$\alpha, \beta, \gamma, \delta, \epsilon, \rho, \sigma$ connected by five equations. The equivalence of the two sets of formulæ may be shown without difficulty.

To the Table 2 of the Quintic Equations, given in the paper, may be added the following result from Legendre's "Théorie des Nombres," Ed. 3, t. ii., p. 213,

p	η^5	η^4	η^3	η^2	η	1	
641	1	+1	−256	−564	+5238	−5120	= 0,

calculated by him for the isolated case $p = 641$.

On the Theory of Matrices. *By* Mr. A. Buchheim, M.A.

[*Read Nov.* 13*th*, 1884.]

Introduction.

The methods used in the following paper are essentially, though not historically, an extension of Hamilton's theory of the linear function of a vector, and the simplest way to connect Grassmann's methods with the theory created by Cayley and Sylvester will be to connect them both with Hamilton's investigations.

It is, or ought to be, well known that the linear and vector function of a vector is simply the matrix of the third order. This is obvious from the definition: for, if ρ is any vector, $\sigma = \phi\rho$ is a vector whose constituents are linear functions of ρ's constituents; that is, if

$$\rho = xi + yj + zk, \quad \sigma = x'i + y'j + z'k,$$

we must have the three equations

$$x' = ax + a'y + a''z,$$

$$y' = bx + b'y + b''z,$$

$$z' = cx + c'y + c''z,$$

that is,
$$(x'y'z') = \left(\begin{array}{ccc} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{array}\right)(x, y, z) \quad \text{......................(A).}$$

That is to say, it is the same thing whether we say that $\sigma = \phi\rho$, or that the constituents of σ are obtained from those of ρ by operating on them with a certain matrix; and we see that in this sense we can identify ϕ with the matrix, and we can say that

$$\phi = \begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix} \quad \text{.........................(B).}$$

Now, in (A), let $\rho = i$, that is, let $(xyz) = (100)$; then

$$(x'y'z') = (abc),$$

that is,

$$\sigma = ai + bj + ck,$$

or say,

$$\phi i = ai + bj + ck = a.$$

In the same way, we get

$$\phi j = a'i + b'j + c'k = a',$$

$$\phi k = a''i + b''j + c''k = a''.$$

And then

$$\begin{aligned} \phi\rho = \phi(xi + yj + zk) = & \ (ax + a'y + a''z)\, i \\ & + (bx + b'y + b''z)\, j \\ & + (cx + c'y + c''z)\, k \\ = & \ x(ai + bj + ck) \\ & + y(a'i + b'j + c'k) \\ & + z(a''i + b''j + c''k) \\ = & \ xa + ya' + za''. \end{aligned}$$

And we can say that (the linear function or matrix) ϕ changes i, j, k into three given vectors a, a', a'', and changes any other vector $xi + yj + zk$ into $xa + ya' + za''$.

Now, on looking at what precedes, it will at once be obvious that we have used none of the special properties of i, j, k: so far as our work is concerned, they might have been any three vectors, provided only that every vector could be expressed in terms of them; and if we call three such vectors asyzygetic, and change the notation, we can say that a linear function, or matrix, changes three given asyzygetic vectors α, β, γ into three given vectors α', β', γ', and changes any vector $x\alpha + y\beta + z\gamma$ into $x\alpha' + y\beta' + z\gamma'$. As regards the word "asyzygetic," I remark that any vector can be expressed in terms of $\alpha\beta\gamma$, provided $S\alpha\beta\gamma$ does not vanish; and we know that $S\alpha\beta\gamma = 0$ is the necessary and sufficient condition that we may have a relation $\lambda\alpha + \mu\beta + \nu\gamma = 0$, where λ, μ, ν are scalars: it is better to use this as a definition of asyzygetic vectors; viz., three vectors are asyzygetic if they are not connected by a linear relation with scalar coefficients.

If we use the notation of the paper, we can write

$$\phi = \frac{\alpha', \beta', \gamma'}{\alpha, \beta, \gamma},$$

$$\phi\,(x\alpha+y\beta+z\gamma) = x\alpha'+y\beta'+z\gamma'.$$

If $$(\alpha'\beta'\gamma') = \left(\begin{array}{ccc} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{array}\right)(\alpha\beta\gamma),$$

$$\phi = \left(\begin{array}{ccc} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{array}\right).$$

Before passing on to matrices of any order, I shall give a simple application of the method as an example. I choose the proof of the identical equation (Hamilton's Symbolic Cubic).

It is known (*cf.* Hamilton's *Elements*, § 353, *seq.*) that for any matrix ϕ there are in general three scalars λ, μ, ν, and three vectors α, β, γ, respectively, such that

$$\left.\begin{array}{lll} \phi\alpha = \lambda\alpha & \text{or} & (\phi-\lambda)\,\alpha = 0 \\ \phi\beta = \mu\beta & \text{or} & (\phi-\mu)\,\beta = \\ \phi\gamma = \nu\gamma & \text{or} & (\phi-\nu)\,\gamma = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{C})$$

and that the three vectors α, β, γ are asyzygetic. Let $\rho = x\alpha+y\beta+z\gamma$ be any vector; then

$$\begin{aligned}(\phi-\lambda)\,\rho &= x\,(\phi-\lambda)\,\alpha+y\,(\phi-\lambda)\,\beta+z\,(\phi-\lambda)\,\gamma \\ &= y\,(\phi-\lambda)\,\beta+z\,(\phi-\lambda)\,\gamma, \text{ by (C)}, \\ (\phi-\mu)(\phi-\lambda)\,\rho &= y\,(\phi-\mu)(\phi-\lambda)\,\beta+z\,(\phi-\mu)(\phi-\lambda)\,\gamma \\ &= y\,(\phi-\lambda)(\phi-\mu)\,\beta+z\,(\phi-\lambda)(\phi-\mu)\,\gamma \\ &= z\,(\phi-\lambda)(\phi-\mu)\,\gamma, \text{ by (C)}, \\ (\phi-\nu)(\phi-\mu)(\phi-\lambda)\,\rho &= z\,(\phi-\lambda)(\phi-\mu)(\phi-\nu)\,\gamma \\ &= 0, \text{ by (C)}. \end{aligned}$$

That is, $(\phi-\lambda)(\phi-\mu)(\phi-\nu)\,\rho$ always vanishes; that is,

$$(\phi-\lambda)(\phi-\mu)(\phi-\nu) = 0.*$$

* This result might, of course, have been obtained in one step, and the general theorem is so obtained in the paper. I have preferred the longer form of the proof because it seemed to show the principle involved more clearly.

We have now to extend this theory to matrices of higher orders. It is fairly obvious that, in the case of matrices of the third order, the success of the method depends on the fact that for three variables (x, y, z) we are able to substitute a single vector $(x\alpha+y\beta+z\gamma)$; and the only property of the vector that we have used is the following:

If $x\alpha+y\beta+z\gamma = x'\alpha+y'\beta+z'\gamma$ (α, β, γ being asyzygetic), then

$$x = x', \quad y = y', \quad z = z'.$$

Now, to extend this to sets of more than three letters, take n *units* $e_1, e_2, e_3, \dots e_n$ (we are not at present concerned with their meaning); and in place of the set of n letters $x_1, x_2, \dots x_n$ consider the *point*

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 + \dots + x_n e_n,$$

and stipulate as before that

$$x_1 e_1 + x_2 e_2 + x_3 e_3 + \dots + x_n e_n = y_1 e_1 + y_2 e_2 + y_3 e_3 + \dots + y_n e_n,$$

say $$x = y,$$

shall *mean* $$x_1 = y_1, \; x_2 = y_2 \dots x_n = y_n.$$

Then we have, for instance,

$$\lambda x + \mu y = (\lambda x_1 + \mu y_1)\, e_1 + \dots + (\lambda x_n + \mu y_n)\, e_n,$$

where λ, μ are scalars.

We now require the theorem,—Every point can be linearly expressed in terms of any n asyzygetic points. Passing over the word *asyzygetic* for the present, it is easy to see the meaning of the theorem, and to convince oneself of its truth. Let x be any point, and let $\alpha, \beta, \gamma \dots$ be n given points; then we are to have

$$x = \lambda\alpha + \mu\beta + \nu\gamma + \dots \quad \dots\dots\dots\dots (d),$$

$\lambda, \mu, \nu \dots$ being scalars, that is

$$\begin{aligned} x_1 e_1 + x_2 e_2 + \dots + x_n e_n = & \;\lambda\,(\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n) \\ & + \mu\,(\beta_1 e_1 + \beta_2 e_2 + \dots + \beta_n e_n) \\ & + \nu\,(\gamma_1 e_1 + \gamma_2 e_2 + \dots + \gamma_n e_n) \\ & + \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ = & \;\; e_1\,(\lambda\alpha_1 + \mu\beta_1 + \nu\gamma_1 + \dots) \\ & + e_2\,(\lambda\alpha_2 + \mu\beta_2 + \nu\gamma_2 + \dots) \\ & + \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

That is, we are to have

$$\left.\begin{array}{l} x_1 = \lambda\alpha_1+\mu\beta_1+\nu\gamma_1+\ldots \\ x_2 = \lambda\alpha_2+\mu\beta_2+\nu\gamma_2+\ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{D}).$$

Now we know that these equations determine λ, μ, $\nu\ldots$, if, and only if,

$$\Delta = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \ldots\ldots \\ \alpha_2 & \beta_2 & \gamma_2 & \ldots\ldots \\ \vdots & \vdots & \vdots & \ldots\ldots \\ \alpha_n & \beta_n & \gamma_n & \ldots\ldots \end{vmatrix}$$

does not vanish. And therefore, if we say that the points α, β, $\gamma\ldots$ are asyzygetic if $\Delta \gtrless 0$, the theorem is proved, and we have also a definition of asyzygetic points. But we can get a better definition: for we know that $\Delta = 0$ is the necessary and sufficient condition that we may be able to solve (D) after putting $x_1 = x_2 \ldots = x_n = 0$; and therefore, if we go back to the equation (d) from which (D) was derived, and write, as we obviously may,

$$0 = 0e_1+0e_2+\ldots\, 0e_n,$$

we see that n points α, β, $\gamma\ldots$ are *not* asyzygetic if it is possible to satisfy a relation of the form

$$0 = \lambda\alpha+\mu\beta+\nu\gamma+\ldots,$$

or, say, if they are connected by a linear relation with scalar coefficients; or, in other words, n points are asyzygetic if they are *not* connected by a linear relation with scalar coefficients. This is the sense in which the word is used in the paper.

Now, suppose we have taken n asyzygetic points $e_1, e_2, \ldots e_n$, and have expressed everything in terms of them, and consider the transformation

$$(y_1, y_2, \ldots y_n) = \left(\begin{array}{cccc} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{array}\right)(x_1, x_2, \ldots x_n),$$

write ϕ to denote the matrix $\| a_{ik} \|$, and denote the transformation by $y = \phi x$.

Now, take $x = e_1$, that is, take

$$(x_1, x_2, \ldots x\ ) = (1, 0, \ldots 0);$$

then we get $(y_1, y_2 \dots y_n) = (a_{11}, a_{21} \dots a_{n1})$,

or $y = \phi e_1 = a_{11} e_1 + a_{21} e_2 + \dots + a_{n1} e_1 = \mathfrak{a}_1$,*

similarly $\phi e_2 = a_{12} e_1 + a_{22} e_2 + \dots + a_{n2} e_2 = \mathfrak{a}_2$.

$\dots \quad \dots \quad \dots \quad \dots \quad \dots$

Moreover,

$$\begin{aligned} \phi x = \; & e_1 (a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n) \\ & + e_2 (a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n) \\ & + \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & + e_n (a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n) \\ = \; & x_1 (a_{11} e_1 + a_{21} e_2 + \dots + a_{n1} e_n) \\ & + x_2 (a_{12} e_1 + a_{22} e_2 + \dots + a_{n2} e_n) \\ & + \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & + x_n (a_{1n} e_1 + a_{2n} e_2 + \dots + a_{nn} e_n) \\ = \; & x_1 \mathfrak{a}_1 + x_2 \mathfrak{a}_2 + \dots + x_n \mathfrak{a}_n. \end{aligned}$$

And we see that we can say that the matrix ϕ changes the points of reference, $e_1, e_2 \dots e_n$ into n given points $\mathfrak{a}_1, \mathfrak{a}_2 \dots \mathfrak{a}_n$, and then changes any other point $(x_1 e_1 + x_2 e_2 + \dots + x_n e_n)$ into $x_1 \mathfrak{a}_1 + x_2 \mathfrak{a}_2 + \dots + x_n \mathfrak{a}_n$. This is the definition of the matrix used in the paper; the relation between $\mathfrak{a}_1$, &c. on the one hand, and the matrix on the other, will be made clear by the following set of equations:

$$(y_1 y_2 \dots y_n) = \left(\begin{array}{cccc} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots\dots & a_{nn} \end{array} \right\rangle\!\!\!\!\big(x_1, x_2 \dots x_n),$$

$$y = \phi x,$$

$$\mathfrak{a}_i = \phi e_i,$$

$$(\mathfrak{a}_1, \mathfrak{a}_2 \dots \mathfrak{a}_n) = \left(\begin{array}{cccc} a_{11} & a_{21} & \dots\dots & a_{n1} \\ a_{12} & a_{22} & \dots\dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots\dots & a_{nn} \end{array} \right\rangle\!\!\!\!\big(e_1, e_2 \dots e_n),$$

* In strict analogy with the rest of the notation, a_1 should of course denote $a_{11} e_1 + a_{12} e_2 + \dots + a_{1n} e_n$; but this inconsistency is unavoidable if we are to keep to the ordinary conventions for matrices. I do not think it need cause any confusion; I have tried to guard against it by using $\mathfrak{a}_1$ instead of a_1.

$$\phi x = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n,$$

$$\phi = \frac{a_1,\ a_2\ \ldots\ a_n}{e_1,\ e_2\ \ldots\ e_n}.$$

It remains to add a few words on the multiplication of points. The laws of the multiplication of all points depend on the laws assumed for the *units;* the law assumed by Grassmann is that known as *polar* multiplication; viz., we have $ab = -ba$, $a^2 = 0$, for the original *units* of reference, and then this law holds for all points.* From this, and the associative law, it follows that any product of points vanishes if a point is repeated. We can use this theorem to interpret the products of points. In all that follows, I use geometrical language. The point x is supposed to be the point in a space of $(n-1)$ dimensions, having as its *homogeneous* (multiplanar) coordinates $(x_1, x_2 \ldots x_n)$; and then we can use the following definitions: let α, β be two points, then, if λ is a variable scalar, the point $\alpha + \lambda\beta$ moves on the straight line $\alpha\beta$; if λ, μ are two variable scalars, the point $\alpha + \lambda\beta + \mu\gamma$ moves in the plane $\alpha\beta\gamma$; if λ, μ, ν are scalars, the point $\alpha + \lambda\beta + \mu\gamma + \nu\delta$ moves in the linear space (*three-point*) $\alpha\beta\gamma\delta$; and generally, if $\lambda_1, \lambda_2 \ldots \lambda_{r-1}$ are scalars, the point

$$\Lambda = \alpha + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \ldots + \lambda_{r-1} \alpha_{r-1}$$

moves in the r-point $(\alpha, \alpha_1 \ldots \alpha_{r-1})$; since Λ can have a ∞^{r-1} series of positions, depending linearly on $(r-1)$ parameters, it is obvious that an r-point is the same as what Clifford calls an $(r-1)$-flat.

I shall follow Grassmann in enclosing all polar products in square brackets. We have to interpret the product $[\alpha\beta]$: we have

$$[(\alpha + \lambda\beta)(\alpha + \lambda'\beta)] = [\alpha\alpha] + \lambda'[\alpha\beta] + \lambda[\beta\alpha] + \lambda\lambda'[\beta\beta]$$
$$= (\lambda' - \lambda)[\alpha\beta].$$

For $$[\alpha\alpha] = [\beta\beta] = 0, \text{ and } [\beta\alpha] = -[\alpha\beta].$$

Therefore the product is unaltered, to a factor *près*, if for α, β we substitute any two points of the straight line $\alpha\beta$; and it will be altered if we substitute any point not on the straight line (this can easily be verified); thus we see that the product appertains to the straight line, and defines it; we may therefore say that $[\alpha\beta]$ *is* the straight line $\alpha\beta$.† Moreover, we see that

$$[\alpha\beta(\alpha + \lambda\beta)] = [\alpha\beta\alpha] + \lambda[\alpha\beta\beta]$$
$$= 0.$$

* This law and the commutative ($ab = ba$) law are the only laws for which this is true; this is proved by Grassmann in his *Ausdehnungslehre.*

† *Cf. Proc. Lond. Math. Soc.*, Vol. XIV., p. 84.

Therefore, the product of two points is the line joining them, and the product of three collinear points vanishes.

In precisely the same way, we can show that the product of three points is the plane containing them, and that the product of four complanar points vanishes; and, generally, the product of r points is the *r-point* determined by them, and the product of $(r+1)$ points contained in the same r-point vanishes.

This last theorem can be put in another form. Suppose the $(r+1)$ points $a_1, a_2 \dots a_{r+1}$ to be in the same r-point; then, since a_{r+1} is in the r-point $(a_1, a_2 \dots a_r)$, we have, by definition,

$$a_{r+1} = \lambda_1 a_1 + \lambda_2 a_2 \dots + \lambda_r a_r.$$

That is, the $(r+1)$ points a are connected by a linear relation, that is, they are not asyzygetic, and, writing r for $r+1$, we can say that the product of r asyzygetic points is the r-point determined by them: if the points are not asyzygetic, the product vanishes. Moreover, it can be proved that, if r points are not asyzygetic, their product will not vanish, and we have, therefore, the important theorem: the necessary and sufficient condition for the existence of a linear relation

$$\overset{r}{\underset{1}{\Sigma}} \lambda_i a = 0,$$

connecting r points, is $[a_1 a_2 \dots a\ ] = 0.$

Lastly, I have to remark that the product of n points $x_1, x_2 \dots x_n$ is

$$\text{Det} \mid x_{ik} \mid [e_1, e_2 \dots e_n];$$

and that $[e_1, e_2 \dots e_n]$ can always be supposed equal to unity.

On p. 241 of the *Ausdehnungslehre* of 1862, Grassmann defines a certain operator, which he calls a quotient: this operator transforms n given points of a space of $(n-1)$ dimensions into n other given points, and then transforms any $(n+1)^{\text{th}}$ point into a determinate point. This operator is, in fact, the general matrix of the n^{th} order; the object of the present paper is to treat the theory of matrices from Grassmann's point of view.* It will be seen that some important parts of the theory are considerably simplified by this treatment. It is hardly necessary to point out that there is not a new theorem in the paper, and that its existence can only be justified, if at all, by the

* *Cf.* Clifford: "A Fragment on Matrices," *Math. Papers*, 337.

methods employed. The language and notations of the paper have been explained in the introduction.

1. Take n asyzygetic points, $e_1, e_2 \dots e_n$, and n points corresponding to them, $a_1, a_2 \dots a_n$; then a matrix ϕ of the n^{th} order is defined as an operator, such that $$\phi e_i = a_i,$$

and that $$\phi \Sigma c_i e_i = \Sigma c_i \phi e_i = \Sigma c_i a_i,$$

the c being scalars; this matrix can be conveniently written as a fraction $$\phi = \frac{a_1, a_2 \dots a_n}{e_1, e_2 \dots e_n},$$

or, more simply, $$\phi = \frac{a_i}{e_i}.$$

We may, if we please, make this notation more definite by adopting a notation of Prof. Cayley's,* and writing

$$\phi = \frac{a_i \rfloor}{\lceil e_i}.$$

Another form is also convenient, and is, in fact, the usual form; let $a_i = \Sigma a_{ji} e_i$, then we write

$$\phi = \left(\begin{array}{llll} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \end{array}\right),$$

viz., we have

$$(a_1, a_2, a_3 \dots) = \left(\begin{array}{llll} a_{11} & a_{21} & a_{31} & \dots \\ a_{12} & a_{22} & a_{32} & \dots \\ a_{13} & a_{23} & a_{33} & \dots \\ \dots & \dots & \dots & \end{array}\right) (e_1, e_2, e_3 \dots),$$

and then $$\phi = \frac{a_i}{e_i}$$

has the form just given.

2. Two matrices, ϕ, ϕ', are said to be equal if $\phi x = \phi' x$, whatever x

---

* $\frac{a \rfloor}{\lceil b} b = a, \quad b \frac{\lfloor a}{b \rceil} = a.$

may be; if e_i' is any set of n asyzygetic points, $\phi = \phi'$ if $\phi e_i' = \phi' e_i'$; for we can express x in the form $\Sigma x_i' e_i'$, and then we have

$$\begin{aligned} \phi x &= \Sigma x_i' \phi e_i' \\ &= \Sigma x_i' \phi' e_i' \\ &= \phi' x. \end{aligned}$$

Hence we can prove that, if $e_i' = \Sigma c_{ij} e_j$ is any asyzygetic set,*

$$\phi = \frac{a_i}{e_i} = \frac{\Sigma c_{ij} a_j}{e_i'},$$

for $$\phi e_i' = \Sigma c_{ij} \phi e_j = \Sigma c_{ij} a_j.$$

Lastly, if $\phi' e_i = \lambda \phi e_i$, where λ is a scalar, we obviously have generally $\phi' x = \lambda \phi x$, or $\phi' = \lambda \phi$, and, if $\phi e_i = \lambda e_i$, $\phi = \lambda$.

5. If we have $$\phi = \frac{a_i}{e_i},$$

$$\chi = \frac{b_i}{a_i},$$

we define the product $\chi\phi$ by the equation

$$\chi\phi = \frac{b_i}{e_i},$$

that is, $$\chi\phi = \frac{b_i}{|a_i} \cdot \frac{a_i}{|e_i} = \frac{b_i}{|e_i}.$$

This product need obviously not be commutative. I proceed to show that it is associative. Let

$$\phi = \frac{a_i}{e_i}, \quad \phi' = \frac{a_i'}{a_i}, \quad \phi'' = \frac{a_i''}{a_i'}.$$

Then $$(\phi''\phi')\,\phi = \left(\frac{a_i''}{a_i'} \cdot \frac{a_i'}{a_i}\right) \frac{a_i}{e_i} = \frac{a_i''}{a_i} \cdot \frac{a_i}{e_i} = \frac{a_i''}{e_i},$$

$$\phi''(\phi'\phi) = \frac{a_i''}{a_i'} \left(\frac{a_i'}{a_i} \cdot \frac{a_i}{e_i}\right) = \frac{a_i''}{a_i'} \cdot \frac{a_i'}{e_i} = \frac{a_i''}{e_i}.$$

and therefore the product is associative.

The following formula is important, but no use is made of it in this paper.†

Let $$\phi = \frac{\Sigma a_{ji} e_j}{e_i}, \quad \phi' = \frac{\Sigma b_{ji} e_j}{e_i}.$$

* *Set* means *set of n points.*

† It is, in fact, the ordinary formula for the multiplication of two matrices.

Then
$$\phi'\phi = \frac{\sum_j e_j \left(\sum_i a_{ik} b_{ji}\right)}{e_k}.$$

We have
$$\phi = \frac{\sum_j b_{ji} e_j}{e_i}$$
$$= \frac{\sum_i a_{ik} \sum_j b_{ji} e_j}{\sum_i a_{ik} e_i} \qquad (k = 1 \dots n)$$
$$= \frac{\sum_j e_j \sum_i a_{ik} b_{ji}}{a_k} \qquad (k = 1 \dots n).$$

Therefore
$$\phi'\phi = \frac{\sum_j e_j \left(\sum_i a_{ik} b_{ji}\right)}{e_k} \qquad (k = 1 \dots n).$$

6. We have now to consider the following problem: Given a matrix ϕ, to find a scalar λ, and a point x, such that
$$\phi x = \lambda x,$$
or
$$(\phi - \lambda) x = 0.$$
Let
$$x = \sum x_i e_i$$
be the required point, then we have
$$0 = (\phi - \lambda) x = \sum x_i (\phi - \lambda) e_i \text{.....................} (A).$$
That is, the n points $(\phi - \lambda) e_i$ are asyzygetic; and their product therefore vanishes, that is, λ must satisfy the equation
$$\left[\prod_i (\phi - \lambda) e_i\right] = 0 \text{ .... } (B).$$
If this equation be multiplied out, we get an expression $f\lambda\,[e_1 \dots e_n]$, and, as the second factor does not vanish, λ must be a root of $f\lambda = 0$; and then the x_i are obtained (by solving a set of linear equations) as the coefficients of the syzygy (A). If there are s unequal roots of the equation $f\lambda = 0$, we obviously get s points x, one such point appertaining to each root: in particular, if the n roots are all unequal, we get n points. It is possible, however, in every case to get n points appertaining in groups to the different roots of $f\lambda = 0$. This I proceed to show.*

* If $\phi = \| a_{ik} \|$, $(\phi - \lambda) e_1 = (a_{11} - \lambda) e_1 + a_{21} e_2 + \dots$ and if we write down the corresponding expressions for $(\phi - \lambda) e_2$, &c., and use the theorem given at the end of the introduction, we shall get (B) in the form $f\lambda\,[e_1 - e_n]$, and it will be seen that $f\lambda = 0$, the well-known determinant equation giving the latent roots.

The whole investigation depends on the fact that, since we might have expressed x in terms of any n asyzygetic points, we can substitute any n asyzygetic points for the e_i, in (B).

Let λ_1 be any root of (B), let $(\phi-\lambda_1)\,e_i = e_i'$; then we have, by (B),

$$[e_1' e_2' \dots e_n'] = 0.$$

It follows, from this, that we have at least one linear relation connecting the e_i'; but there may be more. Let there be r relations,

$$\sum_j A_{ij}\, e_j' = 0 \quad (i = 1, 2 \dots r) \quad \dots\dots\dots\dots(\text{C}),$$

where $$e_j' = (\phi-\lambda_1)\, e_j.$$

Let $$\Sigma A_{ij}\, e_j = a_i.$$

Since the r relations (c) are asyzygetic, by hypothesis, it follows that the r points a are asyzygetic, for, if they were not, and we had $\Sigma\mu_i a_i = 0$, we should, by operating with $\phi-\lambda_1$, get a relation connecting the equations (C).

It follows that we can substitute the points a for r of the e: suppose we substitute them for $e_1 \dots e_r$; then (B) becomes

$$[\bar{a}_1\, \bar{a}_2 \dots \bar{a}_r\, \bar{e}_{r+1} \dots \bar{e}_n] = 0,$$

if $$\bar{a}_i = (\phi-\lambda)\, a_i.$$

But (C) gives $$(\phi-\lambda_1)\, a_i = 0,$$

or $$\phi a_i = \lambda_1 a_i,$$

and therefore $$(\phi-\lambda)\, a_i = (\lambda_1-\lambda)\, a_i,$$

and (B) becomes $$(\lambda_1-\lambda)^r\, [a_1 a_2 \dots a_r\, \bar{e}_{r+1} \dots \bar{e}_n] = 0 \dots\dots\dots\dots\dots\dots(\text{B}).$$

Therefore, if there are asyzygetic relations (C), there are r points a, such that $(\phi-\lambda_1)\, a_i = 0$, and λ_1 is an r-tuple root, at least, of (B). But the multiplicity of λ_1 may be greater than r; if it is, we must

have $$[a_1 a_2 \dots a_r\, e_{r+1}' \dots e_n'] = 0,$$

$$e_i' = (\phi-\lambda_1)\, e_i.$$

Suppose, as before, that there are s asyzygetic relations

$$\sum_1^r B_{ij}\, a_j - \sum_{r+1}^n B_{ij}\, e_j' = 0 \quad (i = 1 \dots s) \quad \dots\dots\dots\dots(\text{C}').$$

Then all the coefficients $B_{i(r+1)} \dots B_{in}$ cannot vanish, since the a are asyzygetic, and we can take

$$b_i = \sum_{r+1}^{n} B_{ij} e_j \quad (i = 1 \dots s),$$

and substitute the s points b in place of, say, $e_{r+1} \dots e_{r+s}$. We have now to see what (B) becomes. In the first place, we get

$$(\lambda_1 - \lambda)^r [a_1 \dots a_r \bar{b}_1 \dots \bar{b}_s \bar{e}_{r+s+1} \dots \bar{e}_n] = 0 \dots\dots\dots\dots (B'),$$

if

$$\bar{b}_i = (\phi - \lambda) b_i.$$

Now (C′) gives

$$\Sigma B_{ij} a_j - (\phi - \lambda_1) b_i = 0,$$

for

$$\Sigma B_{ij} e'_j = (\phi - \lambda_1) b_i.$$

Therefore

$$\phi b_i = \lambda_1 b_i + \Sigma B_{ij} a_j.$$

Therefore

$$(\phi - \lambda) b_i = (\lambda_1 - \lambda) b_i + \Sigma B_{ij} a_j.$$

Therefore

$$[a_1 \dots a_r \bar{b}_1 \dots \bar{b}_s] = [a_1 \dots a_r] \prod_{i=1}^{i=s} [(\lambda_1 - \lambda) b_i + \Sigma B_{ij} a_j]$$

$$= [a_1 \dots a_r] \Pi (\lambda_1 - \lambda) b_i^{*}$$

$$= (\lambda_1 - \lambda)^s [a_1 \dots a_r b_1 \dots b_s],$$

and (B′) becomes

$$(\lambda_1 - \lambda)^{r+s} [a_1 \dots a_r b_1 \dots b_s \bar{e}_{r+s+1} \dots \bar{e}_n] = 0 \dots\dots\dots\dots (B'').$$

It is obvious how we might go on if the multiplicity of λ_1 were greater than $r+s$; we should get t points c, such that

$$(\phi - \lambda_1) c_i = \Sigma c_{ij} a_j + \Sigma c'_{ij} b_j,$$

and then (B″) would become

$$(\lambda_1 - \lambda)^{r+s+t} [a_1 \dots a_r b_1 \dots b_s c_1 \dots c_t \bar{e}_{r+s+t+1} \dots \bar{e}_n] = 0 \dots\dots (B''').$$

We can now enunciate the following theorem :—The equation

$$\Pi (\phi - \lambda) e_i = 0$$

[the left-hand side of which is called the latent function of ϕ] has n roots [called the latent roots of φ]; to a latent root (λ) of multiplicity a appertain a points; these points group themselves into sets, such that, if we call the points of the first set a_i, those of the second set b_i, and so on, we have

$$\phi = \frac{\lambda a_i, \ \lambda b_i + A_i, \ \lambda c_i + A'_i + B_i \dots}{a_i, \quad b_i, \quad c_i},$$

where A_i, B_i, &c. denote syzygies of the a_i, b_i, &c.

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* *Vide* p. 69, l. 8.

Now it is obvious that the A, &c. are asyzygetic: this follows from the way in which they were determined. It follows that their number cannot be greater than that of the a; and obviously $\phi A_i = \lambda A_i$. We can, therefore, substitute the A_i for an equal number of the a, and then we get

$$\phi = \frac{\lambda a_i, \ \lambda b_i + a_i, \ \lambda c_i + A_i + B_i}{a_i, \quad b_i, \quad c_i},$$

where obviously A_i is not the same as before. But now we can substitute $A_i + B_i$ for an equal number of the b. Let $B_i = \Sigma B_{ij} b_j$; then

$$\phi (A_i + B_i) = \lambda (A_i + B_i) + \Sigma B_{ij} a_j.$$

Therefore we must substitute $\Sigma B_{ij} a_j$ for a_i, and then we get

$$\phi = \frac{\lambda a_i, \ \lambda b_i + a_i, \ \lambda c_i + b_i}{a_i, \quad b_i, \quad c_i},$$

and it is obvious how we may proceed.*

The points a_i, b_i, &c. are called the latent points of ϕ appertaining to the latent root λ.

The number of groups a_i, b_i, c_i that we get for any latent root depends on the coefficients of the latent function. There are two cases in which the theory of the latent points is particularly simple:—(1) the case in which no latent root is repeated, so that, for a latent point a_i appertaining to a root λ_i, we have

$$\phi a_i = \lambda_i a_i;$$

and (2) the case in which a latent root is repeated, but all its latent points are a's, so that, again, for all latent points a_{ij} appertaining to λ_i,

$$\phi a_{ij} = \lambda_i a_{ij}.$$

7. We have

$$(\phi - \lambda) a_i = 0,$$

$$(\phi - \lambda) b_i = a_i,$$

and therefore

$$(\phi - \lambda)^2 b_i = (\phi - \lambda) a_i = 0;$$

similarly

$$(\phi - \lambda)^s x_i = 0,$$

if x_i is a latent point in the s^{th} group.

Therefore, if there are s groups of latent points appertaining to λ,

$$(\phi - \lambda)^s e_i = 0,$$

where e_i is any latent point appertaining to λ.

* Jordan, "Traité des Substitutions," 125.

Therefore, if the latent roots are $\lambda_1, \lambda_2 \dots \lambda_r$, and if there are $s_1 \dots s_r$ groups of latent points appertaining to them, then, for any latent point e_i,

$$(\phi-\lambda_1)^{s_1}(\phi-\lambda_2)^{s_2}\dots(\phi-\lambda_r)^{s_r}e_i = 0.$$

But the n points e_i are asyzygetic; therefore, for all points x,

$$(\phi-\lambda_1)^{s_1}(\phi-\lambda_2)^{s_2}\dots(\phi-\lambda_r)^{s_r}x = 0,$$

that is, $$(\phi-\lambda_1)^{s_1}(\phi-\lambda_2)^{s_2}\dots(\phi-\lambda_r)^{s_r} = 0.$$

This is the *identical equation.*

If the roots are all unequal, $r = n$, $s_1 = s_2 = \dots = s_n = 1$, and the equation is

$$(\phi-\lambda_1)(\phi-\lambda_2)\dots(\phi-\lambda_n) = 0.$$

8. We have $$\phi a_i = \lambda a_i,$$

and therefore $$\phi^m a_i = \lambda^m a_i,$$

and generally, if f is any function, not involving matrices other than ϕ,

$$f(\phi)\,a_i = f(\lambda)\,a_i.$$

We have
$$\begin{aligned}\phi b_i &= \lambda b_i + a_i,\\ \phi^2 b_i &= \lambda\phi b_i + \phi a_i\\ &= \lambda(\lambda b_i + a_i) + \lambda a_i\\ &= \lambda^2 b_i + 2\lambda a_i,\end{aligned}$$

and generally $$f(\phi)\,b_i = f(\lambda)\,b_i + f'(\lambda)\,a_i.$$

In the same way, if x_i is in the s^{th} group,

$$f(\phi)\,x_i = f(\lambda)\,x_i + \dots + f^{(s-1)}(\lambda)\,a_i.$$

9. Now let $s_1 = s_2 = \dots = s_r = 1$: then we have, if

$$(\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_r) = \chi\lambda,$$

$$f(\phi) = \Sigma\,\frac{\chi(\phi)}{(\phi-\lambda_i)}\cdot\frac{f\lambda_i}{\chi'\lambda_i}\quad\dots\dots\dots\dots\dots\dots\dots\text{(A)},$$

f being any function not involving any matrix other than ϕ.

The function $$\frac{\chi(\phi)}{\phi-\lambda_i}$$

simply means

$$(\phi-\lambda_1)(\phi-\lambda_2)\dots(\phi-\lambda_{i-1})(\phi-\lambda_{i+1})\dots(\phi-\lambda_r).$$

To prove (A), I denote the right-hand side of it by $F(\phi)$, and I show that $f(\phi)\,e_j = F(\phi)\,e_j$, e_j being any latent point. We have, if e_j appertains to λ_j,

$$\frac{\chi(\phi)}{(\phi-\lambda_i)}\,e_j = \frac{\chi\lambda_j}{\lambda_j-\lambda_i}\,e_j$$

$$= 0 \text{ if } i \gtrless j$$

$$= \chi'\lambda \text{ if } i = j.$$

Therefore $$F(\phi)\,e_j = f(\lambda_j)\,e_j$$

$$= f(\phi)\,e_j,$$

and, therefore, generally $$f(\phi) = F(\phi)^*.$$

If the s are not all equal to unity, the formula is much more complicated.

10. If we have r asyzygetic points e_i, and no more, such that $\phi e_i = 0$, ϕ is said to be of *nullity* r. It follows, from what was proved in (6), that in this case (since $\phi e_i = 0\,.\,e_j$) r, at least, of the latent roots must vanish. But more than r may vanish; and, accordingly, if s latent roots vanish, ϕ is said to be of *vacuity* s. We see that the nullity of a matrix cannot be greater than its vacuity, but may be less. A point x, such that $\phi x = 0$, is called a *null-point* of ϕ: the *r-point* determined by the r asyzygetic null-points of a matrix of nullity r is called the *null-space* of the matrix.

If $e_1 \ldots e_r$ are r asyzygetic null-points of a matrix of nullity r, it is obvious that $\Sigma\lambda_i e_i$ is also a null-point; that is, every point in the null-space of a matrix is a null-point. Moreover, it is easy to see that every null-point must be in the null-space; for, if there were a null-point e_{r+1} not in $[e_1 e_2 \ldots e_r]$, we should have $(r+1)$ asyzygetic null-points, and the nullity of the matrix would be $r+1$.

Applying what was proved in (6) above, to the case $\lambda = 0$, we see

that we get $$\phi = \frac{0 \;\; a_i \;\; b_i \ldots}{a_i \;\; b_i \;\; c_i \ldots}.$$

We see that, if there are s groups $a, b, c \ldots$ and $\alpha, \beta, \gamma \ldots$ points, $a_i, b_i, c_i \ldots$, the nullity of ϕ is α, and its vacuity $\alpha+\beta+\gamma \ldots$; the nullity of ϕ^2 is $\alpha+\beta$ (since $\phi^2 b = 0$, $\phi^2 a = 0$), and its vacuity $\alpha+\beta+\gamma\ldots$; the nullity of ϕ^t is $(\alpha+\beta+\gamma\ldots)$, and its vacuity $\alpha+\beta+\gamma\ldots$ Therefore the nullity of ϕ^t is equal to its vacuity: and, if we apply this to

* (A) is Professor Sylvester's "interpolation formula," giving the standard form to which all functions of a matrix can be reduced: it appears above in what is, I think, a more general form than Professor Sylvester's.

the matrix $\phi-\lambda_i$, where λ_i is a latent root of ϕ having s_i groups, we see that s_i can be defined as the order of the lowest power of $(\phi-\lambda_i)$ of which the nullity is equal to its vacuity.

Consider the equation $y=\phi x$. Given x, this determines y uniquely; given y, x is not determinate unless the nullity of ϕ is zero: for, if we have $y=\phi x=\phi x'$, we have

$$\phi(x-x')=0.$$

Therefore $(x-x')$ must be a null-point of ϕ; and, if ϕ is of nullity r, the solution of the equation $y=\phi x$ contains r arbitrary constants: if x is any solution, the general solution is

$$x+\sum_1^r \lambda_i e_i,$$

the e being r asyzygetic null-points, and the λ arbitrary scalars.

11. Now, take as points of reference r asyzygetic null-points, $e_1 \dots e_r$, and $n-r$ points not in the null-space of ϕ; then, if

$$x=\sum_1^n x_i e_i,$$

$$\phi x=\sum_1^r x_i \phi e_i+\sum_{r+1}^n x_i \phi e_i;$$

$$=\sum_{r+1}^n x_i \phi e_i,$$

since $\phi e_1=\phi e_2=\dots=\phi e_r=0$: therefore the point ϕx is in a certain $(n-r)$ point; viz., the $(n-r)$ point

$$\Pi \equiv [\phi e_{r+1} \,.\, \phi e_{r+2} \dots \phi e_n].$$

This product does not vanish; for, if it did, there would be a relation

$$\sum_{r+1}^n c_i \phi e_i=0,$$

that is, $$\phi \Sigma c_i e_i=0.$$

Therefore there would be a null-point of ϕ in $[e_{r+1} \dots e_n]$, which is contrary to the hypothesis.

The $(n-r)$ point Π is called the *latent* space of ϕ: it is obvious that it contains all the latent points for which λ does not vanish.

If the vacuity of ϕ is greater than r, it follows, from what was proved above, that the latent space of ϕ will contain some or all of its null-points.

12. Let ϕ, χ be two matrices of nullities r, s respectively: it is required to find the nullity of $\phi\chi$. I shall show that, if the null-space

of ϕ intersects the latent space of χ in a t-point, the nullity of $\phi\chi$ is $s+t$.*

I take as points of reference, $e_1 \dots e_{n-s}$, being $(n-s)$ asyzygetic points, not situate in the null-space of χ, and $e_{n-s+1} \dots e_n$ asyzygetic null-points of χ; let the null-space of ϕ cut the latent space of χ in the t-point $[E_1 \dots E_t]$, where the E are, of course, supposed to be asyzygetic; lastly, let $\chi e_i = e'_i$. Let $A = \Sigma A_i e_i$ be any point: then

$$\chi A = \sum_1^n A_i \chi e_i$$

$$= \sum_1^{n-s} A_i e'_i,$$

since $e'_{n-s+1} = \dots = e'_n = 0$; therefore

$$\phi\chi A = \sum_1^{n-s} A_i \phi e'_i.$$

But we can select from $e'_1 \dots e'_{n-s}$, $n-s-t$ points, asyzygetic with the E, and then we have

$$e'_i = \sum_{k=1}^{t} \beta_{ik} E_k + \sum_{j=t+1}^{n-s} a_{ij} e'_j \; (i = 1 \dots t) \quad \dots\dots\dots\dots\dots\dots (\text{X}).$$

Therefore, since $\phi E_k = 0$ (since the E are in the null-space of ϕ),

$$\phi e'_i = \Sigma\, a_{ij} \phi e'_j,$$

and

$$\phi\chi A = \sum_{j=t+1}^{n-s} \phi e'_j \,(A_j + \sum_{i=1}^{t} a_{ij} A_i).$$

Therefore all points ϕ, χ, A are in the $(n-s-t)$ point $[\phi e'_{t+1} \dots \phi e_{n-s}]$; and, to show that this is actually the latent space of $\phi\chi$, we have only to show that these $(n-s-t)$ points are asyzygetic: but if they were not asyzygetic, and we had

$$0 = \sum_1^{n-s-t} c_i \phi e'_{t+i}$$

$$= \phi \Sigma\, c_i e'_{t+i},$$

we must have

$$\Sigma c_i e'_{t+i} = \Sigma \lambda_j E_j,$$

which is contrary to the supposition on which e'_{t+1}, &c. were selected. Therefore the latent space of $\phi\chi$ is an $(n-s-t)$ point, and therefore its nullity is $(s+t)$; and it can be shown without difficulty that its null space is the $(s+t)$ point joining the null space of χ to the t-point in its latent space, which χ transforms into $[E_1 \dots E_t]$.

13. In all that follows, I shall assume that, in the notation of (7), $s_1 = s_2 = s_r = 1$.†

* Cf. *Phil. Mag.*, Nov. 1884.

† That is, that all latent points are a's: Case (2) of (6).

Now let it be proposed to find a non-vacuous* matrix ψ such that $\psi\phi = \phi\psi$, ϕ being a given matrix.

Let $e_1 \ldots e_n$ be the latent points of ϕ. Let e_i appertain to λ_i; suppose $\lambda_1 = \lambda_2 = \ldots = \lambda a_1 = \lambda_1$, &c.; lastly, let

$$\psi e_i = \Sigma a_{ij} e_j.$$

Then
$$\phi\psi e_i = \Sigma a_{ij} \phi e_j$$
$$= \Sigma a_{ij} \lambda_j e_j.$$

But
$$\psi\phi e_i = \psi \lambda_i e_i$$
$$= \lambda_i \Sigma a_{ij} e_j.$$

Therefore, since $\psi\phi = \phi\psi$, and ψ is supposed non-vacuous, we have

$$a_{ij}\lambda_j = a_{ij}\lambda_i.$$

Therefore, unless $\lambda_i = \lambda_j$, $a_{ij} = 0$; and it follows that ψ transforms all latent points appertaining to the same latent root λ_i into points of the same a_i-point, if a_i is the multiplicity of λ_i; we therefore have

$$\psi = \Sigma \psi_i,$$

where ψ_i is a matrix of nullity $n - a_i$, having $[e_1 \ldots e_{a_i}]$ as its latent space, and therefore transforming every latent point appertaining to λ_i into another latent point appertaining to λ_i.

If $a_1 = a_2 = \ldots = a_r = 1$, we can go further than this, and can assign the form of ψ; for in this case it is obvious that ψ must transform every latent point into itself; that is,

$$\psi = \frac{\Lambda_i e_i}{e_i}.$$

But
$$\frac{\Lambda_i e_i}{e_i} = \Sigma \frac{\chi(\phi)}{\phi - \lambda_i} \cdot \frac{\Lambda_i}{\chi'\lambda},$$

using the same notation as in (9). This can be proved by the method there employed. Therefore we can say that, if $\phi\psi = \psi\phi$, and if ψ is non-vacuous, and the latent roots of ϕ are all different, ψ is a function of ϕ, of order $(n-1)$, and with scalar coefficients; viz., we have

$$\psi = \Sigma \frac{\chi(\phi)}{\phi - \lambda_i} \cdot \frac{\Lambda_i}{\chi'(\lambda_i)},$$

* A matrix is vacuous if its vacuity $\overline{\overline{>}}$ 1.

where the Λ are scalars, and $\chi\lambda$ is the latent function of ϕ, viz.,

$$\chi\lambda = (\lambda-\lambda_1)\dots(\lambda-\lambda_n).^*$$

We can apply a similar method to the equation $\phi\psi = k\psi\phi$, k a scalar. We shall find that k is an n^{th} root of unity, and that, if it is a primitive m^{th} root of unity, ψ is equivalent to a substitution operating on the n latent points of ϕ; viz., if $n = mm'$, ψ is the product of m' cyclic substitutions, each cycle being of order m.

14. As a last example of the methods of this paper, I take the solution of the general unilateral equation.

Let the given equation be

$$0 = F(x) = A_0x^m + A_1x^{m-1} + \dots + A_m,$$

where the A are known matrices of order n, and x is an unknown matrix of the same order.

Let $\lambda_1 \dots \lambda_n$ be the latent roots of x; $e_1 \dots e_n$ its latent points. Since $F(x) = 0$, we have

$$\begin{aligned} 0 &= F(x)\, e_i \\ &= F(\lambda_i)\, e_i. \end{aligned}$$

Therefore e_i must be a null-point of $F(\lambda_i)$; $F(\lambda_i)$ must be vacuous, and therefore, if we take n points of reference a_j, we must have

$$\prod_{j=1}^{n} [F(\lambda_i)\, a_j] = 0\dagger \quad (i = 1 \dots n),$$

that is, the latent roots of x must be roots of this equation of order mn, and, if we take any set of n roots, the latent point of x appertaining to a root λ_i is a null-point of $F\lambda_i$, and x is thus completely determined.

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* *Cf.* Clifford's Math. Papers, 339.

† This equation is simply Det $(F\lambda_n) = 0$.