprime\backprime\backprime}C^{\backprime\backprime\backprime}+4FF'-4F^{\backprime\backprime\backprime 2})\\
&-6\mathfrak{F}(B^{\backprime\backprime}C^{\backprime\backprime\backprime}+B^{\backprime\backprime\backprime}C^{\backprime\backprime}-BC^{\backprime}-B^{\backprime}C+4F^{\backprime\backprime}F^{\backprime\backprime\backprime}-4FF^{\backprime})\\
&-6\mathfrak{G}(B^{\backprime\backprime\backprime}C^{\backprime}+B^{\backprime}C^{\backprime\backprime\backprime}-B'C^{\backprime\backprime}-B^{\backprime\backprime}C'+4F^{\backprime\backprime\backprime}F^{\backprime}-4F'F^{\backprime\backprime})\\
&-6\mathfrak{H}(B^{\backprime}C^{\backprime\backprime}+B^{\backprime\backprime}C^{\backprime}-B''C^{\backprime\backprime\backprime}-B^{\backprime\backprime\backprime}C''+4F^{\backprime}F^{\backprime\backprime}-4F''F^{\backprime\backprime\backprime}),
\end{aligned}$$

$$\tfrac{3}{4}\cdot 12\mathbb{L} = \mathbf{AF}+8\mathbf{GH} \qquad ,\ \tfrac{3}{4}\cdot 12\mathbb{M}=\dots,\ \tfrac{3}{4}\cdot 12\mathbb{N}=\dots$$

$$\begin{aligned}
&-6\mathfrak{A}(A'F''+A''F'-2A^{\backprime}F^{\backprime}+2G'H''+2G''H'-4G^{\backprime}H^{\backprime})\\
&-6\mathfrak{B}(A''F+AF''-2A^{\backprime\backprime}F^{\backprime\backprime}+2G''H+2GH''-4G^{\backprime\backprime}H^{\backprime\backprime})\\
&-6\mathfrak{C}(AF'+A'F-2A^{\backprime\backprime\backprime}F^{\backprime\backprime\backprime}+2GH'+2G'H-4G^{\backprime\backprime\backprime}H^{\backprime\backprime\backprime}),\\
&-12\mathfrak{F}(A^{\backprime\backprime}F^{\backprime\backprime\backprime}+A^{\backprime\backprime\backprime}F^{\backprime\backprime}-AF^{\backprime}-A^{\backprime}F+2G^{\backprime\backprime}H^{\backprime\backprime\backprime}\\
&\qquad +2G^{\backprime\backprime\backprime}H^{\backprime\backprime}-2GH^{\backprime}-2G^{\backprime}H)\\
&-12\mathfrak{G}(A^{\backprime\backprime\backprime}F^{\backprime}+A^{\backprime}F^{\backprime\backprime\backprime}-A'F^{\backprime\backprime}-A^{\backprime\backprime}F'+2G^{\backprime\backprime\backprime}H^{\backprime}\\
&\qquad +2G^{\backprime}H^{\backprime\backprime\backprime}-2G'H^{\backprime\backprime}-2G^{\backprime\backprime}H')\\
&-12\mathfrak{H}(A^{\backprime}F^{\backprime\backprime}+A^{\backprime\backprime}F^{\backprime}-A''F^{\backprime\backprime\backprime}-A^{\backprime\backprime\backprime}F''+2G^{\backprime}H^{\backprime\backprime}\\
&\qquad +2G^{\backprime\backprime}H^{\backprime}-2G''H^{\backprime\backprime\backprime}-2G^{\backprime\backprime\backprime}H''),
\end{aligned}$$

$$\begin{aligned}
\tfrac{3}{4}\cdot 4\mathbb{P} = 4\mathbf{BF} &- 6\mathfrak{A}(B'F''+B''F'-2B^{\backprime}F^{\backprime}),\ \tfrac{3}{4}\cdot 4\mathbb{Q}=\dots,\ \tfrac{3}{4}\cdot 4\mathbb{R}=\dots\\
&-6\mathfrak{B}(B''F+BF''-2B^{\backprime\backprime}F^{\backprime\backprime})\\
&-6\mathfrak{C}(BF'+B'F-2B^{\backprime\backprime\backprime}F^{\backprime\backprime\backprime})\\
&-12\mathfrak{F}(B^{\backprime\backprime}F^{\backprime\backprime\backprime}+B^{\backprime\backprime\backprime}F^{\backprime\backprime}-BF^{\backprime}-B^{\backprime}F)\\
&-12\mathfrak{G}(B^{\backprime\backprime\backprime}F^{\backprime}+B^{\backprime}F^{\backprime\backprime\backprime}-B'F^{\backprime\backprime}-B^{\backprime\backprime}F')\\
&-12\mathfrak{H}(B^{\backprime}F^{\backprime\backprime}+B^{\backprime\backprime}F^{\backprime}-B''F^{\backprime\backprime\backprime}-B^{\backprime\backprime\backprime}F'')
\end{aligned}$$

$$\begin{aligned}
\tfrac{3}{4}\cdot 4\mathbb{P}' = 4\mathbf{CF} &- 6\mathfrak{A}(C'F''+C''F'-2C^{\backprime}F^{\backprime}),\quad \tfrac{3}{4}\cdot 4\mathbb{Q}'=\dots,\ \tfrac{3}{4}\cdot 4\mathbb{R}'',\\
&-6\mathfrak{B}(C''F+CF''-2C^{\backprime\backprime}F^{\backprime\backprime})\\
&-6\mathfrak{C}(CF'+C'F-2C^{\backprime\backprime\backprime}F^{\backprime\backprime\backprime})\\
&-12\mathfrak{F}(C^{\backprime\backprime}F^{\backprime\backprime\backprime}+C^{\backprime\backprime\backprime}F^{\backprime\backprime}-CF^{\backprime}-C^{\backprime}F)\\
&-12\mathfrak{G}(C^{\backprime\backprime\backprime}F^{\backprime}+C^{\backprime}F^{\backprime\backprime\backprime}-C'F^{\backprime\backprime}-C^{\backprime\backprime}F')\\
&-12\mathfrak{H}(C^{\backprime}F^{\backprime\backprime}+C^{\backprime\backprime}F^{\backprime}-C''F^{\backprime\backprime\backprime}-C^{\backprime\backprime\backprime}F'').
\end{aligned}$$

Again, the invariant $(a\mathbb{A}+b\mathbb{B}+\ldots)$

$$=\tfrac{3}{4}(a,\ b,\ c,\ f,\ g,\ h,\ f,\ g,\ h,\ l,\ m,\ n,\ l,\ p,\ p',\ m\ q,\ q',\ n,\ r,\ r')(\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ 2\mathbf{F},\ 2\mathbf{G},\ 2\mathbf{H})$$

$$-3\mathfrak{A}(a,\ b,\ \ldots)(A',\ B',\ C',\ 2F',\ 2G',\ 2H')(A'',\ B''\ \ldots)\qquad -\ldots$$

$$+3\mathfrak{A}(a,\ b,\ \ldots)(A^{`},\ B^{`},\ C^{`},\ 2F^{`},\ 2G^{`},\ 2H^{`})^2\qquad +\ldots$$

$$-6\mathfrak{F}(a,\ b,\ \ldots)(A^{``},\ B^{``},\ \ldots)(A^{``},\ B^{```},\ \ldots)\qquad -\ldots$$

$$+6\mathfrak{F}(a,\ b,\ \ldots)(A,\ B,\ \ldots)(A^{`},\ B^{`},\ \ldots)\qquad +\ldots$$

To find whether a quadratic transformation be possible having the following property.

Given $x : y = (a,\ \ldots)(\xi,\ \eta)^2 : (a',\ \ldots)(\xi,\ \eta)^2 = \Upsilon : \Upsilon'$,

and $\xi : \eta = (a,\ \ldots)(x,\ y)^2 : (a',\ \ldots)(x,\ y)^2 = U : U'$.

Then $(\alpha y-\alpha' x,\ \beta y-\beta' x,\ \gamma y-\gamma' x)(U,\ U')^2 = 0$,

i.e., $0 = (\alpha y-\alpha' x)\{a^2x^4+4abx^3y+2(ac+2b^2)x^2y^2+4bcxy^3+c^2y^4$

$+2(\beta y-\beta' x)\{aa'x^4+2(ab'+a'b)x^3y+(ac'+a'c+4bb')x^2y^2+2(bc'+b'c)xy^3+cc'y$

$+\ (\gamma y-\gamma' x)\{a'^2x^2+4a'b'x^3y+2(a'c'+2b'^2)x^2y^2+4b'c'xy^3+c'^2y^4$

i.e., $0 =$

$\{-\ (\alpha',\beta',\gamma')(a,a')^2 \qquad\qquad \}\,a$

$\{-4(\alpha',\beta',\gamma')(a,a')(b,b') \qquad + (\alpha,\beta,\gamma)(a,a')^2 \qquad \}\,a$

$\{-2(\alpha',\beta',\gamma')(a,a')(c,c')-4(\alpha',\beta',\gamma')(b,b')^2+4(\alpha,\beta,\gamma)(a,a')(b,b') \qquad \}\,a$

$\{-4(\alpha',\beta',\gamma')(b,b')(c,c') \qquad +2(\alpha,\beta,\gamma)(a,a')(c,c')+4(\alpha,\beta,\gamma)(b,b')^2\}\,x$

$\{-\ (\alpha',\beta',\gamma')(c,c')^2 \qquad +4(\alpha,\beta,\gamma)(b,b')(c,c') \qquad \}\,a$

$\{ \qquad + (\alpha,\beta,\gamma)(c,c')^2 \qquad \}$

If this is to hold good for all values of $x : y$, we must have

$0 = (\alpha',\beta',\gamma')(a,a')^2$,

$0 = 4(\alpha',\beta',\gamma')(a,a')(b,b') \qquad - (\alpha,\beta,\gamma)(a,a')^2$,

$0 = 2(\alpha',\beta',\gamma')(a,a')(c,c')+4(\alpha',\beta',\gamma')(b,b')^2-4(\alpha,\beta,\gamma)(a,a')(b,b')$,

$0 = 4(\alpha',\beta',\gamma')(b,b')(c,c') \qquad -2(\alpha,\beta,\gamma)(a,a')(c,c')-4(\alpha,\beta,\gamma)(b,l$

$0 = (\alpha',\beta',\gamma')(c,c')^2 \qquad -4(\alpha,\beta,\gamma)(b,b')(c,c')$,

$0 = \qquad (\alpha,\beta,\gamma)(c,c')^2$.

Whence, eliminating $\alpha', \beta', \gamma', \alpha, \beta, \gamma$, we have $0 =$

$$\begin{vmatrix} a^2, & 2aa', & a'^2, & . & . & . \\ 4ab, & 4(ab'+a'b), & 4a'b', & a^2, & 2aa', & a'^2 \\ ac+2b^2, & ac'+a'c+4bb', & a'c'+2b'^2, & 2ab, & 2(ab'+a'b), & 2a'b' \\ 2bc, & 2(bc'+b'c), & 2b'c', & ac+2b^2, & ac'+a'c+4bb', & a'c'+2b'^2 \\ c^2, & 2cc', & c'^2, & 4bc, & 4(bc'+b'c), & 4b'c' \\ . & . & . & c^2, & 2cc', & c'^2 \end{vmatrix}$$

Writing $2a'$ (col. 1)$-a$ (col. 2) for col. 1, we obtain

$$2a'(1)-a(2) = \begin{matrix} 0 \\ -4aC \\ -4bC+aB \\ -4cC-2aA \\ 2cB \\ 0 \end{matrix} \qquad 2a(3)-a'(2) = \begin{matrix} 0 \\ 4a'C \\ 4b'C-a'B \\ 4c'C+2a'A \\ -2c'B \\ 0. \end{matrix}$$

Similarly,

$$2c'(4)-c(5) = \begin{matrix} 0 \\ -2aB \\ 2cC+4aA \\ -cB+4bA \\ 4cA \\ 0 \end{matrix} \qquad 2c(6)-c'(5) = \begin{matrix} 0 \\ 2a'B \\ -2c'C-4a'A \\ c'B-4b'A \\ -4c'A \\ 0. \end{matrix}$$

Hence the whole determinant is

$$= \begin{vmatrix} +4aC, & 4a'C, & -2aB, & -2a'B \\ +4bC-aB, & 4b'C-a'B, & 2cC+4aA, & +2c'C+4a'A \\ +4cC+2aA, & 4c'C+2a'A, & -cB+4bA, & -c'B+4b'A \\ -2cB, & -2c'B, & 4cA, & +4c'A \end{vmatrix}$$

Now taking columns 1 and 2, then the union formed from lines

$$1, 2 = \begin{matrix} 4aC & 4a'C \\ 4bC-aB & 4b'C-a'B \end{matrix} = 16C^3,$$

$$1, 3 = \begin{matrix} 4aC & 4a'C \\ 4cC+2aA & 4c'C+2a'A \end{matrix} = -16BC^2,$$

$$1,4 = \begin{array}{ll} 4aC & 4a'C \\ -2cB & -2c'B \end{array} = 8B^2C,$$

$$2,3 = \begin{array}{ll} 4bC-aB & 4b'C-a'B \\ 4cC+2aA & 4c'C+2a'A \end{array} = \begin{array}{r} 16AC^2+8a'bAC-4c'aBC \\ -8ab'AC+4ca'BC \\ = 8AC^2+4B^2C, \end{array}$$

$$2,4 = \begin{array}{ll} 4bC-aB & 4b'C-a'B \\ -2cB & -2c'B \end{array} = -8ABC-2B^3,$$

$$3,4 = \begin{array}{ll} 4cC+2aA & 4c'C+2a'A \\ -2cB & -2c'B. \end{array} = 4AB^2,$$

Similarly, taking the last two columns,

$$1,2' = \begin{array}{ll} -2aB & -2a'B \\ 2cC+4aA & 2c'C+4a'A \end{array} = 4B^2C,$$

$$1,3' = \begin{array}{ll} -2aB & -2a'B \\ -cB+4bA & -cB+4b'A \end{array} = -2B^3-8ABC,$$

$$1,4' = \begin{array}{ll} -2aB & -2a'B \\ 4cA & 4c'A \end{array} = 8AB^2,$$

$$2,3' = \begin{array}{ll} 2cC+4aA & 2c'C+4a'A \\ -cB+4bA & -c'B+4b'A \end{array} = 8A^2C+4AB^2,$$

$$2,4' = \begin{array}{ll} 2cC+4aA & 2c'C+4a'A \\ 4cA & 4c'A \end{array} = -16A^2B,$$

$$3,4' = \begin{array}{ll} -cB+4bA & -c'B+4b'A \\ 4cA & 4c'A. \end{array} = 16A^3,$$

Hence, for the whole determinant, we have

$$\begin{aligned} 12.34' &= 16^2A^3C^3, \\ -13.24' &= -16^2A^3B^2C^2, \\ +14.23' &= 32AB^2C(B^2+2AC), \\ +23.14' &= 32AB^2C(B^2+2AC), \\ -24.13' &= -4B^2(B^2+4AC)^2, \\ +34.12' &= 16AB^4C. \end{aligned}$$

The sum of which, rejecting the common factor 4, is

$$\begin{aligned}
&= 4^3A^3C^3 - 4\,.\,4^2A^2C^2\,.\,B^2 + 2\,.\,4AC\,.\,B^4 \\
&\qquad + \;\; 4^2A^2C^2\,.\,B^2 + 2\,.\,4AC\,.\,B^4 \\
&\qquad + \;\; 4^2A^2C^2\,.\,B^2 \\
&\qquad - \;\; 4^2A^2C^2\,.\,B^2 - 2\,.\,4AC\,.\,B^4 - B^6 \\
&\qquad\qquad\qquad + \;\; 4AC\,.\,B^4, \\
&= 4^3A^3C^3 - 3\,.\,4^2A^2C^2\,.\,B^2 + 3\,.\,4AC\,.\,B^4 - B^6 \\
&= (4AC - B^2)^3.
\end{aligned}$$

Hence the condition sought is $4AC - B^2 = 0$; *i.e.*, the two quadrics U, U' must have a common factor. Similarly, we should find that Υ, Υ' must have a common factor. In other words, throwing out this common factor in each case, we have

$$\begin{aligned}
\xi : \eta &= uv : uv', \\
x : y &= wv : wv', \\
x : y &= \lambda\xi + \mu\eta : \lambda'\xi + \mu'\eta, \\
\xi : \eta &= lx + my : l'x + m'y.
\end{aligned}$$

The Differential Equations of Cylindrical and Annular Vortices.

By M. J. M. Hill, M.A., Professor of Mathematics at University College, London.

[*Read January 8th,* 1885.]

Abstract.

I. *Cylindrical Vortices.*

The current function Λ of the motion of a fluid in two dimensions being known to satisfy the equation

$$\left(\frac{d}{dt} + \frac{d\Lambda}{dy}\frac{d}{dx} - \frac{d\Lambda}{dx}\frac{d}{dy}\right)\left(\frac{d^2\Lambda}{dx^2} + \frac{d^2\Lambda}{dy^2}\right) = 0,$$