Note on the Representation of a Circle by a Linear Equation.

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[The coordinates x, y, z of the present note have been employed by me to prove various theorems in the geometry of the circle and triangle. See *Proc. Lond. Math. Soc.*, Vols. XXV., XXVI., and other vols. Hence it seems desirable to give a simpler interpretation of them than I have hitherto done.]

1. In the ordinary Cartesian method of determining the position of a point P in a plane, the equation $x = h$ represents a right line, every point on which has the same abscissa h. In like manner, it may be possible to choose a coordinate x, so that $x = h$ shall represent a circle or curve. In other words, the position of P may possibly be fixed by considering it as the intersection not only of straight lines, but also of circles, or curves. For example, suppose a circle to be drawn through the vertices B, C of a triangle ABC to

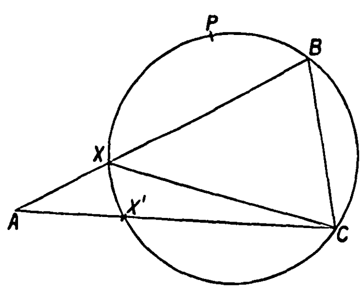

FIG. 1. $x = \frac{b}{AX} = \frac{\sin\theta}{\sin(\theta + A)}$, if $\angle BXC = \pi - \theta$.

intersect the sides AB, AC in X, X'. See Fig. 1. This curve will be completely determined when either of the abscissæ AX, AX' is known. Hence we may take AX or AX', or a function of them, to be the coordinate of a point P on a circle through B, C.

In particular, if we denote either of the equal ratios $\frac{AC}{AX}$, $\frac{AB}{AX'}$ by x, there is no difficulty in expressing x in terms of the trilinear coordinates α, β, γ of P.

Here the function $\alpha . \Sigma a\alpha \div \Sigma a\beta\gamma$ is an invariant. Also, since x is an invariant for every point on the circle, we can assume

$$f(\alpha, \beta, \gamma) = \alpha . \Sigma a\alpha \div \Sigma a\beta\gamma = F(x).$$

This gives $$f(\alpha, \beta, 0) = f(\alpha, 0, \gamma) = F(x),$$

i.e., $$\frac{bc \sin A}{c . AX \sin A} = \frac{bc \sin A}{b . AX' \sin A} = F(x),$$

or $$x = \frac{b}{AX} = \frac{c}{AX'} = F(x) = f(\alpha, \beta, \gamma) = \alpha . \Sigma a\alpha \div \Sigma a\beta\gamma.$$

We thus have one coordinate, viz.,

$$x = \frac{b}{AX} = \alpha . \Sigma a\alpha \div \Sigma a\beta\gamma,$$

for fixing the position of a point $P(\alpha, \beta, \gamma)$.

Similarly, if two circles CPA, APB intersect the sides BC, CA, respectively, in Y and Z, we obtain two other coordinates

$$y = c \div BY = \beta . \Sigma a\alpha \div \Sigma a\beta\gamma;$$

$$z = a \div CZ = \gamma . \Sigma a\alpha \div \Sigma a\beta\gamma.$$

The position of P will then, in general, be uniquely determined by given values of the coordinates x, y, z.

It is, of course, obvious that there must be the fundamental relation $\Sigma ax = \Sigma ayz$ between x, y, z, since three arbitrary circles through BC, CA, AB will not necessarily meet in a common point.

2. Taking x, y, z to mean the three coordinates defined above, viz.,

$$x = b \div AX = \alpha . \Sigma a\alpha \div \Sigma a\beta\gamma,$$

$$y = c \div BY = \beta . \Sigma a\alpha \div \Sigma a\beta\gamma,$$

$$z = a \div CZ = \gamma . \Sigma a\alpha \div \Sigma a\beta\gamma,$$

it follows that a linear non-homogeneous relation $lx + my + nz = k$ will represent a circle, since the equivalent trilinear form is

$$(l\alpha + m\beta + n\gamma) \Sigma a\alpha = k . \Sigma a\beta\gamma.$$

[It is worth noticing here that the relations $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$, between the tricircular and trilinear coordinates of P, can be readily deduced by elementary geometry.]

3. As x, y, z may have any values between $-\infty$ and $+\infty$, it is important to consider what convention is to be adopted with regard

to the signs of the intercepts or abscissæ AX, BY, CZ. Now, since $AX = b \div x$, it is clear that AX is positive or negative according as x is positive or negative. Hence measure off AX in the direction AB or BA according as x is positive or negative.

Similarly, in the case of BY and CZ, we have to take these abscissæ, respectively, in the directions BC, CB; CA, AC, when y and z are positive or negative.

For example, if $x = \frac{b}{c}$, then $AX = c = AB$, *i.e.*, X coincides

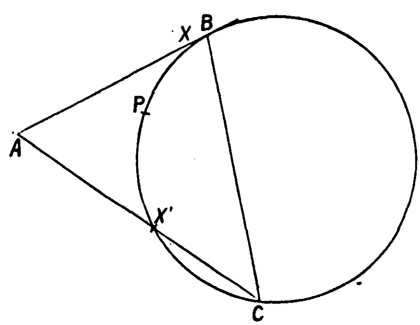

FIG. 2. $x = \frac{b}{c}$. $\angle BXC = \pi - B$, when X coincides with B.

with B, and $x = \frac{b}{c}$ represents the circle which passes through B, C, and touches the side AB at B. See Fig. 2. On the other hand, if

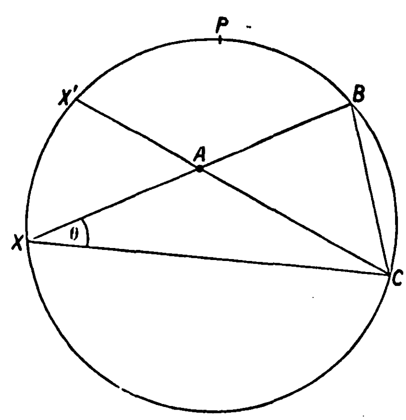

FIG. 3. $x = -\frac{b}{c}$; $AX = -AB = -c$. $x = \frac{\sin\theta}{\sin(\theta - A)} = -\frac{\sin B}{\sin C}$, if $\angle BXC = \theta$.

$x = -\frac{b}{c}$, then $AX = -AB$, so that the circle represented is BXC, where AX is measured off in negative direction BA, and is equal in length to AB or c. See Fig. 3. Again, since $x = \frac{c}{AX'}$, it is clear that $x = \frac{c}{b}$ is the circle (BC) touching AC at C (Fig. 1). We thus easily discover the Brocard points; viz., $\left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$ and $\left(\frac{b}{c}, \frac{c}{a}, \frac{a}{b}\right)$.

4. The above method of determining the position of a point P, by the cointersection of three circles $x = x_1$, $y = y_1$, $z = z_1$, gives an indeterminate result in two exceptional cases; viz., when (1) x_1, y_1, z_1 are all zero, *i.e.*, if P lies on the line of infinity; (2) all infinite, or P is on the circumcircle ABC. To fix the position of P, in either of these cases, we must know the ratios $x : y : z = \alpha : \beta : \gamma$. The vertices A B, C are special points whose tricircular coordinates may be found as follows. For A we have $x = \infty$ and $bz + cy = a$. Here $x = \infty$ denotes the circumcircle ABC, and $bz + cy = a$ is the point-circle at A. The y and z coördinates are thus indeterminate.

5. More generally, the position of a point P can be determined by the cointersection of three curves. If we denote two rational and integral functions of α, β, γ of the nth and $(n-1)$th degrees, respectively, by U_n and V_{n-1}, and write

$$xU_n = \alpha . V_{n-1}, \qquad (1)$$

$$yU_n = \beta . V_{n-1}, \qquad (2)$$

$$zU_n = \gamma . V_{n-1}, \qquad (3)$$

these equations will represent curves, each of the nth degree, having $n^2 - n$ principal points in common, viz., the intersections of

$$U_n = 0, \quad V_{n-1} = 0.$$

Moreover, the remaining common points of (1), (2), and (3), taken in pairs, will lie on the right lines

$$\frac{\beta}{y} = \frac{\gamma}{z}, \quad \frac{\alpha}{x} = \frac{\beta}{y}, \quad \frac{\gamma}{z} = \frac{\alpha}{x},$$

which meet in the point $\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z}$.

In other words, the curves in question have a radical centre P, and the condition for their cointersection is, evidently, that P lies on any one of them; *i.e.*,

$$U_n(x, y, z) = V_{n-1}(x, y, z).$$

In this case also a linear equation

$$lx + my + nz = k$$

represents a curve of the nth degree passing through the n^2-n principal points. It is evident that x is a function of an abscissa AX, or intercept on the line AB; and, similarly, y, z of BY, CZ, respectively.

Let us consider the curve (1), for instance. This passes through n^2 given points, viz., the n^2-n principal points and the n intersections of BC with U_n. One more point, X, will therefore completely determine the curve. We may assume X to be on AB. Hence x is a rational function of AX. In tricircular coordinates, as expounded above, we have seen that $x = \frac{b}{AX}$.

We may sum up, briefly, the results of the present note by saying that here a curve of the nth degree, drawn through n^2-n principal points, is represented by a linear non-homogeneous equation between three coordinates x, y, z, connected by a fundamental non-homogeneous relation of the nth degree; whereas in the ordinary trilinear method the same curve would be represented by a homogeneous equation of the nth degree in α, β, γ, connected by a linear non-homogeneous relation.

Otherwise—A curve of the nth degree W_n can be represented in various ways by a linear equation

$$lx + my + nz = 1.$$

Suppose W_n to be intersected in n^2-n points by an arbitrary curve V_{n-1} of the $(n-1)$th order. Through these and the angular points of a triangle ABC a curve U_n can be drawn, whose trilinear equation is

$$U_n(\alpha\beta\gamma) = (l\alpha + m\beta + n\gamma)\, V_{n-1}(\alpha, \beta, \gamma) - W_n(\alpha, \beta, \gamma) = 0.$$

Here l, m, n are known quantities. For instance,

$$l = \frac{W_n(1, 0, 0)}{V_{n-1}(1, 0, 0)}.$$

Hence W_n will be represented by

$$lx + my + nz = 1,$$

if we write
$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = \frac{V_{n-1}(\alpha, \beta, \gamma)}{U_n(\alpha, \beta, \gamma)}.$$

These equalities give $U_n(x, y, z) = V_{n-1}(x, y, z)$.