December 8th, 1881.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Mr. George Henry Stuart, M.A., late Fellow of Emmanuel College, Cambridge, was elected a Member, and Miss C. A. Scott was admitted into the Society.

The Auditor (Rev. R. Harley) read his Report. On the motion of the Chairman, a vote of thanks was unanimously accorded to the Auditor for his services.

The following papers were read:—

"On the Polar Planes of Four Quadrics:" by Mr. W. Spottiswoode, Pres. R.S.

"On some Forms of Cubic Determinants:" by Mr. R. F. Scott.

"On the Flow of a Viscous Fluid through a Pipe or Channel:" by Prof. Greenhill.

"The Covariant which is the complete Locus of the Vertex of the Involution-pencil of Tangents to a Cubic:" by Mr. J. J. Walker.

The following presents were received:—

"American Journal of Mathematics," Vol. iv., No. 1.

Cartes-de-Visite likenesses from Rev. A. J. C. Allen, and Mr. Donald McAlister.

On the Polar Planes of Four Quadrics. *By* WILLIAM SPOTTISWOODE, M.A., P.R.S.

[*Read Dec. 8th,* 1881.]

Consider the four quadric surfaces U', U_1', U_2', U_3', where $U' = (x', y', z', t')^2$, &c.; and let P', or (x, y, z, t), be any point in space. Then the equation of the tangent plane through the point P to the surface U' may be written thus,

$$u'x+v'y+w'z+s't=0 \quad\text{..........................}(1),$$

where $\qquad (\partial_x, \partial_y, \partial_z, \partial_t)\, U' = u', v', w', s',$

and similarly for the tangent planes through P to the other surfaces. Again, if $U', \ldots u, \ldots$ be the same functions of x, y, z, t, that $U', \ldots u', \ldots$ are of x', y', z', t'; then the equations of the polar plane of the point P with respect to the quadric U' may be written thus:

$$ux'+vy'+wz'+st'=0 \quad\text{..........................}(2),$$

with similar expressions for the polar planes of the point P with respect to the three other quadrics.

The condition that these planes may meet in a point P' will be found by eliminating x', y', z', t' from the four equations of the form (2). It will, in fact, be represented by the Jacobian of the four surfaces, viz., by the equation

$$\begin{vmatrix} u, & u_1, & u_2, & u_3 \\ v, & v_1, & v_2, & v_3 \\ w, & w_1, & w_2, & w_3 \\ s, & s_1, & s_2, & s_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3);$$

and, as the constituents of this determinant are linear in the coordinates of the point P, it follows that, in order that the four polar planes meet in a point, the point P of which they are polars must lie on the quartic surface of which the equation is (3). This surface, and a kindred surface called the symmetroid, have been discussed by Cayley in his first memoir on "Quartic Surfaces," in the *Proceedings of the London Mathematical Society*, Vol. III., p. 19.

The coordinates of the point of intersection P' may be found from any three of the equations represented by (2). Thus, if the developed form of (3) be

$$Au+Bv+Cw+Ds = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

we shall have

$$\left.\begin{array}{l} x' : y' : z' : t' \\ = A : B : C : D \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

in which we may substitute for A, B, C, D [which are the minors formed from the last three columns of the matrix (3)] the minors formed from any three columns of the same matrix.

If the point P' be restricted to a given plane Π, say the plane whose equation is

$$lx'+my'+nz'+pt' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

then, on substituting for x', y', z', t', from (3), we shall have

$$lA+mB+nC+pD = 0,$$

or, writing down, for the sake of brevity, only the upper line of the determinant,

$$|\, l, u_1, u_2, u_3 \,| = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

That is to say, the point P must lie on the curve of intersection of the cubic surface (7) with the quartic surface (3). But, since all the determinants that can be formed from the matrix

$$l, \; u, \; u_1, \; u_2, \; u_3$$

will vanish, if any two of them vanish, it follows that the two equations (3) and (7) may be replaced by any other two, *e.g.*, by the following

$$|\, l, u, u_1, u_2 \,| = 0, \quad |\, l, u, u_1, u_3 \,| = 0 \quad \ldots\ldots\ldots (8).$$

In other words, the point P must lie on the curve of intersection of the two cubic surfaces (8). The order of this curve is 6. See Salmon's "Higher Algebra," 3rd Edition, pp. 243 *et seqq.* In fact, the five surfaces represented by the disjunctive equation

$$\| l,\ u,\ u_1,\ u_2,\ u_3 \| = 0 \text{ ....(9)}$$

have a common curve of intersection.

It may be further observed that the equations (3) and (7) are both satisfied, if the equations

$$\begin{vmatrix} u_1, & v_1, & w_1, & s_1 \\ u_2, & v_2, & w_2, & s_2 \\ u_3, & v_3, & w_3, & s_3 \end{vmatrix} = 0 \text{ ....(10)}$$

are satisfied. In other words, the surfaces (3) and (7) have in common the curve whose equations are (10). But, in the same way as before, the coordinates of the point P' will be proportional to minors formed from any three of the columns of the matrix (9); and, as the equations (10) are equivalent to

$$A = 0,\ \ B = 0,\ \ C = 0,\ \ D = 0,$$

it follows that, for every position of the point P on the curve, the coordinates of the point P' become indeterminate. Similarly, the surfaces (8) have in common the curve whose equations are

$$\begin{vmatrix} l, & m, & n, & p \\ u, & v, & w, & s \\ u_1, & v_1, & w_1, & s_1 \end{vmatrix} = 0 \text{ ....(11),}$$

and, for every position of the point P on this curve, the coordinates of the point P' become indeterminate.

Again, if the point P' be restricted to a given straight line, say the intersection of the plane Π with a plane Π_1, whose equation is

$$l_1x' + m_1y' + n_1z' + p_1t' = 0,$$

then, in addition to the condition (7), we shall have the condition

$$| \, l_1,\ u_1,\ u_2,\ u_3 \, | = 0 \text{ ....(12).}$$

But the three equations (3), (7), (12) may be replaced by any three independent equations derived from the matrix

$$l,\ l_1,\ u,\ u_1,\ u_2,\ u_3,$$

e.g., by the equations

$$| \, l, l_1, u, u_1 \, | = 0, \quad | \, l, l_1, u, u_2 \, | = 0, \quad | \, l, l_1, u, u_3 \, | = 0 \text{...(13);}$$

and consequently the point P must lie at the intersection of the three quadrics (13). In fact, the fifteen surfaces represented by the disjunctive equation $\quad \| l, l_1, u, u_1, u_2, u_3 \| = 0$ ....(14)

intersect in the same common points. The number of these points is 4. See Salmon's "Higher Algebra," *l. c.*

Again, the three equations (13) will be satisfied if the equations

$$\begin{vmatrix} l, & m, & n, & p \\ l_1, & m_1, & n_1, & p_1 \\ u, & v, & w, & s \end{vmatrix} = 0 \quad \text{.........................(15)}$$

are satisfied. In other words, the quadrics have in common the straight line whose equations are (15). If a, b, c, f, g, h, be six coordinates of the line of intersection of the planes Π, Π_1, then the line may be defined by any two of the equations

$$\left.\begin{array}{l} L = \quad . \quad \mathrm{h}v - \mathrm{g}w + \mathrm{a}s = 0 \\ M = -\mathrm{h}u \quad . + \mathrm{f}w + \mathrm{b}s = 0 \\ N = \quad \mathrm{g}u - \mathrm{f}v \quad . + \mathrm{c}s = 0 \\ P = \quad \mathrm{a}u + \mathrm{b}v + \mathrm{c}w \quad . = 0 \end{array}\right\} \quad \text{..................(16).}$$

But, since the coordinates of the point P' are given by the equations

$$\left.\begin{array}{l} x' : y' : z' : t' \\ = L : M : N : P \end{array}\right\} \quad \text{........................(17),}$$

it follows that, for every position of the point P on the line (16), the coordinates of the point P' become indeterminate. In fact, if P lies anywhere on the line $\Pi\, \Pi_1$, then P may lie anywhere on the line (17).

With the quartic surface (3), and with the other surfaces which have similarly occurred in this paper, there are connected certain cubic curves in space; for example, the curve represented by the equations

$$\begin{vmatrix} u_1, & u_2, & u_3 \\ s_1, & s_2, & s_3 \end{vmatrix} = 0 \quad \text{..........................(18).}$$

These, as is well known, represent a cubic curve in space, because the two equations, say $u_1 s_2 - u_2 s_1 = 0$, $u_1 s_3 - u_3 s_1 = 0$, are the equations of two quadric surfaces having a common generator $u_1 = 0$, $s_1 = 0$. We may enquire, in what points does this curve meet the surface (3). If we put (18) into the following form,

$$u_1 : s_1 = u_2 : s_2 = u_3 : s_3 = U : S \text{ suppose} \quad \text{...........(19),}$$

it will be found that the quartic breaks up into two factors, viz., the expression (3) becomes

$$(uS - sU)\,(v_2 w_3 - v_3 w_2 + v_3 w_1 - v_1 w_3 + v_1 w_2 - v_2 w_1) = 0 \quad \text{......(20);}$$

and, if we take the first as the vanishing factor and combine it with (18), we obtain

$$\begin{vmatrix} u, & u_1, & u_2, & u_3 \\ s, & s_1, & s_2, & s_3 \end{vmatrix} = 0 \quad \text{.............................(21);}$$

and, if further we put these equations into the following form

$$u = \lambda s,\ \ u_1 = \lambda s_1,\ \ u_2 = \lambda s_2,\ \ u_3 = \lambda s_3,$$

and eliminate x, y, z, t, we shall obtain a biquadratic in λ. The four roots of this equation, successively substituted in (21), will give four points in which the cubic curve meets the surface (3). Eight more points are to be found by combining (18) with the second factor of (20); and the remainder are to be accounted for by the fact that the quadrics (18) have a common generator.

When the point P' is restricted to the plane Π, the cubic curve in question can meet the curve represented by (3) and (7) only under a condition. The cubic (7) breaks up into the factors

$$(lS-pU)(v_2w_3-v_3w_2+\ldots)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(22).$$

Taking $uS-sU$, $lS-pU$, as the vanishing factors, we obtain

$$u:s=u_1:s_1=u_2:s_2=u_3:s_3=l:p\ldots\ldots\ldots\ldots\ldots(23),$$

and if from these we eliminate x, y, z, t, we obtain the required condition. If, however, we suppose the plane Π to be no longer entirely arbitrary, but to be determined so that the constant $l:p$ shall satisfy the condition in question, it will follow that there are four series of positions of the plane for which the cubic curve (18) will meet the curve of intersection of the surfaces (3) and (7). Beside the constant $l:p$, there are two others $m:p$, $n:p$, which may also be determined so as to satisfy required conditions. We may, in fact, determine them so that the two other cubic curves

$$\begin{vmatrix} v_1, & v_2, & v_3 \\ s_1, & s_2, & s_3 \end{vmatrix}=0, \quad \begin{vmatrix} w_1, & w_2, & w_3 \\ s_1, & s_2, & s_3 \end{vmatrix}=0 \quad\ldots\ldots\ldots\ldots\ldots(24)$$

may meet the curve represented by the equations (3), (7). The plane will be then completely determined.

When the point P' is restricted to the line (Π, Π_1), there will be two conditions to be fulfilled in order that the cubic curve may pass through the points defined by the equations (14). In fact, we shall, on substituting as before, again find the equations (21) and (22); and, in addition to them, also the following:

$$(l_1S-p_1U)(w_2s_3-w_3s_2+\ldots)=0\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

But the equation for determining $l_1:p_1$ will be the same as that for determining $l:p$. If, therefore, we take for $l:p$, and $l_1:p_1$, two different roots of that equation, we may satisfy both conditions.