February 16, 1858.

JOSEPH LOCKE, M.P., President, in the Chair.

No. 975.—"On Submerging Telegraphic Cables."[1] By JAMES ATKINSON LONGRIDGE, M. Inst. C.E., and CHARLES HENRY BROOKS.

THE failure in the first attempt to lay the Atlantic cable has, doubtless, attracted the attention of many Members of this Institution, to the subject of submerging such lines of telegraphic communication; and in the hope that the following attempt to investigate the laws to which such operations are subject will not be unacceptable, it is now submitted by the Authors to their professional brethren—not however claiming to be more than a partial solution of an interesting, and somewhat complicated problem.

Those upon whom the task devolved of laying the cable between Ireland and America have, without doubt, given the whole question careful consideration, and the high attainments of some of them, in mathematical and physical science, lead to the belief, that by them, at any rate, the conditions of the problem have been appreciated. At the same time, it cannot be denied, that much misapprehension does exist, and manifests itself by the various schemes which are, from time to time, proposed to prevent another failure.

Some discussion arose at the meeting of the British Association, at Dublin, in 1857, after the reading of one, or more papers before the Mechanical Section; but the published reports were meagre and unsatisfactory, and one of the reported conclusions appeared to the Authors of this Paper to be so curious and improbable, that their attention was, independently of each other, attracted to the subject. The conclusion just adverted to was thus expressed in some of the Journals:—"During the conversation which arose in the Section, after the reading of this communication, a new light seemed to break upon the members, as it seemed to be universally admitted, that it was mathematically impossible, unless the speed of the

[1] The discussion upon this and the following Paper occupied portions of four consecutive evenings, and was again resumed on the 27th of April, but an abstract of the whole is given consecutively.

vessel, from which the cable was payed out, could be almost infinitely increased, to lay out a cable in deep water (say two miles or more), in such a way as not to require a length much greater than that of the actual distance, as from the inclined direction of the yet sinking part of the cable, the successive portions payed out must, when they reached the bottom, arrange themselves in wavy folds, since the actual length is greater then the entire horizontal distance." It seemed desirable to ascertain how far such an idea as that involved in the above statement was correct, and if correct, what amount of slack ought to be provided, to meet the waste in varying depths of water. This was the primary question which the Authors of this Paper proposed to themselves, and upon which they were each independently engaged. They subsequently communicated their ideas to each other. The inquiry branched off in other directions, and the results are now jointly submitted to the consideration of the Members of the Institution.

The following are the questions which are discussed, generally, in the body of the Paper, the calculations being, for the most part, given in an Appendix :—

I. Is it possible to lay a cable straight along the bottom, in deep water, free from the action of currents?

II. If possible, what degree of tension is required, in paying out, so as to lay the cable straight?

III. What is the effect on the cable, as regards strain, of varying

(*a.*) The depth of the water,
(*b.*) The specific gravity of the cable, and
(*c.*) The velocity of the paying-out vessel.

IV. What are the relative velocities of the cable and of the paying-out vessel, requisite to reduce the strain, or tension, to any given amount, and what will be the consequent waste of cable?

V. What is the effect of currents, and the consequent waste of cable?

VI. How far is it necessary and safe to check the velocity of paying out, in passing currents, so as to avoid, as far as possible, waste of cable?

VII. Is it safe, and if so, under what circumstances, to stop the paying out, and to attempt to haul in the cable from great depths?

VIII. What is the effect of the vessel pitching in a heavy sea?

IX. What are the desiderata in the paying-out apparatus?

X. What would be the effect of floats, or resisters?

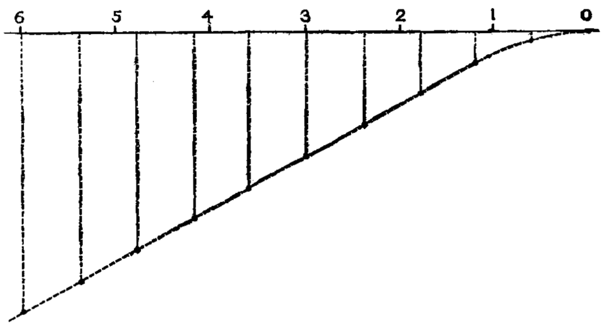
XI. What are the best means for saving the cable, in case of fracture?

XII. What is the best mechanical construction of a submarine telegraphic cable?

I.—Is it possible to lay a Cable straight along the bottom, in Deep Water, free from the Action of Currents?

If a vessel moving uniformly forward drop, at equal intervals of time, balls of equal size, and of the same material, it may be shown, that in water, and with a material of the specific gravity of the Atlantic cable, the motion of each ball vertically will soon become uniform, and the line drawn through the whole of the balls, at any instant of time, will be very nearly a straight line, descending obliquely from the ship to the bottom. Fig. 1 shows

Fig. 1.



the form of the line drawn through such a series of balls, dropped from a vessel moving at the rate of 6 feet per second. The equations of motion are (Appendix, Problem I.)—

$$v = \sqrt{n^2 - (n^2 - \nu^2)\,\varepsilon^{-2cx}} \quad \ldots \ldots \quad (a).$$

$$t = \frac{1}{2\,c\,n} \log \frac{n + \sqrt{n^2 - (n^2 - \nu^2)\,\varepsilon^{-2cx}}}{n - \sqrt{n^2 - (n^2 - \nu^2)\,\varepsilon^{-2cx}}} \cdot \frac{n - \nu}{n + \nu} \quad \ldots \quad (b).$$

which, when x is large, becomes, very nearly,

$$v = n \quad \ldots \ldots \quad (c).$$

$$t = \frac{x}{n} + \frac{1}{c\,n} \log \frac{2\,n}{n + \nu} \quad \ldots \ldots \quad (d).$$

The velocity, consequently, ultimately becomes n, which is a quantity depending on the form and specific gravity of the sinking body. Although this ultimate velocity is only attained at an infinite depth, yet it is rapidly approached from the beginning of the motion. If a sphere of the same diameter and specific gravity as the Atlantic cable is placed in water, and is allowed to descend, it can be shown, that in six-tenths of a second it will have acquired a velocity of 3·27766 feet per second, only differing from its ultimate velocity by one three-thousandth part. The depth passed through, in acquiring this velocity, is not quite 2 feet, so that in dealing with considerable depths of water, the terminal velocity may be assumed as the true velocity throughout.

From these equations, the time of descent of a sphere of the specific gravity and diameter of the Atlantic cable, through 2,000 fathoms of water, is found to be 48 minutes; and if the ship is moving at the velocity of 6 feet per second, it would have moved forward 17,449 feet whilst the body sank to the bottom; consequently the length of the imaginary line passing through the series of balls would be 21,177 feet. But it is evident, that each ball would sink vertically. If then each particle of a cable followed the same law, and no tension was applied at the top, there would be 21,177 fathoms of cable paid out for each 17,449 fathoms run by the ship, or a loss of cable of about 21 per cent. It will be readily seen, that this loss would increase with any increase in the specific gravity of the cable, and *vice versâ*. A continuous cable would not, however, sink in the same way as a series of unconnected balls. If a rod, of any material heavier than water, is placed in water in an inclined position, it will be found, on letting it go, that it does not sink vertically, but moves diagonally downwards to the lower side. In fact, it runs down an inclined plane, lying between the vertical and its own direction. The equations of motion, under these circumstances, of a body, such as the cable, descending obliquely without tension, are (Appendix, Problem II.)—

$$v = \sqrt{w\left(\frac{\cos\alpha}{q'} + \frac{\sin\alpha}{q}\right)} \quad \ldots\ldots\ldots \quad (e).$$

$$\tan(\alpha - \beta) = \sqrt{\frac{q'}{q}} \tan\alpha, \text{ or}$$

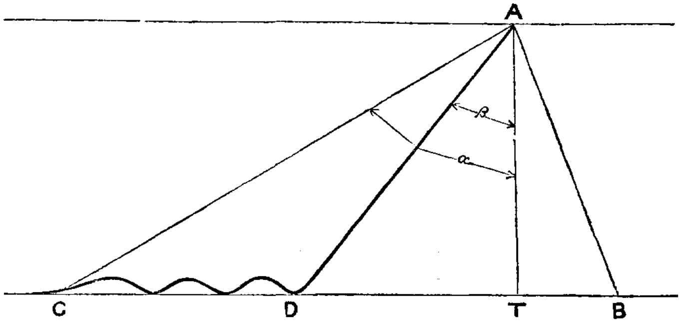
$$\beta = \alpha - \tan^{-1}\sqrt{\frac{q'}{q}} \tan\alpha \quad \ldots\ldots\ldots \quad (f).$$

β being the angle of the direction of motion, and α the angle of the cable with the vertical.

If the cable runs out free from tension at the top, these equations give the circumstances of its descent, and by calculating the angle β, (Fig. 2,) the distance which each particle would run before it reached the bottom may easily be found.

The particle at A instead of coming to B, so as to make CB = AC, would arrive at D, so that the whole length AC would be deposited in folds, or coils, between C and D, and the waste of cable would be the difference between AC and CD divided by AC. The formulæ for calculating the angle β and the waste of cable are given in the Appendix, Problem II., and by means of them, the following tables have been calculated for two descriptions of

Fig. 2.



cable. The first is the Atlantic cable, with a specific gravity of 3·489; the other is a lighter cable, with a specific gravity of 1·50:—

ATLANTIC CABLE.

Velocity of the paying-out vessel, in feet per second	0	2	4	6	8	10	12	15
Inclination of the cable to the horizon	90°	68° 37′	41° 44′	28° 45′	21° 47′	17° 31′	14° 38′	11° 44′
Angle of motion with, or inclination of line of descent of each particle to, the vertical	0	16 50	40 35	51 30	56 49	59 43	62 22	62 39
Velocity of the cable, in feet per second	24·201	24·13	22·72	22·06	22·15	22·82	23·77	25·60
Waste per cent. of cable payed out	100	92	83	73	64	56	50	41

THE LIGHT CABLE.

Velocity of the paying-out vessel, in feet per second	0	..	4	6	8	10	12	15
Inclination of the cable to the horizon	90°	..	19° 56′	13° 21′	10° 2′	8° 2′	6° 43′	5° 22′
Angle of motion with, or inclination of line of descent of each particle to, the vertical	0	..	58 7	62 00	63 7	63 14	62 55	62 4
Velocity of the cable, in feet per second	11·024	..	10·20	9·83	12·48	14·02	15·70	18·31
Waste per cent. of cable payed out	100	..	61	39	36	29	24	18

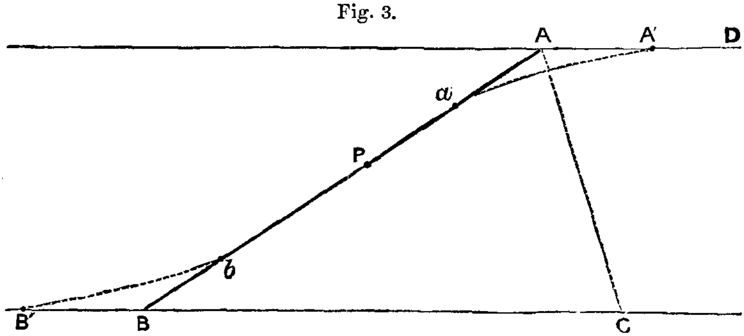
It has been proposed, that in order to lay a cable safely in a considerable depth of water, it should be suffered to run out freely from a drum without tension, whilst the paying-out vessel should be kept at the highest possible speed. An inspection of the above tables shows with what a waste of cable such a proceeding would be attended; for, with the Atlantic cable, and a velocity of the paying-out vessel of 15 feet per second, or about 10 miles per hour, the waste of cable would be 41 per cent. With a light cable it would certainly be less; but even with the above-mentioned cable, having a specific gravity of 1·5, it would be 18 per cent. at this high speed of the ship. The above formulæ also give the velocity of sinking of the cable, in a vertical and horizontal position, in feet per minute, to be as follows:—

	Vertical.	Horizontal.
Atlantic cable . . .	24·201	3·082
Light cable. . . .	11·024	1·404

Lastly, they give the direction of motion and velocity of the end of the cable in case of fracture, and so may be useful in estimating any proposed means of catching it under such circumstances.

The waste consequent on the tendency to run backwards may be prevented, by paying out the cable under a certain amount of tension. This naturally leads to an extension of the same idea, and gives rise to the inquiry before proposed, whether it be possible so to adapt the tension, that a forward motion may be given to each particle, sufficient to cause the cable to lie in a straight line along the bottom, free from tension on the one hand, and from bends and coils on the other.

Fig. 3.



If the cable assumes the form of a straight line A B (Fig. 3), it is evident, that to bring it in order to lie straight at the bottom, the point A must move in the direction A C, bisecting the angle B A D, and that each point between A and B must move parallel to A C; that is to say, the motion of every point in the cable must be in a direction bisecting the angle formed by the cable and the horizon, and this condition may easily be shown to be necessary, whatever be the form taken by the cable. (Appendix, Problem III.)

In the case of a straight line, all that would be necessary to ascertain the tension, would be to find what force would bring an inflexible rod A B, into the position B C, against the resistance of the water. But there is no reason '*à priori*' for concluding, that the line of the cable is straight. Indeed, a little consideration leads to an opposite conclusion; for if A B is the direction of the cable sinking without tension, and a force is applied at A, in order to compel A to move forward so as to arrive at C whilst it sinks to the bottom, it is to be expected, that the top part of the cable will be drawn forward, and assume some new position, such as A′ *a*. Again, the tension in A B, unless it vanishes at B, will prevent the existence of an angle at that point, and another curve B′ *b* will result. It is, therefore, necessary to ascertain the form of the curve assumed by the cable under tension, in order to estimate the resistance of the water to its motion; and as this depends at any point, upon the inclination of the cable to the horizon at that point, the problem becomes somewhat complicated. As it is not yet supposed to be known, whether it is possible to lay the cable straight without some amount of tension at the bottom, it is necessary to frame the equations on the supposition of such a tension existing, and by making it = 0 in the final result, the effect will then be known. The problem to be solved, then, is, to find the equation to the curve A′ B′. The simplest mode of proceeding seems to be, to consider the cable and the ship at rest, and the water moving, not horizontally, nor in any one direction, but in such a manner that its direction at any point P, bisects the angle made at P by the cable and the horizon, and with a velocity equal to the actual velocity of the point P in that same direction.

This mode of considering the question enables it to be treated statically, as regards the form of the curve; and the following are the forces:—

1. Its own weight in water, acting vertically.
2. The tension at the vessel, acting in the direction of the curve at the surface of the water.
3. The tension at the bottom, acting horizontally in a direction opposite to the motion of the vessel.
4. The resistance of the water, acting at every point of the cable, in a direction bisecting the angle formed at that point by the cable and the horizon.
5. The friction of the water on the cable.

By resolving these forces into vertical and horizontal components, differential equations are obtained, and from them is deduced the following expression:—

$$\frac{1}{w - m' \frac{(p - \sqrt{1 + p^2})^2}{\sqrt{1 + p^2}}} \log \frac{w\,a}{w\,a + \left\{ w - m' \frac{(p - \sqrt{1 + p^2})^2}{\sqrt{1 + p^2}} \right\} x}$$

$$= \frac{1}{\sqrt{w^2 + 4\,m^2}} \log \left\{ \frac{2\,m \sqrt{1 + \frac{1}{p^2}} - w - \sqrt{w^2 + 4\,m^2}}{2\,m \sqrt{1 + \frac{1}{p^2}} - w + \sqrt{w^2 + 4\,m^2}} \cdot \frac{2\,m - w + \sqrt{w^2 + 4\,m^2}}{2\,m - w - \sqrt{w^2 + 4\,m^2}} \right\}.$$

This, it will be observed, is not an equation between x and y, but between x and $p = \frac{d\,y}{d\,x}$, or the co-tangent of the angle formed by the curve and the ordinate corresponding to the abscissa x; but from it the form of the curve may be found, without further integration, which would be required to obtain the ordinate in terms of the abscissa, and would be very complicated, if not impracticable. It is shown, in the Appendix, that the effect of increasing the friction of the cable is to diminish the radius of curvature near the bottom. In the case of friction being disregarded, the equation above given has been integrated, and x obtained in terms of y. It is further shown, that if the tension at the bottom be = zero, the cable takes the form of a straight line.

Having thus shown that it is quite possible to lay the cable straight along the bottom, the Authors proceed to investigate the amount of the tension, which forms the second head of the inquiry.

II. What degree of tension is required in paying out, so as to lay the Cable straight?

The general differential equation for the tension cannot be exactly integrated; but the supposition upon which the integration has been effected, is one that will not materially influence the result, and in fact becomes strictly true, if the tension at the bottom be = 0.

The equation is—

$$t = w\,(x + a) - m' \frac{(1 - \cos \mathrm{A})^2}{\sin \mathrm{A}}\, x$$

It shows that the tension at any point is equal to the weight in water of a length of cable equal to the depth of the water below that point, plus the tension at the bottom, less an amount due to the friction of the water against the cable.

The next heads of the inquiry are—

III. What is the effect on the Cable, as regards strain, of varying

(*a.*) The depth of the water,
(*b.*) The specific gravity of the cable, and
(*c.*) The velocity of the paying-out vessel?

IV. What are the relative velocities of the Cable and of the paying-out Vessel, requisite to reduce the strain, or tension, to any given amount, and what will be the consequent waste of Cable?

These questions being connected in their nature, are considered together.

It is evident, from the above equation, that the tension increases uniformly as the depth, and as the weight of the cable in water; from which it follows, that the less the specific gravity of the cable, the less is the risk from overstrain. It is further apparent, that the strain is diminished by any increase of the coefficient (m') of friction, and it is therefore a subject for inquiry, how far this can be practically accomplished. This question comes properly under the tenth head of the inquiry, and it is, therefore, laid aside for the present.

Another method of diminishing the tension is by increasing the velocity of paying out, beyond that of the paying-out vessel. The effect of this is investigated in the Appendix, where it is shown that if

v be the velocity of the cable,
ν that of the ship, the tension

$$t' = w\,x - \frac{m'\left(\frac{v}{\nu} - \cos A\right)^2 x}{\sin A}$$

from which the values of t' may be found for any values of x, v, and ν. These values have been calculated for two cables, viz.:—the Atlantic, and one of a specific gravity of 1·50,—and Tables are given, showing for each, the decrease of tension attendant upon increased ratios of the speed of the cable to that of the ship.

These results are exhibited in Figs. 4 and 5. In Fig. 4, the vertical column on the left shows the tension of the cable in lbs.,

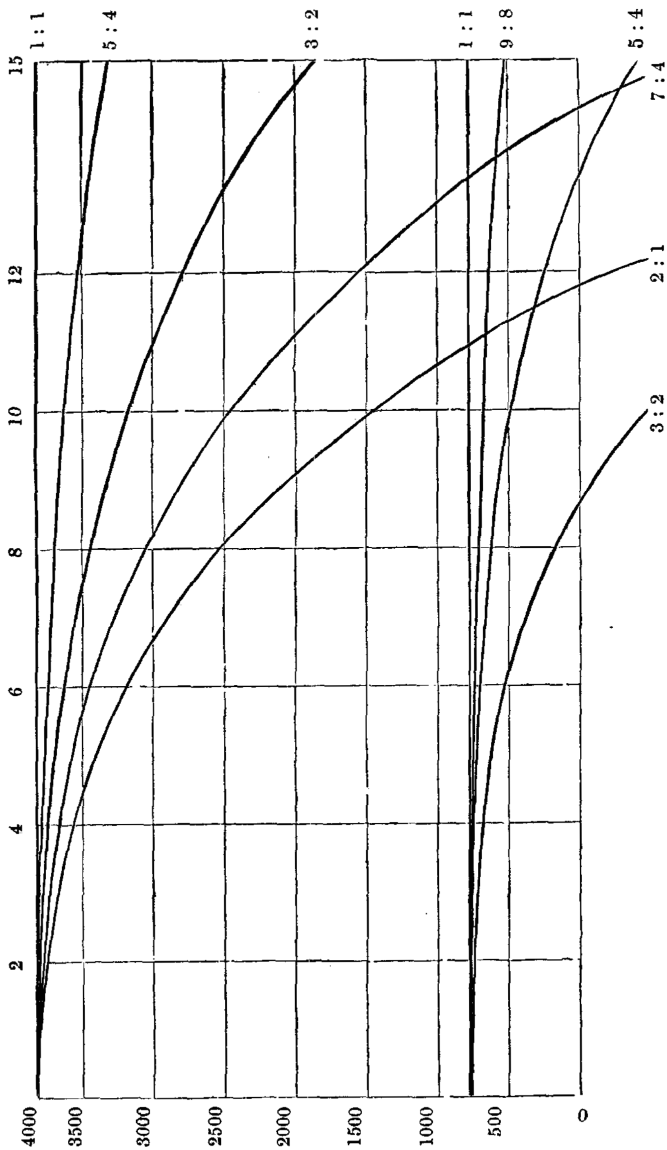
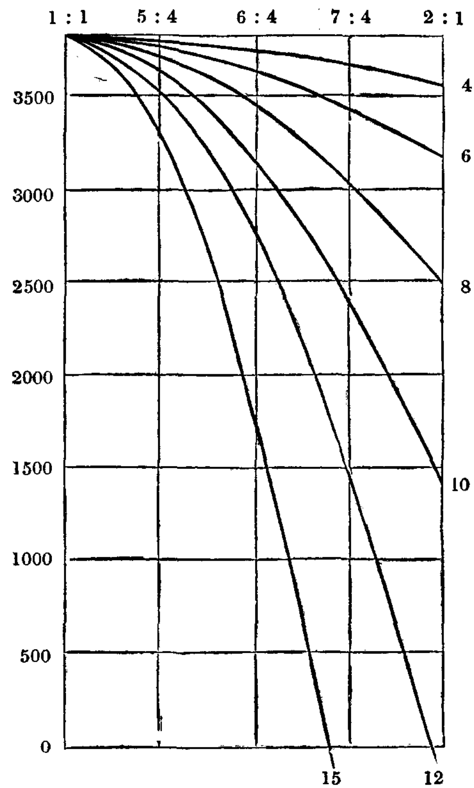


Fig. 4.

the numbers at the top show the rate of the paying-out vessel, and the ratio at the end of each curved line is that which the velocity of the cable bears to that of the vessel. The upper series of curves refers to the Atlantic cable, and the lower series to the light one. In Fig. 5, which refers to the Atlantic cable only, the number

Fig. 5.



at the end of each curve is the rate of the vessel in feet per second; the ratio at the top is that of the velocity of the cable to that of the vessel, and the numbers at the left side show the corresponding tension. It is found, that the diminution of tension due to an increased rate of paying out is comparatively small, unless the velocity of the ship itself is considerable. In fact, the decrease of tension, arising from letting the Atlantic cable run out at twice the speed of the ship, is when the—

Speed of the ship is	4	feet per second,	251 lbs.	= $6\frac{1}{2}$	per cent.
,, ,,	6	,,	622 ,,	= 16	,,
,, ,,	8	,,	1,301 ,,	= 34	,,
,, ,,	10	,,	2,393 ,,	= 62	,,

and at 12 feet per second the tension would be nil.

Again, if the speed of the cable is only one-fourth greater than that of the ship, the decrease of tension is, when the

Speed of the ship is	4	feet per second,	32 lbs.	= 0·8	per cent.
,, ,,	6	,,	69 ,,	= 1·8	,,
,, ,,	8	,,	117 ,,	= 3·0	,,
,, ,,	10	,,	192 ,,	= 5·0	,,
,, ,,	12	,,	299 ,,	= 7·8	,,

Taking the light cable, it appears, that if the velocity of the cable is one-fourth greater than that of the ship, the decrease of tension is, when the

Speed of the ship is	4	feet per second,	30 lbs.	= 3·7	per cent.
,, ,,	6	,,	78 ,,	= 9·8	,,
,, ,,	8	,,	166 ,,	= 20·8	,,
,, ,,	10	,,	318 ,,	= 39·9	,,
,, ,,	12	,,	546 ,,	= 63·9	,,

From this it appears, that the relief obtained by this method of decreasing the tension is, at the ordinary velocity of paying out, very inconsiderable; whilst the waste of cable is very great. It is submitted, that the true remedy for the evil of great tension is the employment of a cable of small specific gravity. If, for instance, the two cables above mentioned are taken, it is seen that there is a remarkable difference in the tension, the light cable having a tension, in 2,000 fathoms, of 879 lbs., against 3,849 lbs. in the case of the Atlantic cable. It is further to be observed, that so long as the velocity of paying out does not exceed that of the ship, no advantage is derived from increasing the speed of the ship, but that, on the contrary, a slight increase of tension must result. The great waste of cable attendant on a slight deficiency of tension, as indicated by these tables, seems to point to the desirability of laying cables with some moderate amount of tension at the bottom; because, it is evident, that a very moderate increase in the depth of the water would be attended with a great waste of cable. If, for instance, the ship was moving with a velocity of 6 feet per second, the tension on the cable, at a depth of 2,000 fathoms, would be 3,849 lbs. If, now, the depth is increased by 100 fathoms, the increase of tension due to this depth would be about 190 lbs. In order to balance this, the same extra resistance must either be applied by the breaks, or the cable must run out at a velocity one-half greater than that of the ship, or at 9 feet per second, and consequently with a waste of 33 per cent. In the case of the light cable, a similar increase of depth would, if not resisted by the breaks, give rise to a velocity of cable of about $7\frac{1}{5}$ feet per second, thus involving a waste of $16\frac{1}{2}$ per cent. of the cable. These Tables will serve to account for the sudden increase of velocity, which has been mentioned in laying heavy cables, when the depth of water has increased; and they show, how desirable it is to be prepared with the fullest information respecting the depth to be traversed, and to have in readiness efficient means, under the control of vigilant and intelligent men, so that by a proper and gradual increase of resistance by the breaks, the cable may be prevented from acquiring any undue velocity.

V. What is the effect of Currents, and the consequent Waste of Cable?

The depth to which the ocean currents extend, their breadth and their velocity are difficult to ascertain; but the Authors enter upon this part of the subject, in the hope that their investigations will be of practical advantage, in so far as they serve to point out, the danger and inutility of attempting to check the running out of cable due to a current. The action of currents upon a cable increases with the length exposed at any moment of time, and as the extent of the currents is much greater horizontally than vertically, it is obviously desirable that the cable should traverse them in the shortest possible direction, consistent with other necessary conditions. In this respect then a heavy cable is to be preferred, as it descends at a higher angle; and it is worthy of consideration, how far it may be practicable to increase the velocity of sinking, by attaching weights to the cable whilst passing through a current. This, the Authors think, might be accomplished without much difficulty; but, before giving any opinion upon its desirability, they prefer to examine, what would be the approximate loss which a current of a given extent and velocity might occasion.

Let Figs. 6 and 7 represent a ground-plan and section of the ship's course, $A\ A_{,}\ A_{,,}$, Fig. 6, and of the position of the cable $A'\ B'$, Fig. 7, and let the commencement of the current be at X Y. When the ship moves past A' it will, unless prevented, drift with the current in some direction such as $A_{,}\ a$ (Fig. 6), and the first effect will be, to give the sinking cable an apparent motion in a contrary direction; but after the action of the current comes upon the whole suspended portion of the cable, it will go on depositing in the altered line of the ship's course, as if no current existed.

But if the ship is kept to a true course, the action will be entirely different; and it is now to be considered, what would be the effect on the cable, as regards running out and tension. If the current

Fig. 6.

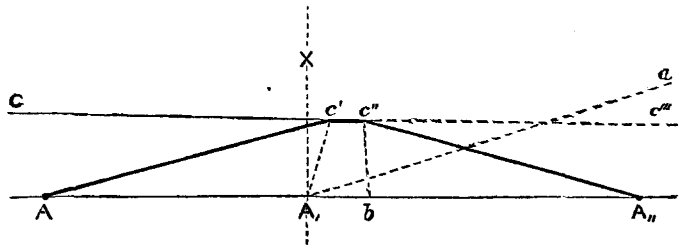
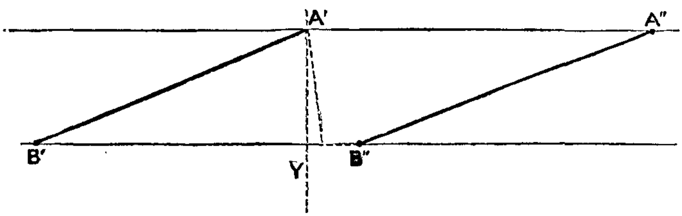


Fig. 7.



flows with such a velocity as to move a cable laterally, to a distance represented by the line $c' c'' c'''$, whilst it sinks from A′ A″ to B′ B″, and if the lateral resistance of the water behind X Y is neglected, the cable would come into the position A c', a straight line along the bottom equal in length to A′ B′. If, on the other hand, the lateral resistance behind X Y is very great, the cable will take a direction approaching to A $A_{,}$ c'; but, since at the velocity of any ordinary current, the resistance would be small, and since, also, the transition from still water to a current would be gradual, the line A c' may be taken as virtually a straight line. The distance to which a current would transport a particle freely suspended, is that due to the velocity of the current itself. Since there is no tension at the bottom, the cable, at that depth, would be free to move laterally with the current, and the whole cable would assume a diagonal position from some point at the bottom, such as c'', to the ship at $A_{,,}$. When the ship has arrived at $A_{,,}$, the distance from $A_{,}$ due to its velocity whilst the cable is sinking, the cable will be in the following position:—A c' c'' along the bottom, and from c'' to $A_{,,}$ rising at its usual angle to the horizon, the projection of this part in the horizontal plane being represented by c'' $A_{,,}$ (Fig. 6). The extra length of cable which must be paid out whilst the ship moves from A to $A_{\prime\prime}$ is shown in the Appendix, Problem V., to be given by the formula

$$\frac{d}{\nu \sin A}\left\{\nu - \sqrt{\nu^2 - v^2} + \cos A\left(\nu - \sqrt{\nu^2 - \frac{v^2}{\cos^2 A}}\right)\right\}.$$

After this there is no further waste of cable, because the suspended portion c'' $A_{\prime\prime}$ then moves on parallel to itself, and the point c'' moves forward parallel to, and at the same velocity as, the ship at $A_{\prime\prime}$.

In order to give some idea of the amount of waste under the action of a current, it has been calculated, that for a current of 100 fathoms deep, running at right angles to the ship's course, at a velocity of $1\frac{1}{2}$ foot per second, the ship moving at the rate of 6 feet per second, the waste of the Atlantic cable would be 14 fathoms, and of light cable 28 fathoms. This shows a slight advantage, in this respect, in using the heavy cable, which descends at an angle of 28° 45′, as compared with the light cable, descending at an angle of 13° 21′. It should be observed, that, practically, the ultimate waste would not be quite so great; because a portion of it would be recovered on quitting the current.

The next point for investigation is the amount of strain due to the action of the current. In order to arrive at this, it is shown in the Appendix, Problem VI., that the curve assumed by a flexible line stretched across a current is a catenary, but differing from the common catenary in this, that the tension is constant throughout. Now, any tension which may come upon the cable from the action of the current must follow this law, and from this fact two remarkable results proceed, viz., that the current produces no catenarian strain upon the cable, and that the line from c'' to $A_{\prime\prime}$ is a straight line. For, in the first place, there is no tension at the bottom; but by the nature of the curve, if it is a curve, the tension is constant, consequently it has the same value at the top as at the bottom, where it is evidently zero. Again, if it is not a straight line, but a curve, it must be a catenary; but it cannot be a catenary without tension at the bottom; and as there is no tension at the bottom, consequently the line is not a catenary, but a straight line. That there is no tension at the bottom is one of the conditions of the problem; and it is evident that it must be so, because if the velocity of the cable exceeds, by ever so little, the velocity of the ship, the cable must be deposited in folds, or coils, and consequently without tension.

The same reasoning will apply to the case of a current whose depth is less than the total depth of the water; because each particle after passing through the current descends through the rest of its course without any further lateral deflection, beyond what is due to the portion above it in the current.

The result, that a current causes no catenarian strain on the cable, is an important one, and removes what appeared, for a long time, to the minds of the Authors, an objection to light cables, whose sole disadvantage now appears to be confined to a little extra waste on entering currents.

If, however, during the passage through a current, the paying out is stopped, or retarded, a strain will immediately arise, for the cable will then take the catenarian form from c'' to $A_{''}$. Under these circumstances the strain may be calculated, as shown in the Appendix, Problem VI., and will be additive to the vertical strain arising from the weight. A slight additional strain is brought upon a cable by a current caused by the friction of the water, but it is so slight that practically it may be disregarded. The method of estimating this is given in the Appendix, Problem VII., and the amount is shown to be quite insignificant.

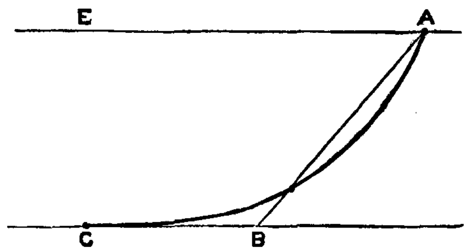
VI. How far is it necessary and safe to check the velocity of paying out in passing Currents, so as to avoid, as far as possible, Waste of Cable?

Since the waste of cable is confined to the first entrance into the current, it would seem advisable not to check it by increasing the tension, for the amount lost must, under ordinary circumstances, be very inconsiderable, even with a much lighter cable than the Atlantic cable. For instance, if the current of the Gulf Stream is assumed to extend to a depth of 200 fathoms, which is probably beyond the truth, and to flow with a velocity of $1\frac{1}{2}$ foot per second, and if the rate of the paying-out vessel is 6 feet per second, the waste of cable, of the specific gravity of the Atlantic cable, would not be more than 28 fathoms; and this is, as shown above, independent of the width of the stream, and only occurs at the first entrance upon it. Again, in the case of a much lighter cable, having a specific gravity of 1·5, the waste would not be more than 56 fathoms, and this might be diminished, by attaching sinkers, if desired. It is, therefore, maintained, that any attempt to check the running of the cable due to a current, by an increase of tension, is alike injudicious and unnecessary.

VII. Is it safe, and if so, under what circumstances, to stop the paying out, and to attempt to haul in the Cable from great Depths?

In order to solve this question, it is necessary to find the curve which the cable will assume, when the paying out is stopped, and then to calculate the tension at the vessel. It is evident, that the curve is the common catenary. Let A B (Fig. 8) represent the position of the cable at any moment of time during the paying out, and let the ship and the paying out stop simultaneously. The cable which was in the position A B C will immediately begin to

Fig. 8.



rise at B, and to assume a catenarian form. It is shown in the Appendix, Problem VIII., that the following relations subsist:—

$$\frac{x \sin \alpha}{1 - \cos \alpha} = s - \sqrt{s^2 - x^2} + \frac{x \cos \alpha}{1 - \cos \alpha} \log \frac{\cos \alpha}{1 - \sin \alpha}$$

and

$$\sqrt{2\,a\,x + x^2} = s - \sqrt{s^2 - x^2} + a \log \frac{a + x + \sqrt{2\,a\,x + x^2}}{a}$$

when x, y and s are the respective abscissa, ordinate, and length of the curve, α the angle formed with it and the surface, and a the tension at the bottom. Having obtained the value of a, the total tension is equal to a weight of cable of the depth x plus a.

The following Tables have been calculated from the Formula 5, Problem VIII., to show the amount of strain which would be brought upon the cable, by a stoppage of the paying-out apparatus, in a depth of water of 2,000 fathoms, and also the length which must be paid out, if the ship is stopped, in order to produce a minimum tension:—

ATLANTIC CABLE.

Velocity of the ship in feet per second	2	4	6	8	10	12	15
Tension in lbs. when the cable is stopped	4,704	8,624	15,736	25,760	38,050	54,460	83,100
Length to run out for minimum tension, in case of stoppage; and distance to move back, to bring the cable vertical, in feet	3,812	7,426	8,924	9,689	10,140	10,459	10,767

LIGHT CABLE.

Velocity of the ship in feet per second	..	4	6	8	10	12	15
Tension in lbs. when the cable is stopped	..	6,328	13,140	23,880	35,990	52,530	79,111
Length to run out for minimum tension, in case of stoppage; and distance to move back, to bring the cable vertical, in feet	..	9,892	10,596	10,947	11,158	11,296	11,437

From these Tables it appears, that the result of a stoppage of the paying-out apparatus, in a depth of 2,000 fathoms, whilst the vessel was proceeding at the rate of 6 feet per second, would be to bring the following strains on the cables:—

Atlantic cable . . . $140\frac{1}{2}$ cwt.
Light cable . . . . $117\frac{1}{4}$ cwt.

The time in which the change of form would take place would be a difficult problem to determine, but it might be considerably less than the hypothesis above stated would give, because the ship could not be stopped instantaneously, although the paying out might be so arrested, through an accident to the apparatus. The ultimate form of the cable would be the same; but the time being less, the danger would be greater. If, then, it is necessary to stop the paying out, or if any accident should occur, involving the stoppage of the paying-out apparatus, the engines ought to be immediately reversed, and the ship be backed, as quickly as possible, until it has arrived at a position which will allow the cable to hang vertically from the stern. Again, if it is requisite to stop to repair any part of the cable, or to splice it, the ship should be put into the same position before stopping the paying out. The distances to be moved back are given in the above Tables. In attempting to haul in, the same relative position of the ship and the cable should be maintained throughout the operation, which might perhaps be accomplished by reversing the ship's course, when the cable has taken a vertical position, and then moving backwards at the same rate as the cable is hauled in. This operation cannot be regarded as otherwise than hazardous, in great depths of water, on account of the practical difficulty of keeping the vessel vertically above the cable; for it is evident, that any departure from that position must give rise to a catenarian strain; and any considerable amount of this, especially in a rough sea, would undoubtedly prove fatal.

VIII. What is the effect of the Vessel pitching in a Heavy Sea?

The Authors are of opinion, that if the paying-out apparatus is not too heavy, and if it works freely, no danger need be apprehended from its use. Whilst the vessel's stern is rising, the cable will be drawn out more quickly, and whilst falling, more slowly, than its ordinary rate; but no extra tension will arise, except that due to the inertia of the paying-out drums, and of the cable upon them. The amount of this it would not be difficult to estimate, knowing the details of the apparatus, and the form and velocity of the wave; but into this it is unnecessary to enter, further than to remark, that it furnishes an argument for a light and free-working apparatus. It is true, that the vertical rise of the ship's stern would, to some extent, call into action the catenarian strain, but only so far as the paying out was influenced by the causes just mentioned. If the paying out was retarded, or even entirely stopped, the only effect on the cable would be to increase the abscissa of the catenary, by the amount due to half the height of the wave, an amount quite inconsiderable, in proportion to the whole catenarian strain, in such depths as from 1,000 fathoms to 2,000 fathoms.

IX. What are the Desiderata in the Paying-out Apparatus?

With reference to the paying-out apparatus, the Authors would limit themselves to the expression of an opinion, of what ought to be its characteristics. The principal one is, that its inertia shall be as small as possible; and this affords an argument against the views of those who advocate the use of drums, upon which the cable should be coiled. Another suggestion has been to coil the cable upon a huge turn-table; but if either of these plans were practicable, it would still be liable to the objection, that the inertia of the mass would be so great, that the effect of the pitching of the vessel would then be felt upon the cable, almost as much as if the rate of paying out was kept strictly uniform. The only argument in favour of such plans is the prevention of kinks; but it does not appear that any difficulty arose from this cause in the paying out of the Atlantic cable. A very important part of the paying-out apparatus is the break, and its essential characteristic should be, the impossibility of any strain arising from it, under any circumstance, beyond that which is intentionally imposed. No increase of velocity should produce increase of strain, and it is only by this condition being rigidly adhered to, that such operations can be conducted with safety. The Authors urge this point the more earnestly, as they can only account for the failure of the late attempt, on the hypothesis of the paying-out apparatus having in some way got out of order, and a strain having arisen far beyond that recorded by the indicator. The cable was said to be running out freely, the tension indicated was under 4,000 lbs., and yet it parted, although the actual strength is given as not less than $4\frac{1}{2}$ tons. It is not stated where the cable parted; but it is the opinion of the Authors, that the stoppage of the paying-out apparatus was the cause, and not the consequence, of its parting; for it has already been shown, that such a stoppage would, in this instance, bring an ultimate strain of 7 tons upon the cable.

X. What would be the effect of Floats, or Resisters?

Floats, buoys, and resisters have severally been proposed, as a means of diminishing the risk from tension. The action of the former would be better accomplished by using a cable of less specific gravity, because then the relief would not be partial, but would be felt at every point. It has been shown, that the angle of sinking has no effect on the tension, except in so far as this is modified by the friction of the water, and therefore the mere action of floats, as tending to keep the cable more horizontal, goes for little. Their only use would be to reduce, virtually, the specific gravity of the cable.

Resisters would act in relief of the tension, and be equivalent to increasing the coefficient m' in the equations. Possibly, it might be practicable so to apply them, as to make the tension equal to zero; but the length to which this Paper has extended does not permit the Authors to enter further into the detail of such an arrangement.

The next question which the Authors have proposed is—

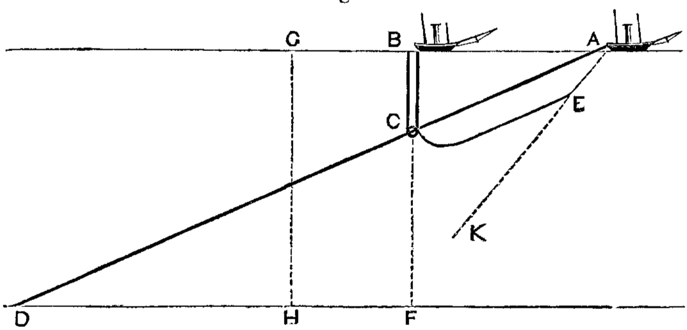
XI. What are the Best Means for saving the Cable, in case of Fracture?

In spite of every precaution, and the most complete and well-devised system of paying-out apparatus, some accident may occur, even when the task is all but accomplished, resulting in delay, and the loss, probably, of the cable itself. A fracture may take place from some unlooked-for accident, even when within a few miles of the destination, and 2,000 miles of cable lie uselessly, and perhaps irrecoverably, at the bottom of the sea.

A simple and effective plan for recovering the end of the cable is, therefore, a very important subject for consideration. The arrangement about to be described appears to the Authors likely to succeed in securing the end of the cable in case of fracture. It is proposed, that a second vessel shall follow in the wake of the paying-out vessel, at such a distance that it may always be about 200 fathoms above the cable. From this vessel is to be suspended a link, through which the cable should pass freely. This link should be suspended in such a manner, that at any moment, upon a signal being given from the paying-out vessel, it might be reversed, or caused to rotate in a vertical plane, so as to make a single, double, or threefold hitch in the cable. By this means the further passage of the cable through the link would be arrested. The link and cable are then to be lowered to the bottom, and the following vessel is to put about, and be moved gently in the opposite direction, at such a rate as to insure the link and the cable attached to it rising vertically to the surface, when the link is hauled up.

This is best illustrated by Fig. 9, in which A and B are the

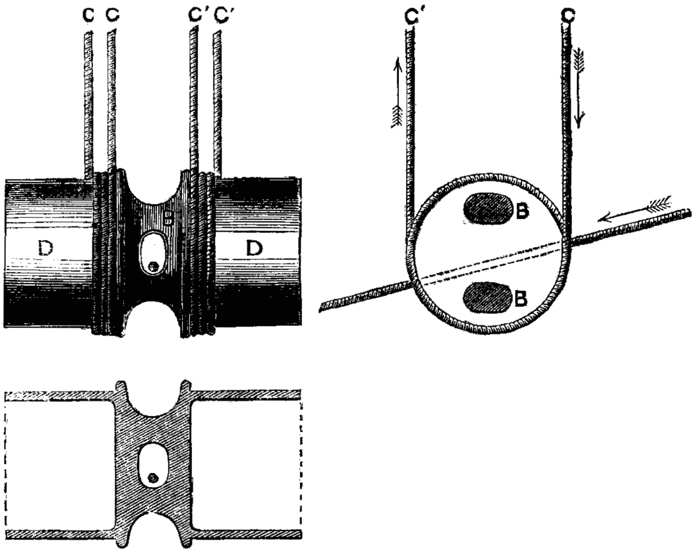
Fig. 9.



ordinary positions of the vessels, and A C D the line of cable, C being the link. When the link is reversed, the motion of the cable through C is momentarily arrested, and the portion of the cable A C comes into some position such as C E, the link C is then lowered to F. Then the vessel B is turned round, and goes slowly back to G, during which time, the link is gradually raised, and finally the cable is brought into the position G H, where it may be spliced, and the paying out proceed as before. The form of the link, and the method by which it is proposed to make it rotate, are shown in Fig. 10, in which D D are two drums of cast-iron, connected by an intermediate piece B B, forming the link through which the cable passes. The whole is suspended by wire ropes C C′, which are coiled round each drum, in such a manner, that when C′ is hauled up C descends, and thus the drums and link are caused to rotate, and the cable is jammed between the cheeks B B, by as many turns of the apparatus as may be deemed necessary.

The strain on the cable being greatest at A (Fig. 9), it is probable, that the fracture would take place near that point, and by the formula given in the Appendix, it is easy to calculate the

Fig. 10.



direction of motion of the point A, which would not be vertical, but in an oblique direction, shown by the dotted line A K.

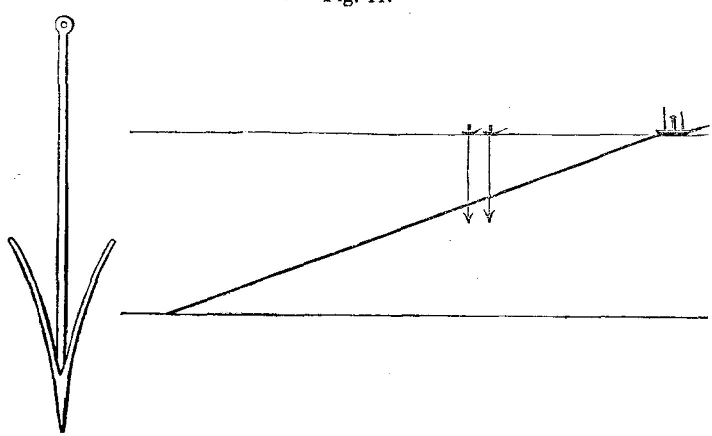
For greater security, two, or more following-vessels might be employed, each carrying its check-link, suspended at the depth due to its position.

The Authors, whilst suggesting the above plan, freely admit that it is open to objection. Indeed they feel strongly, that any apparatus connected with the cable, after it leaves the paying-out vessel is undesirable, as introducing another possible cause of accident, from some unforeseen derangement, or unlooked-for neglect.

The objections which occur to them, to the apparatus above described, will now be pointed out, and it must be left to others to decide, whether they are of sufficient importance to lead to the rejection of the plan. First, there is a possibility of the cable fouling, either with the suspended drums, or with the suspending-ropes. This may be guarded against by care in paying out, so as not to allow the cable too much slack. The risk of the catching apparatus twisting, may, the Authors think, be avoided, by the suspending-ropes being brought up to the opposite sides of the following-vessel, so that any tendency to twist would be resisted, on the principle of the bifilar mode of suspension in the torsion balance. Secondly, a danger might occur if a kink went overboard. This, perhaps, might be met by increasing the opening through the link between the drums, so as to allow a kink to pass; but if not, it does not follow that a fracture would take place. The suspending-ropes of the link should be so arranged, as not to admit of any considerable increase of tension, and in case of fouling, or of kinks which could not pass through the apparatus, the paying out should be stopped and the link lowered to the bottom, the cable being afterwards drawn up, as before detailed, in case of fracture. Doubtless, this might involve some loss of time, and it necessitates an additional amount of personal attention; but the question is, whether a better mode can be devised, and if not, is the possible loss of time, and the extra attention compensated for, by the probable safety of the cable in case of fracture?

A different method of catching the cable might, perhaps, be adopted with success, if the cable were not too heavy. At a certain distance behind the paying-out vessel, two small steamers might follow, keeping as nearly as possible in the line of the sinking cable. From each of them might be suspended one, or more grapnels, with four, or six arms, rising upwards at an acute angle (Fig. 11). These grapnels should be suspended in the water at a

Fig. 11.



depth a little below the sinking cable. In case of fracture, upon a signal from the paying-out vessel, the two small vessels should at once steam at right angles to their former course, in opposite directions, so that one of them should cross the line of cable, which would probably be caught and jammed in the acute angle of the grapnel. As soon as this was done, which would be known by the increase of tension on the suspending-rope of the grapnel, this rope should be slacked out so as to avoid fracture from any undue catenarian strain. The ship's course should then be so altered, as to cause the suspending-rope to hang vertically, with only such tension as is due to its own weight, to that of the grapnel, and to that of the cable between it and the bottom. The hauling in would then be proceeded with, care being taken always to direct the ship's course, so that the suspending-rope might hang vertically. With a light cable this method might succeed. It would not be so certain in its action as the first proposed method, but, on the other hand, it is less liable to objection, as a cause which might lead to fracture, from fouling, kinks, &c. The depths at which the grapnels should hang, and the distances of the following-vessels, would be regulated by the specific gravity of the cable, and the speed of the paying-out vessel. Tables are given in the Appendix, to show the position of the cable relatively to the following-vessels, at the end of certain intervals of time after fracture. These Tables further show the time after fracture, when the end of the cable would pass beneath the following-vessel; first, on the supposition of that vessel continuing her course, and secondly, on the supposition of her immediately crossing the line of cable when fracture was signalled. These Tables are calculated from the formula in the Appendix, Problem IX., for two different cables, and could of course be extended to any other conditions of a particular case. Instead of suspending the grapnels below the cable, it might perhaps be preferable to suspend them a little above it, and lower them upon the signal of fracture being given.

XII. What is the best Mechanical Construction of a Submarine Telegraphic Cable?

In venturing a few remarks upon the construction of the cable itself, the Authors beg to say, that they only claim for their opinions, the merit of being legitimate deductions from the foregoing investigations, entered upon without prejudice, and followed out to the best of their ability. If the investigations can be shown to be defective in principle, or to be inapplicable, the opinions based on them will be of comparatively little importance; but on the other hand, if the reasoning, which is given in the Appendix, be correct, the Authors claim for these deductions a positive, practical value.

The information which has been published respecting the recent great experiment, is but scanty, and there is much that is contradictory in the various accounts. The Authors have, therefore, refrained from entering into any review of the published statements, or from attempting to give a reason for the failure, beyond a passing suggestion, that the stoppage of the paying-out apparatus might be the cause. They would, before going further, only suggest that a minute detail of all the proceedings, and of the phenomena observed, both electrical and mechanical, would form a very interesting and valuable communication to the Institution.

In the construction of a submarine cable, there are three principal matters to be attended to;—

1st. Its conducting power.
2nd. Its insulation.
3rd. Its mechanical structure and condition.

The two former do not enter within the province of this Paper, but the latter does so eminently.

There are two descriptions of cable which have come under the notice of the Authors. In the first, the conducting medium is placed in the centre; next to it comes the insulating medium, and outside of all, that which is to give it at once protection and strength. In the other, the strength and the conducting medium are one and the same, and are placed in the centre, being surrounded by the insulating medium. The former may be called the heavy, the latter the light system of cable. The investigations of the Authors lead them to give a decided preference to the light system. Taking the Atlantic cable as a type of the heavy system, and as a very perfect one, inasmuch as those who have had to lay it, declare that they can offer no suggestion for its improvement, it is found that with a weight in water of $15\frac{1}{2}$ cwts. to the mile, it offers a tensile strength of about $4\frac{1}{4}$ tons; that is to say, it will support a length of $5\frac{1}{2}$ miles of itself, hanging vertically in water. Taking, again, the cable proposed by Mr. Allan, as a type of the light cable, it is found that with a weight in water of $3\frac{1}{4}$ cwts. to the mile, it offers a tensile strength of 2 tons, or will support a length of $12\frac{1}{2}$ miles of itself hanging vertically in water. Consequently the light cable has to the heavy cable a relative strength of 25 to 11.

In the next place, the weights in air are about $21\frac{1}{2}$ cwts. and 10 cwts. to the mile respectively, being an economy, as regards transport, of 54 per cent. in favour of the light cable.

Another question for consideration is, the effect of tension and compression on the two descriptions of cable. The heavy cable is composed of two independent metallic portions, separated by the insulating medium. The outer casing is wound spirally round, and it is clear that the effect of tension must be to stretch it, and so to strain both the insulating medium and the inner core. This extension is greater than that ordinarily due to tension, from the following cause. In great depths of water, the pressure upon the cable is very considerable; for instance, in 2,000 fathoms, it is about $2\frac{1}{2}$ tons per square inch. Now, under such a pressure, it is to be expected that the insulating medium will be compressed. This being so, the cable will be reduced in diameter, and the spiral strands on the outside will adjust themselves to the new diameter, and the angle of the spirals becoming more acute, the outer shell will increase in length. The proportion of this increase may be calculated, as shown in the Appendix, Problem X., whence it appears it is $\frac{\pi^2 d \delta}{l^2}$ nearly, where d is the original diameter from centre to centre of the spiral strands, δ the decrease of diameter due to compression, and l the length in which the strands make an entire turn round the cable. In the Atlantic cable d is about 0·5 of an inch, l is 9 inches, and if $\delta = 0{\cdot}1$ of an inch, the increase of length would be about one-eighteenth part of its length, and this, it must be borne in mind, is altogether independent of the stretching due to tension.

The Authors are not aware to what extent gutta-percha is compressible, nor have they any information respecting the amount of stretching of the Atlantic cable under a given tension. They, therefore, content themselves with pointing out the two causes of tension and compression, as both resulting in a stretching of the outer shell, whilst the insulation and the inner core are not thus acted on. It is possible, that the amount of such stretching, and its consequent strain on the inner part of the cable, may not be of serious moment, within the limits of the tension due to the laying of the cable; but it must not be forgotten, that undue strains may easily be brought on, by any fouling, or imperfect action of the paying-out apparatus, and it has been shown, that even at a rate of 6 feet per second, such strain may easily amount to 7 tons. It is, therefore, perhaps not going too far to say, that this structure of cable may, possibly, have its conducting power, or its insulation, seriously injured by the stretching of the outer shell, although no absolute fracture may take place. From such a contingency the other description of cable is free, and it would probably remain uninjured by any amount of compression, or by any tension, within the limits of its tensile force.

As regards the protection given by the outer metallic casing, it has been stated, that it is only designed to protect the inner core from mechanical violence, and to confer on the cable a convenient amount of proportionate weight during the process of submergence, and that when once laid at the bottom, the rust may eat up the external coat. The Authors submit, that there can be no practical difficulty in protecting the lighter cable from mechanical violence, by giving it an outer coating of hempen cord, as has been proposed, and their investigations have led them to the conclusion, that the increase of weight due to the outer casing of the other cable is not an advantage, or convenience, but quite the contrary, inasmuch as it necessitates the application of a greatly increased resistance of the breaks, whilst being laid down. But even if weight is desirable, they would observe, that what they have termed the light system admits of its being made of any desired specific gravity; and if made as heavy as the Atlantic cable, it would still possess the advantage of having the whole of the metallic material in the centre, instead of partly in the centre, and partly in the circumference.

The relative cost of the two kinds of cable is also a matter for consideration. The Authors are informed, that on a light cable of the same power as the Atlantic cable, there would be a saving of about thirty per cent. in first cost.

In every point of view, which has fallen under their notice, the Authors feel bound to give their decided opinion in favour of the light system of cable. Whether there are any objections to it electrically, they are not prepared to say; but finding that, so long ago as 1853, a cable of this description was proposed and advocated, for the express purpose of crossing the Atlantic, curiously enough under the very title of "The Atlantic Telegraph," the Authors cannot but think, that there must exist some reason, unknown to them, why a cable more expensive, and more difficult to manage, was adopted. They venture to hope, that this Paper may be the means of eliciting some information on the subject, which cannot but be interesting to the profession at large.

In now bringing this communication to a close, the Authors desire to state, that though they have referred to the Atlantic cable and to Mr. Allan's cable, as types of two distinct systems of construction, they must disclaim any intention of imputing carelessness to those who undertook the late experiment, which so unfortunately failed. The magnitude of the operation removes it, to a great extent, beyond the pale of previous experience, and as such, those engaged in it are entitled to the sympathy of all generous minds. This they doubtless have received, and the Authors trust, that there will not be ascribed to them any disposition to cavil, or blame, when as the result of their investigation they have felt themselves compelled to express the opinion freely, that though the Atlantic cable is a step in the right direction, as compared with the heavier cables of former days, it yet falls far short, in mechanical structure, and in condition, of the light system.

Still less would the Authors be thought to deny the practicability of submerging the present Atlantic cable; on the contrary, they have no hesitation in saying, that with proper precautions, and a due attention to what is required in the construction of the paying-out apparatus, the submerging may probably be successfully accomplished; but they cannot too earnestly repeat their conviction, that with the present cable, the success of the operation will mainly depend upon the nature of the paying-out machinery and the general mechanical arrangements.

The Authors hope, that the free expression of their own opinions, and of the grounds on which they are based, will lead to a like free expression of the opinions and experience of others, in the discussion that will probably ensue; and that however imperfect may be the present treatment of the subject, it may contribute to diffuse a better knowledge of the principles, upon the following out of which the successful result of such undertakings mainly depends.

The Paper is illustrated by a series of diagrams, from which Figs. 1 to 16 are compiled; and is accompanied by the mathematical investigations, at length, of which an abstract is given in the APPENDIX.

APPENDIX.

Problem I.

Equations of motion of a body descending in a resisting medium.

Let a = the area of the horizontal section of the unit of length.
A = the volume of the body for the unit of length.
c = the coefficient of resistance, depending on the form of the body.
s = the specific gravity of the medium.
s' = the specific gravity of the sinking body.
ν = the initial velocity of the descent.
v = the velocity of the descent at any time.
t = the corresponding time.
x = the corresponding space.
g = the accelerating force of gravity = 32·2.
ε = the base of the Napierian logarithms.

Then, the accelerating force of the body $= g\dfrac{s'-s}{s'}$,

$$\text{the resistance at } v = \frac{c\,a\,s\,v^2}{2g},$$

$$\text{the retarding force} = \frac{c\,a\,s\,v^2}{2g} \times \frac{g}{A\,s'},$$

and, in the case of a cable laid horizontally, and when the diameter $= d$,

$$A = \frac{d^2\pi}{4},\ a = d,\ c = \frac{2}{3};$$

$$\text{therefore the retarding force} = \frac{c\,a\,s\,v^2}{\frac{2\,d^2\pi\,s'}{4}} = \frac{4\,s\,v^2}{3\,\pi\,d\,s'}$$

and, therefore, the actual force of the descent $= g\dfrac{s'-s}{s'} - \dfrac{4\,s\,v^2}{3\,\pi\,d\,s'}$

Also, for a sphere whose diameter $= d$,

$$A = \frac{d^3\pi}{6},\ a = \frac{d^2\pi}{4},\ c = \frac{1}{2};$$

whence, the actual force of the descent $= g\dfrac{s'-s}{s'} - \dfrac{3\,s\,v^2}{8\,d\,s'}$

Now, $v\,d\,v = f\,d\,s$, therefore, for the cable, $v\,d\,v = \left(g\dfrac{s'-s}{s'} - \dfrac{4\,s\,v^2}{3\,\pi\,d\,s'}\right)d\,x$;

and making $c = \dfrac{4\,s}{3\,\pi\,d\,s'}$ and $n = \sqrt{\dfrac{3\,g\,\pi\,d}{4}\cdot\dfrac{s'-s}{s}}$,

and integrating and correcting, by making $v = \nu$, when $x = o$,

$$v = \sqrt{n^2 - (n^2 - \nu^2)\,\varepsilon^{-2cx}} \quad \ldots\ldots\ldots\ldots \quad (1).$$

Again, $d\,t = \dfrac{d\,x}{v} = \dfrac{1}{c}\,\dfrac{d\,v}{n^2 - v^2}$,

and integrating and correcting, by making $v = \nu$, when $t = o$,

$$t = \frac{1}{2\,c\,n}\log_{\varepsilon}\left\{\frac{n+v}{n-v}\cdot\frac{n-\nu}{n+\nu}\right\} \quad \ldots\ldots\ldots \quad (2).$$

and substituting (1) in (2),

$$t = \frac{1}{2\,c\,n} \log_{\varepsilon} \left\{ \frac{n + \sqrt{n^2 - (n^2 - \nu^2)\,\varepsilon^{-2cx}}}{n - \sqrt{n^2 - (n^2 - \nu^2)\,\varepsilon^{-2cx}}} \cdot \frac{n - \nu}{n + \nu} \right\} \quad . \quad . \quad (3).$$

The equations for a sphere are similar in form, but the constants are

$$c = \frac{3\,s}{8\,d\,s'}$$

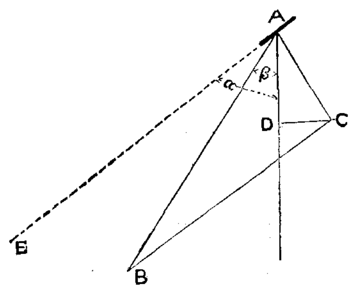
$$n = \sqrt{\frac{8\,g\,d}{3} \cdot \frac{s' - s}{s}}.$$

Cor. From (1), it appears that when x is large, $v = n$ very nearly.

Problem II.

Equations of motion of a body descending uniformly, in an oblique position, through a resisting medium.

Fig. 12.



Let A be a particle of the body.

α the angle made by A and the vertical.
β the angle made between the path of the descent A B, and the vertical A D.
v = the velocity in A B.
q = the coefficient for the resistance of the water to the unit of length, at the velocity of 1 foot per second.
q' = the coefficient for friction in the unit of length at 1 foot per second.
w = the weight of A in water.

Then, the resistance of the water in B A may be resolved into A C at right angles to A, which may be again resolved into two forces, viz.,

$$\text{Force in A D} = q\,v^2 \sin^2(\alpha - \beta) \sin\alpha \text{ up;}$$
$$\text{Force in C D} = q\,v^2 \sin^2(\alpha - \beta) \cos\alpha \text{ to left.}$$

Also friction in A $= q'\,v^2 \cos^2(\alpha - \beta)$;
which may be resolved into

$$q'\,v^2 \cos^2(\alpha - \beta) \cos\alpha \text{ vertically up;}$$
$$q'\,v^2 \cos^2(\alpha - \beta) \sin\alpha \text{ to the right.}$$

Therefore, since the motion is uniform,

$$w - q\,v^2 \sin^2(\alpha - \beta) \sin\alpha - q'\,v^2 \cos^2(\alpha - \beta) \cos\alpha = o,$$
$$q\,v^2 \sin^2(\alpha - \beta) \cos\alpha - q'\,v^2 \cos^2(\alpha - \beta) \sin\alpha = o;$$

from which equations,

$$v = \sqrt{w\left(\frac{\cos\alpha}{q'} + \frac{\sin\alpha}{q}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

$$\beta = \alpha - \tan\sqrt{\frac{q'}{q}\tan\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Cor. 1. The velocity of running out vertically, without tension, is found by making $\alpha = o$, whence $v = \sqrt{\frac{w}{q'}}$.

Cor. 2. The angle at which the cable would run out with the greatest velocity may be found by making $\sqrt{w\left(\frac{\cos\alpha}{q'} + \frac{\sin\alpha}{q}\right)}$ a maximum, from which $\tan\alpha = \frac{q'}{q}$, and $\beta = o$.

Cor. 3. By this problem may also be found the waste of cable when it runs out at any given angle (α) free from tension:

$$\text{Waste per cent.} = 100\,\frac{\sec\alpha - \tan\alpha + \tan\beta}{\sec\alpha}.$$

Cor. 4. Also, the angle of motion (β) of the end, in case of fracture, may be found, for

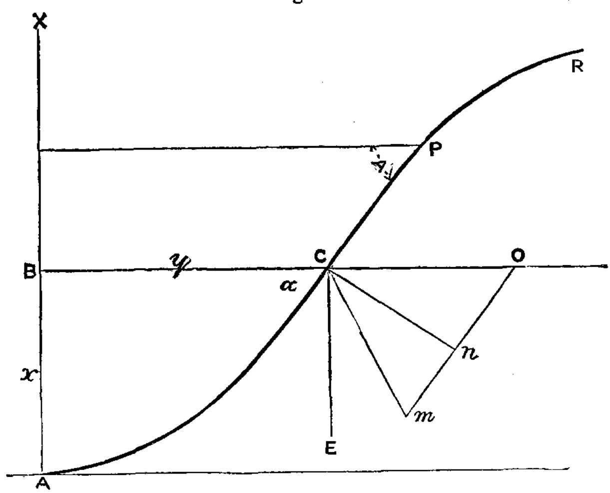
$$\tan(\alpha - \beta) = \sqrt{\frac{q'}{q}}\tan\alpha,$$

when α is given.

Problem III.

General equation to the curve assumed by the sinking cable.

Fig. 13.



Let A C P R be a portion of the curve.

x = any abscissa.
y = the corresponding ordinate.
s = the corresponding length of the curve.
α = the angle with the horizon at C.
A = the angle at any other point, P.
a = the length of the cable whose weight in water is equal to the horizontal tension at A.
t = the tension at P, in the direction of the curve towards R.

v = the velocity of the paying-out vessel.
m = the coefficient of the resistance of the water at the velocity v.
m' = the coefficient of friction at the velocity v.
w = the weight in water of the unit of length.
d = the diameter of the cable in the same unit.

Then, it may be shown, in order that the length AC should be laid without slack along the bottom, that the point C must move in a direction Cm, bisecting the angle formed at C with the horizon; and this is true of every point in the curve.

Therefore, the resistance of the water may be replaced at every point, by a force of a stream acting in the direction bisecting the angle with the horizon, and with a velocity equal to the velocity of the point in that direction.

Taking any point C,

$$\text{the velocity in } Cm = 2\, v \sin \frac{\alpha}{2},$$

$$\text{therefore, the resistance in } Cm = 4\, m \sin^2 \frac{\alpha}{2};$$

but the cable is not at right angles to Cm, as it forms an angle $= \frac{180 - \alpha}{2}$ with it; so that the resistance must be resolved into Cn and nm; the former at right angles to the curve, and the latter parallel to it, and only acting by friction. Whence,

$$\text{the resistance in } n\,C = 4\, m \sin \frac{2\,\alpha}{2} \sin^2 \frac{180 - \alpha}{2}\, d\,s = m \sin^2 \alpha\, d\,s;$$

which must be again resolved into

$m \sin^2 \alpha \cos \alpha\, d\,s$ vertically up;
$m \sin^3 \alpha\, d\,s$ horizontally to the left.

Again, the velocity in the direction $mn = 2\, v \sin^2 \frac{\alpha}{2}$, therefore the friction for the unit of length at $c = m'\, 4 \sin^4 \frac{\alpha}{2}$;

which must be resolved into (for $d\,s$)

$4\, m' \sin^4 \frac{\alpha}{2} \sin \alpha\, d\,s$ vertically up;

$4\, m' \sin^4 \frac{\alpha}{2} \cos \alpha\, d\,s$ horizontally to the right.

Also, the tension at P, or t, must be resolved into

$t \sin$ A vertically up;
$t \cos$ A horizontally to the right.

The remaining forces are

$w\,s$ vertically down;
$w\,a$ horizontally to the left.

Therefore, collecting the forces,

$$t \sin A + m \int_o^s \sin^2 \alpha \cos \alpha\, d\,s + m' \int_o^s 4 \sin^4 \frac{\alpha}{2} \sin \alpha\, d\,s - w\,s = o \quad . \ (1),$$

and

$$t \cos A - m \int_o^s \sin^3 \alpha\, d\,s + m' \int_o^s \sin^4 \frac{\alpha}{2} \cos \alpha\, d\,s - w\,a = o \quad . \ . \ . \ (2);$$

from which, by taking s up to P,

$$\frac{d\,t}{d\,s} - w \sin A + m'\, 4 \sin^4 \frac{A}{2} = o \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (3),$$

$$t \frac{dA}{ds} - w \cos A + m \sin^2 A = o \quad . \quad . \quad . \quad (4).$$

Now, $\sin A = \frac{dx}{ds}$, and by substituting in (3), integrating and correcting,

$$t = w(x + a) - m' \frac{(1 - \cos A)^2}{\sin A} x \text{ nearly} \quad . \quad . \quad . \quad (5),$$

which is the equation for tension.

Again, from (4),

$$t \frac{dA}{ds} - w \cos A + m \sin^2 A = o; \text{ but } A = \cot^{-1} \frac{dy}{dx};$$

therefore, $t \frac{dA}{ds} = - t \frac{d^2 y}{dx^2} \cdot \left(\frac{dx}{ds}\right)^3$,

and $- t \frac{d^2 y}{dx^2} - w \frac{dy}{dx} \left(\frac{ds}{dx}\right)^2 + m \frac{ds}{dx} = o$;

and writing, $p = \frac{dy}{dx} = \cot A$,

$$- t \frac{dp}{dx} - wp(1 + p^2) + m\sqrt{1 + p^2} = o;$$

and by writing $\frac{1}{q} = p$, and $\sqrt{1 + q^2} = z$ successively,

$$- \frac{dx}{t} = \frac{dz}{mz^2 - wz - m} \quad . \quad . \quad . \quad (6).$$

But (5), $t = w(x + a) - m' \frac{(1 - \cos A)^2}{\sin A} x$;

therefore

$$- \frac{dx}{t} = \frac{- dx}{w(x + a) - m' \frac{(1 - \cos A)^2}{\sin A} x} \quad . \quad . \quad . \quad (7).$$

Equating (6) and (7),

$$- \frac{dx}{w(x + a) - m' \frac{(1 - \cos A)^2}{\sin A} x} = \frac{dz}{mz^2 - wz - m}$$

and integrating and correcting, and writing for z its value $\sqrt{1 + \frac{1}{p^2}}$ it is found, finally, that

$$\frac{1}{w - m' \frac{(p - \sqrt{1 + p^2})^2}{\sqrt{1 + p^2}}} \log \frac{wa}{wa + \left\{ w - m' \frac{(p - \sqrt{1 + p^2})^2}{\sqrt{1 + p^2}} \right\} x} = \frac{1}{\sqrt{w^2 + 4m^2}}$$

$$\log \left\{ \frac{2m\sqrt{1 + \frac{1}{p^2}} - w - \sqrt{w^2 + 4m^2}}{2m\sqrt{1 + \frac{1}{p^2}} - w + \sqrt{w^2 + 4m^2}} \cdot \frac{2m - w + \sqrt{w^2 + 4m^2}}{2m - w - \sqrt{w^2 + 4m^2}} \right\} . \quad (8).$$

An equation between x and $\frac{dy}{dx} = \cot A$, from which the form of the curve may be derived.

If m' be neglected, or made $= o$, y may be obtained in terms of a series containing x, or

$$y = \frac{x}{\sqrt{\tau}} - \frac{a \cdot \frac{\tau + 2}{\tau \sqrt{\tau}} \cdot \frac{S - 1}{D + 1} \cdot \left(\frac{a}{a + x}\right)^{\sqrt{1 + \frac{4 m^2}{w^2}} - 1}}{\sqrt{1 + \frac{4 m^2}{w^2}} - 1}$$

$$- \frac{a \frac{4 \tau + 12 - \tau \theta}{2 \tau^2 \sqrt{\tau}} \cdot \left(\frac{S - 1}{D + 1}\right)^2 \left(\frac{a}{a + x}\right)^{2 \sqrt{1 + \frac{4 m^2}{w^2}} - 1}}{2 \sqrt{1 + \frac{4 m^2}{w^2}} - 1} - \text{\&c. \&c.} + C_1 \quad (9).$$

When $S = \frac{w + \sqrt{w^2 + 4 m^2}}{2 m}$, $D = \frac{- w + \sqrt{w^2 + 4 m^2}}{2 m}$,

$\tau = S^2 - 1$,
and $\theta = D^2 - 1$.

When $x =$ infinity, this gives $y = \frac{x}{\sqrt{\tau}} + C_1$, whence $\frac{d x}{d y} = \sqrt{\tau}$, or the inclination of the asymptote to the horizon $= \tan^{-1} \sqrt{\tau} = \cos^{-1} \frac{\sqrt{w^2 + 4 m^2} - w}{2 m}$ (10).

When $a = o$,

$y = \frac{x}{\sqrt{\tau}} + C_1$, which is the equation to a straight line inclined to the horizon at an angle $\cos^{-1} \frac{\sqrt{w^2 + 4 m^2} - w}{2 m}$ . . . . (11).

Which value might also have been derived from equation (8).

From this it appears, that when a cable is laid without tension at the bottom, it will descend in a straight line, inclined at the above angle to the horizon.

Problem IV.

Equation for tension.

From (5) Problem III., $t = w (x + a) - m' \frac{(1 - \cos A)^2}{\sin A} x$,
which when $a = o$ becomes

$$t = w x - m' \frac{(1 - \cos A)^2}{A} x.$$

If v is the velocity of the cable running out, and ν the velocity of the paying-out vessel, then

$$t = w x - m' \frac{\left(\frac{v}{\nu} - \cos A\right)^2}{\sin A} x \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1);$$

but m' is a function of ν, say $= q' \nu^2$,
whence

$$t = w x - q' \nu^2 \left(\frac{v}{\nu} - \cos A\right)^2 \operatorname{cosec} A \, . \, x. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

The ratio of v to ν, which is required in order to obtain any given amount of tension t', will be

$$\frac{v}{\nu} = \cos A + \sqrt{\frac{w x - t'}{q' \nu^2 x} \sin A}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3).$$

The following Tables have been calculated for two cables, the one being the Atlantic cable, and the other a cable of the same diameter, but with a specific gravity of 1·5.

Atlantic Cable.

Specific gravity = 3·489, Diameter = $\frac{5}{8}$ths of an inch, Weight of 1 foot in water = 0·3208 lbs., Depth of water = 2000 fathoms, and Tension, when vertical, = 3849·6 lbs.

		Tensions in lbs.						
Velocity of the Vessel in Feet per Second		2	4	6	8	10	12	15
Angle of Inclination to the Horizon		68° 37′	41° 44′	28° 45′	21° 47′	17° 31′	14° 38′	11° 44′
Ratio of paying out to Velocity of Vessel	1	3839	3839	3842	3844	3845	3846	3846
Ditto	$\frac{5}{4}$	3828	3806	3781	3733	3658	3550	3316
Ditto	$\frac{6}{4}$	3813	3760	3658	3481	3198	2763	1876
Ditto	$\frac{7}{4}$	3795	3690	3474	3085	2464	1556	−483
Ditto	2	3774	3599	3227	2548	1457	−143	..

Light Cable.

Specific gravity = 1·500, Diameter = $\frac{5}{8}$ths of an inch, Weight of 1 foot in water = 0·06578 lbs., Depth of water = 2000 fathoms, and Tension, when vertical = 798·9 lbs.

		Tensions in lbs.					
Velocity of the Vessel in Feet per Second		4	6	8	10	12	15
Angle of Inclination to the Horizon		19° 56′	13° 21′	10° 2′	8° 2′	6° 43′	5° 22′
Ratio of paying out to velocity of Vessel	1	798	798·2	798·3	798·4	798·5	798·6
Ditto	$\frac{9}{8}$	788	775	752	714	655	534
Ditto	$\frac{5}{4}$	769	720	633	481	253	—227
Ditto	$\frac{6}{4}$	702	514	158	—424	..	..

Problem V.

To find the waste of cable from currents.
It is shown that if

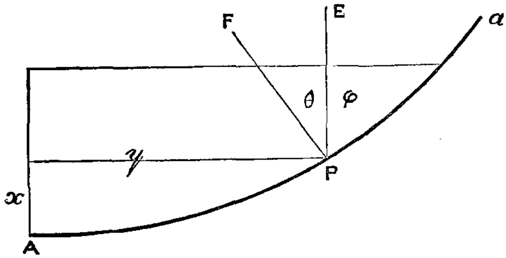
d = the depth to which the current extends,
v = the velocity of the current,
ν = the velocity of the ship,
A = the angle of inclination of the cable to the horizon, the

$$\text{waste} = \frac{d}{\nu \sin A}\left\{ \nu - \sqrt{\nu^2 - v^2} + \cos A \left(\nu - \sqrt{\nu^2 - \frac{v^2}{\cos^2 A}} \right) \right\}$$

Problem VI.

Equation to the curve assumed by a flexible line stretched across a current.

Fig. 14.



Let ϕ = the angle between the curve and the direction of the current at any point P.
θ = the angle between the direction of the current and the normal to the curve at P.
t = the tension at P.
a = the tension at A.
f = the force of the current on an unit of surface at right angles to its direction.

Then, at any point P, f may be resolved into

$f \sin^2\phi \cos\theta$ in the direction of the current.
$f \sin^2\phi \sin\theta$ perpendicular to the current.

Also t may be resolved into

$t \cos\phi$ in the direction of the current.
$t \sin\phi$ perpendicular to the current.

Therefore,

$$\int_o^s f \sin^2\phi \cos\theta \, d s - t \cos\phi = o \quad \ldots\ldots\ldots\ldots \quad (1).$$

$$\int_o^s f \sin^2\phi \sin\theta \, d s + t \sin\phi - a = o \quad \ldots\ldots\ldots\ldots \quad (2).$$

From which, as $\phi + \theta = \frac{\pi}{2}$,

$$\frac{d}{d s} = o \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

$$f \sin^2\phi \, d s = -t \, d\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

From (3) it appears that the tension is constant, and since it is $= a$ at the vertex, that must be its value at any other point;

therefore from (4) $f \sin^2\phi \, d s = a \, d x$, but $\sin\phi = \frac{d y}{d s}$,

and substituting and integrating

$$f s = a \frac{d x}{d y} + c;$$

but when $s = o, \frac{d x}{d y} = o$, therefore $c = o$,

and $\frac{a}{f s} = \frac{d y}{d x}$, which is the equation to the common catenary.

It therefore appears, that the form of the curve is the common catenary, and that the tension is uniform throughout.

Problem VII.

To find the tension due to the friction of the water in a current.

In Problem VI. the effect of friction was neglected. It may, however, easily be obtained,

If d is the depth of the current,
v the velocity of the current,
ν = the velocity of the ship,
q' = the coefficient of friction for one lineal foot of cable, at 1 foot per second,
A = the angle of the cable with the horizon ;

then, the total friction on the cable $= q' \, d \frac{v^4}{\nu^2}$ cosec A.

In the case of the Atlantic cable, if

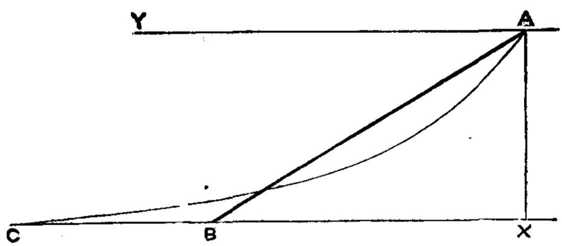
ν = 6 feet per second,
v = $1\frac{1}{2}$ foot per second,
d = 12,000 feet,
q' = ·00054778 lbs.,
A = 28° 45′;

then, the total friction = 0·1922 lbs., showing that the effect is quite insignificant.

Problem VIII.

To find the form the cable will assume, and the tension, in case the paying-out vessel and the paying out should be suddenly stopped.

Fig. 15.



Let A B be the line of the cable whilst the paying out proceeds.

And let A B = s.
A X = x, the depth of the water.
X C = the distance horizontally from X to C, the origin of the curve assumed by the cable after the stoppage.
l = the length of the curve A C, when equilibrium is established.
a = the tension at C.
t = the tension at A.
α = the angle formed by the curve A C with the horizon at A.

Then, $l = AB + BC = s + y - \sqrt{s^2 - x^2}$ . . . . (1).

But, by the equations to the catenary,

$l = t \sin \alpha$ . . . . (2).

$x = t(1 - \cos \alpha)$ . . . . (3).

$y = t \cos \alpha \log \frac{\cos \alpha}{1 - \sin \alpha}$ . . . . (4).

Equating (1) and (2)

$$t \sin \alpha = s + y - \sqrt{s^2 - x^2},$$

but (3) $t = \frac{x}{1 - \cos \alpha}$, therefore

$$\frac{x \sin \alpha}{1 - \cos \alpha} = s + y - \sqrt{s^2 - x^2}$$

$$= s - \sqrt{s^2 - x^2} + \frac{x}{1 - \cos \alpha} \cos \alpha \log \frac{\cos \alpha}{1 - \sin \alpha} \quad . \quad . \quad . \quad (5).$$

But s and x are given; therefore, from this equation α may be found, and then

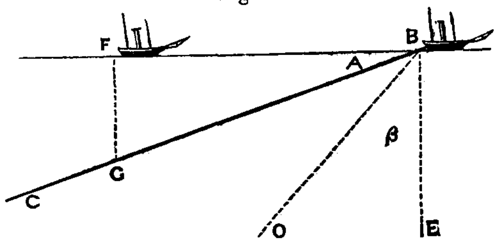
$$t = \frac{x}{1 - \cos \alpha}$$

$$y = \frac{x}{1 - \cos \alpha} \cos \alpha \log \frac{\cos \alpha}{1 - \sin \alpha}.$$

Problem IX.

To find the depth of the cable below a following vessel, at any interval after fracture.

Fig. 16.



Let B C be the line of the cable paying out,
B O, the direction of motion after fracture,
d = B F = the distance of the vessels apart.

Then F G = $d \sin A$ = the depth before fracture,
v = the velocity of the end after fracture, as obtained from Problem II.; then

$v \cos \beta$ = the vertical component of the velocity, and the depth at the end of t seconds will be

$$= d \sin A + v \cos \beta . t;$$

but in t'' the vessel has moved forward νt, and decreased the distance between it and the cable by $\nu t \sin A$; therefore the depth of the cable below F, at the end of t'',

$$= d \sin A + t (v \cos \beta - \nu \sin A).$$

The following Tables have been calculated, by means of this formula, for two cables, at various speeds of paying out.

Atlantic Cable; Velocity of the Vessel 6 Feet per Second.

Time after Fracture .		0"	10"	20"	30"	40"	50"	60"	70"	80"	90"	100"	120"	A[1]	B[1]
Distance between the Vessels .	ft. 300	144	222	299	.	.	.	.	.	.	.	.	.	13"	23"
Ditto . . .	600	288	366	443	520	.	.	.	.	.	.	.	.	27	46
Ditto . . .	900	433	510	587	664	741	.	.	.	.	.	.	.	41	70
Ditto . . .	1200	577	654	731	808	886	963	.	.	.	.	.	.	55	93

[1] Column A gives the time, in seconds, after fracture, when the end of the cable passes beneath the following vessel, the following vessel continuing her course. Column B gives the time of passing beneath the following vessel, on the supposition of the vessel being stopped at the instant of fracture.

ATLANTIC CABLE; Velocity of the Vessel 8 Feet per Second.

Time after Fracture .		0''	10''	20''	30''	40''	50''	60''	70''	80''	90''	100''	120''	A[1]	B[1]
Distance between the Vessels .	ft. 300	111	164	217	.	.	.	.	.	.	.	.	.	13''	24''
Ditto . . .	600	223	275	327	380	.	.	.	.	.	.	.	.	27	48
Ditto . . .	900	334	387	439	492	544	.	.	.	.	.	.	.	40	72
Ditto . . .	1200	445	498	550	603	656	709	.	.	.	.	.	.	54	95

LIGHT CABLE; Velocity of the Vessel 6 Feet per Second.

Time after Fracture .		0''	10''	20''	30''	40''	50''	60''	70''	80''	90''	100''	120''	A[1]	B[1]
Distance between the Vessels .	ft. 300	69	81	93	105	.	.	.	.	.	.	.	.	30''	62''
Ditto . . .	600	139	151	163	175	187	199	211	.	.	.	.	.	61	124
Ditto . . .	900	208	220	232	244	256	267	279	291	303	315	.	.	92	186
Ditto . . .	1200	277	289	301	313	325	337	349	361	373	385	397	421	122	250

LIGHT CABLE; Velocity of the Vessel 8 Feet per Second.

Time after Fracture .		0''	10''	20''	30''	40''	50''	60''	70''	80''	90''	100''	120''	A[1]	B[1]
Distance between the Vessels .	ft. 300	52	60	68	.	.	.	.	.	.	.	.	.	24''	70''
Ditto . . .	600	104	112	120	128	136	144	.	.	.	.	.	.	48	140
Ditto . . .	900	157	165	173	181	189	197	205	213	.	.	.	.	72	210
Ditto . . .	1200	209	217	225	233	241	249	257	265	273	281	289	.	96	280

LIGHT CABLE; Velocity of the Vessel 10 Feet per Second.

Time after Fracture .		0''	10''	20''	30''	40''	50''	60''	70''	80''	90''	100''	120''	A[1]	B[1]
Distance between the Vessels .	ft. 300	42	47	53	.	.	.	.	.	.	.	.	.	21''	79''
Ditto . . .	600	84	90	95	101	106	.	.	.	.	.	.	.	43	158
Ditto . . .	900	126	132	137	143	148	153	159	.	.	.	.	.	64	237
Ditto . . .	1200	168	173	179	185	190	196	201	207	213	218	.	.	85	316

[1] Column A gives the time, in seconds, after fracture, when the end of the cable passes beneath the following vessel, the following vessel continuing her course. Column B gives the time of passing beneath the following vessel, on the supposition of the vessel being stopped at the instant of fracture.

Problem X.

To find the extension of length due to the compression of the inner core.

If d = the original diameter from centre to centre of the outer strands,
δ = the decrease due to compression,
l = the length of the cable in which the strands make an entire turn,
L = the length of the strand making an entire turn,

then $l^2 = L^2 - \overline{\pi d}|^2$;
and if d becomes $d - \delta$, and l' the new value of l,

$$l'^2 = L^2 - \overline{\pi d}|^2 + 2\pi^2 d\delta - \pi^2\delta^2,$$
$$l' - l = \sqrt{l^2 + 2\pi^2 d\delta - \pi^2\delta^2} - l;$$

and expanding the part under the radical sign,

$$l' - l = \frac{\pi^2(2d\delta - \delta^2)}{2l} \text{ nearly,}$$
$$= \frac{\pi^2 \delta d}{l} \text{ nearly;}$$

$$\text{therefore } \frac{l' - l}{l} = \frac{\pi^2 d\delta}{l^2}.$$

Problem XI.

To find the variation of tension due to the motion caused by waves.

Let R = the normal tension due to the depth at the mean level, as given by Problem IV.,
T = the tension at any time t,
t = the time reckoned from the moment when the vessel is at the mean level,
t' = the total time of transit of a wave under the ship,
θ = the additional angular rotation caused by the rise and fall of the ship, and which may be either positive or negative,
x = the height of the ship above the mean level at t,
r = the radius of the sheaves,
ϱ = the radius of gyration of the sheaves,
g = the accelerating force of gravity = 32·2,
h = the total height of the wave, and
w = the weight of the rotative machinery.

Since the moments of the impressed forces are equal to the moments of the effective forces,

$$(T - R)r = \Sigma m z^2 \frac{d^2\theta}{dt^2}, \text{ but } \Sigma m z^2 = \frac{w}{g}\cdot\varrho^2;$$

$$\text{therefore } (T - R)\,r = \frac{w}{g}\varrho^2\frac{d^2\theta}{dt^2}, \text{ and } \frac{d^2\theta}{dt^2} = \frac{(T - R)\,r\,.\,g}{w\varrho^2} \quad . \; . \; . \; . \quad (1).$$

Now if A is the angle of paying out at the mean level, it is shown that, approximately, the extra length paid out whilst the vessel rises to x, is $x \sin A$; therefore

$$x \sin A = r\theta,$$
$$\text{and } r\frac{d^2\theta}{dt^2} = \sin A\frac{d^2x}{dt^2}, \text{ and } \frac{d^2\theta}{dt^2} = \frac{\sin A}{r}\cdot\frac{d^2x}{dt^2}.$$

Hence, from (1),

$$T - R = \frac{w \sin A\,\varrho^2}{g r^2}\cdot\frac{d^2x}{dt^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

It is now necessary to find, or assume, x as some function of t, and it will probably be tolerably near the truth, if it is assumed that

$$x = \frac{h}{2} \sin \left(2 \pi \frac{t}{t'}\right),$$

$$\text{whence } \frac{d x}{d t} = \frac{\pi h}{t'} \cos \left(2 \pi \frac{t}{t'}\right),$$

$$\frac{d^2 x}{d t^2} = - \frac{2 \pi^2 h}{t'^2} \sin \left(2 \pi \frac{t}{t'}\right);$$

therefore, from (2),

$$T - R = - \frac{2 w \sin A \varrho^2 \pi^2 h}{g r^2 t'^2} \sin \left(2 \pi \frac{t}{t'}\right);$$

or, writing $m = \frac{2 w \sin A \varrho^2 \pi^2 h}{g r^2 t'^2}$,

$$T = R - m \sin \left(2 \pi \frac{t}{t'}\right) \quad \ldots \ldots \ldots \ldots \ldots (3).$$

Hence the variation of tension is $= 2 m$, because $\sin \left(2 \pi \frac{t}{t'}\right)$ may be either $+ 1$ or $- 1$.

From this the following corresponding values may be derived:—

t	x	Velocity of the Vessel Vertically.	Accelerating Force.
0	0	$\frac{\pi h}{t'}$	0
$\frac{t'}{4}$	$\frac{h}{2}$	0	$- \frac{2 \pi^2 h}{t'^2}$
$\frac{t'}{2}$	0	$- \frac{\pi h}{t'}$	0
$\frac{3 t'}{4}$	$- \frac{h}{2}$	0	$\frac{2 \pi^2 h}{t^2}$
t'	0	$\frac{\pi h}{t}$	0