"Bulletin de la Société Mathématique de France," Tome xx., No. 6; Paris, 1892.

"Bulletin des Sciences Mathématiques," Tome xvi.; November, 1892.

"Transactions of the Texas Academy of Science," Vol. i., No. 1; November, 1892.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. i., Fasc. 10, 11, 2° Sem.

"Educational Times," January, 1893.

"Journal für die reine und angewandte Mathematik," Bd. cx., Heft iv.

"Annals of Mathematics," Vol. vii., No. 1; Virginia, November, 1892.

"Indian Engineering," Vol. xii., Nos. 21-25.

"The Stresses in Statistically Intermediate Structures"; Reprint from "Indian Engineering."

"Memoirs of the National Academy of Sciences," Vol. v.; 4to, Washington, 1891.

"Cayley's Collected Mathematical Papers," Vol. v.; 4to, 1892. Two copies.

On the Evaluation of a certain Surface-Integral, and its application to the Expansion, in Series, of the Potential of Ellipsoids. *By* E. W. Hobson. Received and read January 12th, 1893.

If V be any function of x, y, z, the coordinates of a point, the function being finite and continuous throughout a sphere of radius R whose centre is the origin, it is known that

$$\iint V dS = 4\pi R^2 \sum_{n=0}^{n=\infty} \frac{R^{2n}}{(2n+1)!} \nabla^{2n} V,$$

the integration being taken over the whole surface of the sphere, and $\nabla^{2n}V$ having its value at the origin; ∇^2 denotes Laplace's operator. This theorem has been applied by Mr. W. D. Niven* to the evaluation of a number of important definite integrals involving spherical harmonics, and to the development, in series, of the potentials of a uniform solid ellipsoid and of a homœoid.

I propose here to investigate a more general surface-integral theorem which includes the above, and which also furnishes a proof

* *Phil. Trans.*, 1879.

and an extension of an important surface-integral theorem due to Maxwell. The theorem is then applied to the determination of the expression for an external ellipsoidal harmonic in a series of spherical harmonics. I have then shown how to obtain expansions of the potentials of ellipsoidal shells, solid ellipsoids, and elliptic discs of variable density, the law of force being any given function of the distance.

The formulae given by most writers on the subject of the attraction of ellipsoids, express the potentials in the form of definite integrals; such formulae have been given by Dr. Ferrers,* and recently in a very elegant form by Mr. Dyson.† The formulae given in the present communication are of such a character that approximate values of the potentials may be obtained by taking as many terms of the series as may be necessary, whereas the definite integral formulae do not lend themselves readily to such approximation.

1. It is known‡ that the expansion of $e^{\iota r\cos\theta}$ in a series of zonal harmonics $P_n(\cos\theta)$ is given by

$$e^{\iota r\cos\theta} = \sum_{n=0}^{n=\infty} \frac{(\iota r)^n}{3.5.7\ldots(2n-1)} \left\{1 - \frac{r^2}{2.2n+3} + \frac{r^4}{2.4.2n+3.2n+5} - \ldots\right\} \times P_n(\cos\theta) \quad\ldots\ldots(1).$$

This expansion may be conveniently obtained as follows:—

The differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + V = 0 \quad\ldots\ldots(2)$$

is satisfied by the expressions

$$r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(r)\, P_n(\cos\theta), \qquad r^{-\frac{1}{2}}\, Y_{n+\frac{1}{2}}(r)\, P_n(\cos\theta),$$

where $J_{n+\frac{1}{2}}(r)$, $Y_{n+\frac{1}{2}}(r)$ denote the two Bessel's functions of order $n+\frac{1}{2}$; the functions $r^{-\frac{1}{2}}J_{n+\frac{1}{2}}$ and $r^{-\frac{1}{2}}Y_{n+\frac{1}{2}}$ are of the forms

$$Ar^n \frac{d^n}{d(r^2)^n} \frac{\sin r}{r}, \quad Br^{-n-1} \frac{d^n}{d(r^2)^n} \frac{\cos r}{r},$$

respectively, A and B being constants. Now (2) is satisfied by

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* *Quarterly Journal*, Vol. XIV.
† *Quarterly Journal*, Vol. XXV.
‡ See Heine's *Kugelfunctionen*, Vol. I., p. 82.

$V = e^{\iota z} = e^{\iota r \cos\theta}$; thus, if $e^{\iota r\cos\theta}$ be expanded in a series of the harmonics $P_n(\cos\theta)$, we should expect the general term to be

$$\left\{ A_n r^n \frac{d^n}{d(r^2)^n}\,\frac{\sin r}{r} + B_n r^{-n-1} \frac{d^n}{d(r^2)^n}\,\frac{\cos r}{r} \right\} P_n(\cos\theta).$$

It is clear that we must have $B_n = 0$, as the expression must be finite, when $r = 0$; thus

$$e^{\iota r\cos\theta} = \Sigma A_n r^n \frac{d^n}{d(r^2)^n}\,\frac{\sin r}{r} P_n(\cos\theta),$$

or

$$e^{\iota r\cos\theta}$$

$$= \Sigma \frac{(-1)^n n!}{(2n+1)!} A_n r^n \left\{1 - \frac{r^2}{2.2n+3} + \frac{r^4}{2.4.2n+3.2n+5} - \dots\right\} P_n(\cos\theta).$$

Equating the coefficients of the term $r^n \cos^n\theta$ on both sides of the equation, we have

$$\frac{\iota^n}{n!} = \frac{(-1)^n n!}{(2n+1)!}\,\frac{(2n)!}{2^n n!\, n!} A_n,$$

or

$$\frac{(-1)^n n!}{(2n+1)!} A_n = \frac{\iota^n}{3.5\dots(2n-1)},$$

and thus the expansion (1) is proved. In (1), change r into $-\iota\rho$; we thus obtain the expansion

$$e^{\rho\cos\theta} = \sum_{n=0}^{n=\infty} (2n+1) \frac{\rho^n}{3.5\dots 2n+1}$$

$$\times \left\{1 + \frac{\rho^2}{2.2n+3} + \frac{\rho^4}{2.4.2n+3.2n+5} + \dots\right\} P_n(\cos\theta) \dots (3).$$

2. Let $Y_n(x, y, z)$ denote a spherical harmonic of positive integral degree n, and suppose it is required to evaluate

$$\iint e^{\alpha x+\beta y+\gamma z}\, Y_n(x, y, z)\, dS,$$

where dS is an element of surface of the sphere of radius R, whose centre is the origin, the integral being taken over the whole surface of the sphere. Using the expansion (3), we have

$$e^{\alpha x+\beta y+\gamma z}$$

$$= \Sigma (2n+1) \frac{R^n(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}n}}{3.5\dots 2n+1} \left\{1 + \frac{R^2(\alpha^2+\beta^2+\gamma^2)}{2.2n+3} + \dots\right\} P_n(\cos\theta),$$

where

$$\cos\theta = \frac{\alpha x+\beta y+\gamma z}{R(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}}.$$

If we substitute this expression in the definite integral, since

$$\iint P_m(\cos\theta)\, Y_n(x, y, z)\, dS$$

is zero, unless $m = n$, we have

$$\iint e^{\alpha x+\beta y+\gamma z}\, Y_n(x, y, z)\, dS$$

$$= (2n+1)\,\frac{R^n(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}n}}{3.5\ldots 2n+1}\left\{1+\frac{R^2(\alpha^2+\beta^2+\gamma^2)}{2.2n+3}+\ldots\right\}$$

$$\times\iint P_n(\cos\theta)\, Y_n(x, y, z)\, dS.$$

Now $\displaystyle\iint P_n(\cos\theta)\, Y_n(x, y, z)\, dS = \frac{4\pi}{2n+1}\, R^{n+2}\, Y_n\left(\frac{\alpha}{A}, \frac{\beta}{A}, \frac{\gamma}{A}\right)$

$$= \frac{4\pi}{2n+1}\cdot\frac{R^{n+2}}{A^n}\, Y_n(\alpha, \beta, \gamma),$$

A denoting $(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}$; we thus obtain the expression

$$\iint e^{\alpha x+\beta y+\gamma z}\, Y_n(x, y, z)\, dS$$

$$= 4\pi R^{2n+2}\,\frac{2^n n!}{(2n+1)!}\left\{1+\frac{R^2(\alpha^2+\beta^2+\gamma^2)}{2.2n+3}+\frac{R^4(\alpha^2+\beta^2+\gamma^2)^2}{2.4.2n+3.2n+5}+\ldots\right\}$$

$$\times\, Y_n(\alpha, \beta, \gamma) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Now put for α, β, γ the operators $\dfrac{\partial}{\partial x_0}, \dfrac{\partial}{\partial y_0}, \dfrac{\partial}{\partial z_0}$, respectively, and let each side operate upon a function $f(x_0, y_0, z_0)$, where $f(x, y, z)$ is a function which is finite and continuous throughout the volume of the sphere, and where x_0, y_0, z_0 are all put zero after the operations are performed; then, since

$$e^{x.\partial/\partial x_0+y.\partial/\partial y_0+z.\partial/\partial z_0} f(x_0, y_0, z_0)$$

$$= f(x+x_0, y+y_0, z+z_0) = f(x, y, z),$$

we have the following surface-integral theorem:—

$$\iint Y_n(x, y, z) f(x, y, z)\, dS$$

$$= 4\pi R^{2n+2}\,\frac{2^n n!}{(2n+1)!}\left\{1+\frac{R^2\nabla^2}{2.2n+3}+\frac{R^4\nabla^4}{2.4.2n+3.2n+5}+\ldots\right\}$$

$$Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f(x, y, z) \quad\ldots\ldots\ldots(5),$$

where, on the right-hand side, x, y, z are all put equal to zero after the operations have been performed, and ∇^2 denotes the operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$. The only restriction to which the function $f(x, y, z)$ is subject, is that it must be finite and continuous throughout the sphere.

3. I now proceed to consider some particular cases of the theorem (5).

Putting $n=0$, in which case we can put $Y_n=1$, the theorem reduces to that employed by Mr. W. D. Niven,

$$\iint f(x, y, z)\, dS = 4\pi R^2 \left\{1+\frac{R^2\nabla^2}{3!}+\frac{R^4\nabla^4}{5!}+\dots\right\} f(x_0, y_0, z_0) \dots (6).$$

Next suppose that $f(x, y, z)$ is a rational homogeneous function of degree m; in that case the integral vanishes, unless $m-n$ is a positive even number; the theorem then becomes

$$\iint Y_n(x, y, z)\, f_m(x, y, z)\, dS$$

$$= 4\pi R^{m+n+2}\, 2^n \frac{\left(\frac{m+n}{2}\right)!}{\left(\frac{m-n}{2}\right)!\,(m+n+1)!} \nabla^{m-n}\, Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f_m(x, y, z) \quad \dots\dots\dots\dots\dots (7),$$

since all the other terms on the right hand vanish.

A particular case of (7) is

$$\iint x^\alpha y^\beta z^\gamma\, Y_n(x, y, z)\, dS$$

$$= 4\pi R^{n+\alpha+\beta+\gamma+2}\, 2^n \frac{\left(\frac{n+\alpha+\beta+\gamma}{2}\right)!}{\left(\frac{\alpha+\beta+\gamma-n}{2}\right)!\,(n+\alpha+\beta+\gamma+1)!}$$

$$\nabla^{\alpha+\beta+\gamma-n}\, Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) x^\alpha y^\beta z^\gamma \dots\dots\dots\dots (8),$$

where $\alpha+\beta+\gamma-n$ is an even integer.

In (7), put $m = n$; we then have

$$\iint Y_n(x, y, z) f_n(x, y, z)\, dS = 4\pi R^{2n+2} \frac{2^n n!}{(2n+1)!} Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f_n(x, y, z) \quad \text{.............(9)}.$$

This last theorem (9) includes, as a particular case, Maxwell's theorem, giving the surface-integral of the product of two surface harmonics of the same degree n. If $h_1, h_2 \ldots h_n$ are the axes of Y_n, we have*

$$Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{(2n)!}{2^n n!\, n!} \frac{\partial^n}{\partial h_1 \partial h_2 \ldots \partial h_n} + \text{a multiple of } \nabla^2,$$

and thus (9) becomes, in the case in which $f_n(x, y, z)$ is a spherical harmonic,

$$\iint Y_n(x, y, z) f_n(x, y, z)\, dS = \frac{4\pi R^{2n+2}}{2n+1} \frac{1}{n!} \frac{\partial^n}{\partial h_1 \partial h_2 \ldots \partial h_n} f_n(x, y, z) \quad \text{...........(10)},$$

which is Maxwell's theorem.† The theorem (9) is more general than Maxwell's, since $f_n(x, y, z)$ is not restricted to being a spherical harmonic, but may be any homogeneous function of degree n.

An important case of (5) is that in which $f(x, y, z)$ is of the form $F(\xi - x, \eta - y, \zeta - z)$, where ξ, η, ζ are the coordinates of a point outside the sphere. In that case, we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^t Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) F(\xi - x, \eta - y, \zeta - z)$$

$$= (-1)^n \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}\right)^t Y_n\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right) F(\xi - x, \eta - y, \zeta - z),$$

and, when $x = 0$, $y = 0$, $z = 0$, the expression on the right-hand side becomes

$$(-1)^n \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}\right)^t Y_n\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right) F(\xi, \eta, \zeta).$$

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* See my paper on "A Theorem in Differentiation," p. 68 of the present volume.
† See *Electricity and Magnetism*, second edition, Vol. I., p. 186.

We thus obtain the following theorem :—

$$\iint Y_n(x, y, z)\, F(\xi-x, \eta-y, \zeta-z)\, dS$$

$$= 4\pi R^{2n+2}(-1)^n \frac{2^n n!}{(2n+1)!}\left\{1+\frac{R^2\nabla^2}{2.2n+3}+\frac{R^4\nabla^4}{2.4.2n+3.2n+5}+\dots\right\}$$

$$Y_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right) F(\xi, \eta, \zeta) \quad\dots\dots\dots\dots\dots\dots(11),$$

where $$\nabla^2 = \frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}.$$

If $$\rho^2 = (\xi-x)^2+(\eta-y)^2+(\zeta-z)^2,$$

and $$F(\xi-x, \eta-y, \zeta-z) = \phi(\rho),$$

we obtain the theorem

$$\iint Y_n(x, y, z)\, \phi(\rho)\, dS$$

$$= 4\pi R^{2n+2}(-1)^n \frac{2^n n!}{(2n+1)!}\left\{1+\frac{R^2\nabla^2}{2.2n+3}+\dots\right\} Y_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\phi(u)$$

$$\dots\dots\dots\dots(12),$$

where $$u^2 = \xi^2+\eta^2+\zeta^2.$$

This theorem can be applied to the determination of the potential of a surface distribution on the sphere at an external point, under any law of force; I shall however consider this application in the more general case of a distribution on the surface of an ellipsoid.

If dv is an element of volume of a shell contained between the spheres of radii R and $R+dR$, we have $dv = dS.dR$; hence (12) may be written

$$\iint Y_n(x, y, z)\, \phi(\rho)\, dv$$

$$= 4\pi R^{2n+2} dR\,(-1)^n \frac{2^n n!}{(2n+1)!}\left\{1+\frac{R^2\nabla^2}{2.2n+3}+\dots\right\} Y_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\phi(u).$$

Multiply both sides by $\psi(R)$, and integrate with respect to R from

$R=0$ to $R=a$; we then obtain the formula

$$\iiint \psi(R)\, Y_n(x, y, z)\, \phi(\rho)\, dv$$

$$= 4\pi(-1)^n \frac{2^n n!}{(2n+1)!}\left\{\int_0^a R^{2n+2}\psi(R)dR + \int_0^a R^{2n+4}\psi(R)dR\,\frac{\nabla^2}{2.2n+3}+\dots\right\}$$

$$Y_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\phi(u) \quad \dots\dots\dots\dots\dots(13).$$

This volume-integral can be used to obtain the potential of a solid sphere of density $\psi(R)\, Y_n(x, y, z)$ at an external point, under any given law of force.

4. In the fundamental formula (5), put $R=1$, change x, y, z into $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ respectively; then the surface integral will be replaced by one taken over the surface of the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$$

instead of dS, we must write $\frac{p\,dS}{abc}$, where the new dS denotes an element of area of the ellipsoidal surface, and p is the perpendicular from the centre upon the tangent plane containing the element. We thus obtain the formula

$$\iint Y_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) f\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) p\, dS$$

$$= 4\pi abc\frac{2^n n!}{(2n+1)!}\left\{1+\frac{D^2}{2.2n+3}+\dots\right\} Y_n\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right)$$

$$f\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right),$$

or, changing $f\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$ into $f(x, y, z)$,

$$\iint Y_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) f(x, y, z)\, p\, dS$$

$$= 4\pi abc\,\frac{2^n n!}{(2n+1)!}\left\{1+\frac{D^2}{2.2n+3}+\frac{D^4}{2.4.2n+3.2n+5}+\dots\right\}$$

$$Y_n\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right) f(x, y, z) \dots\dots(14),$$

where D^2 denotes the operator $a^2\frac{\partial^2}{\partial x^2}+b^2\frac{\partial^2}{\partial y^2}+c^2\frac{\partial^2}{\partial z^2}$; as in (5), x, y, z are put equal to zero when the operations on the right-hand side have been performed.

Corresponding to (11), we obtain, by putting

$$f(x, y, z) = F(\xi-x,\ \eta-y,\ \zeta-z),$$

where ξ, η, ζ are the coordinates of an external point,

$$\iint Y_n\left(\frac{x}{a},\ \frac{y}{b},\ \frac{z}{c}\right) F(\xi-x,\ \eta-y,\ \zeta-z)\, p\, dS$$

$$= 4\pi abc\,(-1)^n\frac{2^n n!}{(2n+1)!}\left\{1+\frac{D^2}{2.2n+3}+\ldots\right\}$$

$$Y_n\left(a\frac{\partial}{\partial\xi},\ b\frac{\partial}{\partial\eta},\ c\frac{\partial}{\partial\zeta}\right) F(\xi, \eta, \zeta)\ldots\ldots(15),$$

where D^2 now denotes the operator $a^2\frac{\partial^2}{\partial\xi^2}+b^2\frac{\partial^2}{\partial\eta^2}+c^2\frac{\partial^2}{\partial\zeta^2}$.

Corresponding to (12), we obtain

$$\iint Y_n\left(\frac{x}{a},\ \frac{y}{b},\ \frac{z}{c}\right)\phi(\rho)\, p\, dS$$

$$= 4\pi abc\,(-1)^n\frac{2^n n!}{(2n+1)!}\left\{1+\frac{D^2}{2.2n+3}+\ldots\right\}$$

$$Y_n\left(a\frac{\partial}{\partial\xi},\ b\frac{\partial}{\partial\eta},\ c\frac{\partial}{\partial\zeta}\right)\phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right)\ldots(16),$$

where $$\rho^2 = (\xi-x)^2+(\eta-y)^2+(\zeta-z)^2.$$

In (16), change a, b, c into $\epsilon a, \epsilon b, \epsilon c$; then

$$\delta p\,.\,dS = \frac{\delta\epsilon}{\epsilon}\,.\,p\,dS = dv$$

is the element of volume of a shell bounded by the two ellipsoids whose semi-axes are $\epsilon a, \epsilon b, \epsilon c$ and $(\epsilon+d\epsilon)\,a$, $(\epsilon+d\epsilon)\,b$, $(\epsilon+d\epsilon)\,c$ respectively. Multiplying both sides of the equation by $\epsilon^{n-1}\psi(\epsilon)\,d\epsilon$, and integrating from $\epsilon = 0$ to $\epsilon = 1$, we obtain the formula

$$\iiint Y_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) \psi(\epsilon)\, \phi(\rho)\, dv$$

$$= 4\pi abc(-1)^n \frac{2^n n!}{(2n+1)!}\left\{\int_0^1 \epsilon^{2n+2}\psi(\epsilon)d\epsilon + \int_0^1 \epsilon^{2n+4}\psi(\epsilon)\, d\epsilon \frac{D^2}{2.2n+3} + \dots\right\}$$

$$Y_n\left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right)\phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right) \dots (17),$$

where the volume-integral is taken throughout the volume of the ellipsoid, and ϵ denotes $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^{\frac{1}{2}}$.

5. The first application that I shall make of the formulae of the last section is to express an external ellipsoidal harmonic in a series of spherical harmonics; I use throughout the notation in Mr. W. D. Niven's memoir* on ellipsoidal harmonics, in which memoir the expression is found by other methods.

At the surface of the ellipsoid, the ellipsoidal harmonic

$$G_n(x, y, z) \text{ or } \left\{\begin{matrix} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{matrix}\right\} \Pi\left(\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} - 1\right)$$

is equal to

$$\Pi(-\theta)\left\{\begin{matrix} & a^{-1}x & b^{-1}c^{-1}yz & \\ 1 & b^{-1}y & c^{-1}a^{-1}zx & a^{-1}b^{-1}c^{-1}xyz \\ & c^{-1}z & a^{-1}b^{-1}xy & \end{matrix}\right\} \times \left\{\begin{matrix} & a & bc & \\ 1 & b & ca & abc \\ & c & ab & \end{matrix}\right\}$$

$$\times \Pi\left(\frac{x^2}{a^2 . a^2+\theta} + \frac{y^2}{b^2 . b^2+\theta} + \frac{z^2}{c^2 . c^2+\theta}\right),$$

or

$$\left\{\begin{matrix} & a & bc & \\ 1 & b & ca & abc \\ & c & ab & \end{matrix}\right\} \Pi(-\theta)\, H_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right),$$

where $H_n(x, y, z)$ is a spherical harmonic.

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* *Philosophical Transactions*, 1891.

In (16), put $Y_n = H_n$, $\phi(\rho) = \frac{1}{\rho}$; we then have

$$\iint H_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)\frac{1}{\rho}\, p\, dS$$

$$= 4\pi abc\,(-1)^n \frac{2^n\, n!}{(2n+1)!}\left\{1+\frac{D^2}{2.2n+3}+\dots\right\}$$

$$H_n\left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right)\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}};$$

now
$$H_n\left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right)\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}}$$

$$= \left\{\begin{matrix} & a & bc & \\ 1 & b & ca & abc \\ & c & ab & \end{matrix}\right\} \Pi(-\theta)\, H_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}},$$

and thus we have

$$\iint \frac{1}{\rho}\, G_n(x, y, z)\, p\, dS$$

$$= 4\pi abc\kappa^2 \left\{\Pi(\theta)\right\}^2 \frac{2^n\, n!\,(-1)^n}{(2n+1)!}\left\{1+\frac{D^2}{2.2n+3}+\dots\right\}$$

$$\times H_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}},$$

where κ denotes the bracket containing a, b, c.

Now, if $\mathfrak{G}_n(\xi, \eta, \zeta) = G_n(\xi, \eta, \zeta)\, I_n(\xi, \eta, \zeta)$ denotes an external harmonic, where I_n is an integral of the form

$$\int_\epsilon^\infty \frac{d\lambda}{(\theta_1-\lambda)^2(\theta_2-\lambda)^2\dots(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}},$$

ϵ being the parameter of the confocal ellipsoid through the point ξ, η, ζ, the surface density σ of a distribution on the ellipsoid which will produce an external potential $\mathfrak{G}_n(\xi, \eta, \zeta)$, is given by

$$4\pi\sigma = -\frac{\partial\mathfrak{G}_n}{\partial\nu} + I_n\frac{\partial G_n}{\partial\nu},$$

where $\partial\nu$ is an element of normal, and I_n has its value at the surface of the ellipsoid; we thus obtain

$$4\pi\sigma = G_n(x, y, z)\frac{\partial\epsilon}{\partial\nu}\cdot\frac{1}{\{\Pi(\theta)\}^2}\cdot\frac{1}{abc\kappa^2},$$

and it is easily shown that $\dfrac{\partial \epsilon}{\partial \nu} = 2p$;

hence $$\sigma = \frac{pG_n(x, y, z)}{2\pi \{\Pi(\theta)\}^2} \cdot \frac{1}{abc\kappa^3}.$$

We obtain therefore the formula

$$\mathfrak{G}_n(\xi, \eta, \zeta) = (-1)^n \frac{2^{n+1} n!}{(2n+1)!} H_n\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right)$$
$$\times \left\{1 + \frac{D^2}{2 \,.\, 2n+3} + \frac{D^4}{2 \,.\, 4 \,.\, 2n+3 \,.\, 2n+5} + \ldots\right\} \frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(18),$$

which is Mr. Niven's expression for an external ellipsoidal harmonic in a series of spherical harmonics. It will be observed that in the above proof the distance of the point ξ, η, ζ from the origin is not necessarily greater than the greatest semi-axis of the ellipsoid, for, since (12) holds for all points external to the sphere, it follows that (16) and consequently (18) holds for all points external to the ellipsoid.

6 When it is required to find the potentials of ellipsoidal shells or of solid ellipsoids of variable density, at an external point, it is in ordinary cases better not to use the ellipsoidal harmonics, but to use the theorems (16) and (17). The formula (16) gives an expression for the potential of a surface distribution of which the density is $pY_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$, when the law of force is $-\phi'(\rho)$; in order to find the potential of a surface distribution of density $pF(x, y, z)$, it is necessary to express $F(x, y, z)$ as the sum of a number of functions of the form $Y_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$.

The formula (17) gives the potential, at an external point, of a solid ellipsoid whose density is

$$Y_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) \psi\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}\right),$$

and thus, as in the case of a shell, the potential of an ellipsoid whose density is

$$F(x, y, z)\, \chi\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

can be found, the law of attraction being $-\phi'(\rho)$. I shall obtain the formulae for the potential in one or two simple cases, as an example of the general method.

(*a*) To find the potential of a homeoid whose density is μxyz, the law of force being that of the inverse square: in this case xyz is a harmonic of degree $n = 3$; we thus obtain, from (16),

$$V = -\frac{4\pi a^3b^3c^3\mu}{105}\left\{1 + \frac{D^2}{2.9} + \frac{D^4}{2.4.9.11} + \dots\right\}\frac{\partial^3}{\partial\xi\partial\eta\partial\zeta}\cdot\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}},$$

where $$D^2 = a^2\frac{\partial^2}{\partial\xi^2} + b^2\frac{\partial^2}{\partial\eta^2} + c^2\frac{\partial^2}{\partial\zeta^2}.$$

(*b*) To find the potential of a solid gravitating ellipsoid whose density is

$$\mu z^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^m,$$

where m is any positive quantity: in this case we write μz^2 in the form

$$\frac{\mu c^2}{3}\left\{\left(\frac{2z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+\epsilon^2\right\},$$

the quantity $\frac{2z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}$ being of the form

$$Y_2\left(\frac{x}{a},\frac{y}{b},\frac{z}{c}\right).$$

The required potential is the sum of the potentials of the ellipsoids whose densities are $\frac{\mu c^2}{3}\left(\frac{2z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\epsilon^{2m}$, and $\frac{\mu c^2}{3}\epsilon^{2m+2}$; we thus obtain, from (17), for the potential required,

$$V = \tfrac{4}{3}\pi\mu abc^3\,\Sigma\,\frac{1}{(2t+3)(2t+5)(2t+2m+5)(2t+1)!}$$
$$D^{2t}\left(2c^2\frac{\partial^2}{\partial\zeta^2}-a^2\frac{\partial^2}{\partial\xi^2}-b^2\frac{\partial^2}{\partial\eta^2}\right)\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}}$$
$$+\tfrac{4}{3}\pi\mu abc^3\,\Sigma\,\frac{1}{(2t+2m+5)(2t+1)!}D^{2t}\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}}.$$

(*c*) To find the potential of a solid gravitating ellipsoid whose density is

$$\mu\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)^{m-1},$$

where m is a positive quantity: we have

$$\int_0^1 \epsilon^{2t+2} (1-\epsilon^2)^{m-1} \, d\epsilon = \frac{\Gamma(t+\frac{3}{2})\,\Gamma(m)}{2\Gamma(m+t+\frac{3}{2})},$$

so that the required potential is

$$V = 2\pi abc \sum_{t=0}^{t=\infty} \frac{\Gamma(t+\frac{3}{2})\,\Gamma(m)}{\Gamma(m+t+\frac{3}{2})(2t+1)!} D^{2t} \frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}}.$$

7. I shall now proceed to modify the formula (17), so that it may be adapted to the case in which the ellipsoid becomes an elliptic disc; we put $c = 0$, and in this case we must suppose that $Y_n(x, y, z)$ does not contain z, or that $Y_n(x, y)$ is one of the harmonics

$$(x+\iota y)^n + (x-\iota y)^n, \quad \iota\left\{(x+\iota y)^n - (x-\iota y)^n\right\}.$$

The mass of a prismatic section of the ellipsoid of which the base is the element $dxdy$, is

$$2c\,dxdy \int_a^1 \frac{\psi(\epsilon)\,\epsilon}{\sqrt{\epsilon^2-a^2}} \, d\epsilon,$$

where
$$a^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

If we put $\chi(a)$ for the value of the definite integral, we find, on dividing both sides of the equation (17) by $2c$,

$$\iint \left(\frac{x}{a} \pm \iota \frac{y}{b}\right)^n \chi(a)\,\phi(\rho)\,dxdy$$

$$= 2\pi ab(-1)^n \frac{2^n n!}{(2n+1)!} \left\{ \int_0^1 \epsilon^{2n+2} \psi(\epsilon)\,d\epsilon + \int_0^1 \epsilon^{2n+4} \psi(\epsilon)\,d\epsilon \frac{D^2}{2,2n+3} + \dots \right\}$$

$$\left(a\frac{\partial}{\partial \xi} \pm \iota b \frac{\partial}{\partial \eta}\right)^n \phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right) \dots (19),$$

where
$$D^2 = a^2 \frac{\partial^2}{\partial \xi^2} + b^2 \frac{\partial^2}{\partial y^2}.$$

This formula gives the potential of an elliptic disc of density $\left(\frac{x}{a} \pm \iota \frac{y}{b}\right)^n \chi\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}\right)$ at an external point, the law of force being $-\phi'(\rho)$.

As an example of the use of (19), suppose

$$\psi(\epsilon) = (1-\epsilon^2)^{m-\frac{1}{2}};$$

then

$$\chi(a) = \int_a^1 \frac{(1-\epsilon^2)^{m-\frac{1}{2}}}{\sqrt{\epsilon^2-a^2}}\,\epsilon\,d\epsilon.$$

Changing the variable in the integration to v, where

$$(1-\epsilon^2) = v\,(1-a^2),$$

we have

$$\chi(a) = \tfrac{1}{2}(1-a^2)^m \int_0^1 v^{m-\frac{1}{2}}(1-v)^{-\frac{1}{2}}\,dv;$$

thus

$$\chi(a) = \tfrac{1}{2}(1-a^2)^m \frac{\Gamma(m+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(m+1)} = \frac{\pi}{2}(1-a^2)^m \frac{(2m)!}{2^{2m}(m!)^2}.$$

Putting $n=0$, in (19) we find, for the potential of an elliptic disc of uniform thickness, and of density $\mu\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^m$, the value

$$V = 2^{2m+2}\frac{(m!)^2}{(2m)!}\mu ab \sum_{t=0}^{t=\infty}\int_0^1 \epsilon^{2t+2}(1-\epsilon^2)^{m-\frac{1}{2}}\,d\epsilon\,\frac{D^{2t}}{(2t+1)!}\,\phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right)$$

$$= m!\,\mu\pi ab \sum_{t=0}^{t=\infty} \frac{1}{(t+m+1)!\,t!\,2^{2t}}\left(a^2\frac{\partial^2}{\partial\xi^2}+b^2\frac{\partial^2}{\partial\eta^2}\right)^t \phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right).$$

As another example, suppose it is required to find the potential of a disc of density $\mu xy\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^m$; in this case we put $n=2$, since xy is a harmonic of the second degree. The values of $\psi(\epsilon)$ and $\chi(a)$ are the same as before; we have therefore

$$V = 2^{2m+2}\frac{(m!)^2}{(2m)!}\mu a^3 b^3 \frac{2^2.2!}{5!}\sum_{t=0}^{t=\infty}\int_0^1 \epsilon^{2t+4}(1-\epsilon^2)^{m-\frac{1}{2}}\,d\epsilon$$

$$\frac{15D^{2t}}{(2t+1)!\,(2t+3)(2t+5)}\,\frac{\partial^2}{\partial\xi\,\partial\eta}\,\phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right)$$

$$= \tfrac{1}{2}m!\,\mu\pi a^3 b^3 \sum_{t=0}^{t=\infty}\frac{1}{t!\,(t+m+2)!\,(2t+5)\,2^{2t}}$$

$$\times\left(a^2\frac{\partial^2}{\partial\xi^2}+b^2\frac{\partial^2}{\partial\eta^2}\right)^t \frac{\partial^2}{\partial\xi\,\partial\eta}\,\phi\left(\sqrt{\xi^2+\eta+\zeta^2}\right).$$

8. A line-integral round the circumference of a circle, analogous to the surface integral in (4), may be found.

We have $$e^{\iota\rho\cos\phi} = J_0(\rho) + 2\sum_{n=1} \iota^n J_n(\rho)\cos n\phi,$$

where $$J_n(\rho) = \frac{\rho^n}{2^n n!}\left\{1 - \frac{\rho^2}{2.2n+2} + \frac{\rho^4}{2.4.2n+2.2n+4} - \dots\right\};$$

thence we have

$$e^{ax+\beta y} = \left\{1 + \frac{R^2(a^2+\beta^2)}{2^2} + \frac{R^4(a^2+\beta^2)^2}{2^2.4^2} + \dots\right\}$$

$$+\Sigma\frac{2R^n(a^2+\beta^2)^{\frac{1}{2}n}}{2^n n!}\left\{1 + \frac{R^2(a^2+\beta^2)}{2.2n+2} + \dots\right\}\cos n\phi,$$

where $x^2+y^2 = R^2$, and $\cos\phi = \dfrac{ax+\beta y}{R\sqrt{a^2+\beta^2}} = \cos(\theta-\beta)$,

where $x = R\cos\theta$, $y = R\sin\theta$, $\cos\beta = \dfrac{a}{\sqrt{a^2+\beta^2}}$, $\sin\beta = \dfrac{\beta}{\sqrt{a^2+\beta^2}}$.

The value of the integral $\int e^{ax+\beta y}(x\pm\iota y)^n\, ds$, taken round the circumference of a circle of radius R, whose centre is at the origin, is easily seen to be

$$2\pi R^{2n+1}\frac{1}{2^n n!}\left\{1 + \frac{R^2(a^2+\beta^2)}{2.2n+2} + \frac{R^4(a^2+\beta^2)^2}{2.4.2n+2.2n+4} + \dots\right\}(a\pm\iota\beta)^n.$$

As before, we obtain from this result the theorem

$$\int (x\pm\iota y)^n f(x, y)\, ds$$

$$= 2\pi R^{2n+1}\frac{1}{2^n n!}\left\{1 + \frac{R^2\nabla^2}{2.2n+2} + \frac{R^4\nabla^4}{2.4.2n+2.2n+4} + \dots\right\}$$

$$\left(\frac{\partial}{\partial x} \pm \iota\frac{\partial}{\partial y}\right)^n f(x, y) \quad\dots\dots(20),$$

where, on the right-hand side, x and y are put equal to zero after the operations are performed; $f(x, y)$ must be finite and continuous within the circle, and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

If $$f(x, y) = F(\xi - x, \eta - y),$$

where ξ, η, ζ are the coordinates of an external point, (20) becomes

$$\int (x \pm \iota y)^n F(\xi - x, \eta - y, \zeta)\, ds$$

$$= 2\pi R^{2n+1} \frac{(-1)^n}{2^n n!} \left\{ 1 + \frac{R^2 \nabla^2}{2.2n+2} + \dots \right\} \left(\frac{\partial}{\partial \xi} \pm \iota \frac{\partial}{\partial \eta} \right)^n F(\xi, \eta, \zeta) \dots (21),$$

where $$\nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}.$$

The theorem for the ellipse which can be derived from (20), is

$$\int \left(\frac{x}{a} \pm \iota \frac{y}{b} \right)^n f(x, y)\, p\, ds$$

$$= 2\pi ab \frac{1}{2^n n!} \left\{ 1 + \frac{D^2}{2.2n+2} + \frac{D^4}{2.4.2n+2.2n+4} + \dots \right\}$$

$$\times \left(a \frac{\partial}{\partial x} \pm \iota b \frac{\partial}{\partial y} \right) f(x, y) \dots (22),$$

where $$D^2 = a^2 \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^2}{\partial y^2},$$

and, as before, the value of the expression on the right-hand side is taken at the origin. In the special case

$$f(x, y) = F(\xi - x, \eta - y, \zeta);$$

this becomes $$\int \left(\frac{x}{a} \pm \iota \frac{y}{b} \right)^n F(\xi - x, \eta - y, \zeta)\, p\, ds$$

$$= 2\pi ab \frac{(-1)^n}{2^n n!} \left\{ 1 + \frac{D^2}{2.2n+2} + \dots \right\} \left(a \frac{\partial}{\partial \xi} \pm \iota b \frac{\partial}{\partial \eta} \right)^n F(\xi, \eta, \zeta) \dots (23),$$

where $$D^2 = a^2 \frac{\partial^2}{\partial \xi^2} + b^2 \frac{\partial^2}{\partial \eta^2}.$$

We might proceed to obtain, from these last results, integrals taken over the area of the ellipse: such integrals we have, however, obtained as a special case of the ellipsoidal volume-integrals; it is therefore unnecessary to proceed further in this direction.