line L_1 in the plane Π_1 always passes through a fixed point of the plane Π. And secondly, if a fixed circle C_1 in the plane Π_1 always touches a fixed line L in the plane Π, this is equivalent to the condition that a fixed point in the plane Π_1 is always situate in a fixed line L_1 in the plane Π_1. The different forms of condition therefore are—

(α) A fixed circle C_1 in the plane Π_1 always touches a fixed circle C in the plane Π (where, as above, either circle indifferently may be reduced to a point).

(β) A fixed line L_1 in the plane Π_1 always passes through a fixed point C in the plane Π.

(γ) A fixed point C_1 in the plane Π_1 is always situate in a fixed line L of the plane Π.

Hence, if the motion of the plane Π_1 satisfy any two such conditions (of the same form or of different forms, viz., the conditions may be each α, or they may be α and β, &c.), then the motion of the plane Π_1 will depend on a single variable parameter, and the question arises as to the locus described by a given point or enveloped by a given line of the plane Π; and again of the locus traced out or enveloped on the moving plane Π_1 by a given point of the plane Π. The case considered in the present paper is of course a particular case of the two conditions being each of them of the form (α).

It may be remarked, that if the two conditions be each of them (β), then there will be in the plane Π_1 a fixed point C_1 which describes a circle; and similarly, if the two conditions be each of them (γ), then there will be in the plane Π_1 a fixed point C_1 which describes a circle;* that is, the combination ($\beta\beta$) is a particular case of ($\alpha\beta$), and the combination ($\gamma\gamma$) a particular case of ($\alpha\gamma$).

In a discussion on the paper, Mr. Roberts gave some additional results bearing on the subject, and Mr. Spottiswoode stated that many of the curves drawn (and exhibited) were recognized by him as having come under his notice in the course of experiments he had recently made with elastic strings.

Mr. Roberts then read his Paper

On the Ovals of Descartes.

1. As a contribution to the theory of Cartesian Ovals, I wish (1) to fix the interpretation of the polar equation; (2) to explain a method of description by points, which does not imply the existence of more than one real and finite single axial focus, and is therefore applicable to Cartesians generally, whether the axial single foci are all real, or contain an

* The theorem is, that if an isosceles triangle, on the base AA′ and with angle $= 2\omega$ at the vertex C, slide between two lines OA, OA′ inclined to each other at an angle ω, in such manner that C is the centre of the circle circumscribed about OAA′, then the locus of C is a circle having O for its centre.

imaginary pair; and, by way of illustration, (3) to state some conclusions which afford general interpretations of analytical results incompletely explained or overlooked in consequence of the ambiguity of the polar equation.

Since for a class of Cartesians, viz. those to which I have alluded as having two imaginary axial foci, the vectorial equations so effectively employed by Mr. Crofton (Proc. No. 6, p. 5) become unreal,—that is to say, the coefficients are imaginary in equations representing real curves,—an attempt to interpret the polar equation, which is not subject to this drawback, will not be without utility. I am aware that imaginary vectorial equations, considered as forms derived from real equations, may be interpreted; yet, as remarked by M. Chasles, it is desirable sometimes to use forms which do not imply more than one real axial single focus given. In any case, I may justify myself by another remark of his apropos of these curves,—"On ne saurait avoir trop de moyens differens de décrire une même courbe, parceque chacun exprime une propriété caractéristique de la courbe d'où dérivent naturellement plusieurs autres propriétés qui n'apparaissent pas aussi aisément dans les autres modes de description."

I do not, of course, aim at giving a complete account of the curves in question as far as they are known, and I am afraid that some of the points touched upon are treated with too great conciseness.

2. According to their most general definition, Cartesians are quartic curves having the circular points at infinity for cusps. Their class being 6 and their order 4, they possess only one double tangent. It immediately follows that they consist of not more than two ovals; and, if real, the two ovals lie one within the other. The double tangent also must touch the outer oval only, if the points of contact are real.

Since the circular points at infinity are cusps, the system of Cartesians having the same triple focus* and the same double tangent touching at the same points, real or imaginary, may be represented by $S^2+k(x-l)=0$, where $S=0$ is the equation of a circle, whose centre is at the triple focus and which passes through the points of contact of the double tangent $x-l=0$. Hence, evidently, the curves are symmetrically formed on each side of an axis.

This involves the property, that there are three axial single foci, and that the axis passes through the triple focus. For, generally, if a curve is symmetrical with regard to an axis, and p tangents can be drawn

* In employing the term "triple focus" to designate the intersection of the cuspidal tangents, I follow Dr. Salmon and others. ("Higher Plane Curves," p. 128.) In the case of simply circular curves, the intersections of tangents at the circular points at infinity are analogously called "double foci." In the case of Cartesians, the triple focus is also the centre, and has been so designated. The name "cuspo-focus" has been suggested to me as more satisfactory; this terminology however seems to be incapable of extension.

from a circular point at infinity to the curve, there are p corresponding axial foci; since the two circular points at infinity are symmetrical with regard to any axis.

I have long ago pointed out ("Quarterly Journal of Pure and Applied Mathematics," vol. ii. p. 196) that if any curve has four concyclic foci, there are four sets of concyclic foci lying on four circles which cut one another orthogonally. Consequently, in the present case, since there is in a certain sense a fourth focus on the axis, but infinitely distant,* there are three circles (one of them imaginary) which have their centres on the axis and contain the non-axial single foci of the curve in fours, only 6 of these foci however being finite. Encountering these circles as limiting cases of triconfocal Cartesians, Mr. Crofton has called them confocal circles. It will be convenient to call the three single and (in the case of proper Cartesians) finite axial foci, simply, the axial foci.

3. We may conveniently take the axis of the curve for the axis of x, and a line perpendicular to it through a focus for the axis of y.

Writing ρ^2 for x^2+y^2, we have for the equation of a complete Cartesian, a focus being the pole,

$$[\rho^2-2Bx+C^2]^2-4A^2\rho^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

The equivalent form

$$\{(x-B)^2+y^2-B^2-2A^2+C^2\}^2-4A^2(A^2-C^2+2Bx)=0 \quad \ldots\ (A')$$

shows that ($x=B$, $y=0$) are the coordinates of the triple focus, and the line $x=\dfrac{C^2-A^2}{2B}$ is the double tangent. The radius of the circle which passes through the points of contact of the double tangent, and has its centre at the triple focus, is $\sqrt{(B^2+2A^2-C^2)}$. The radius of the corresponding circle, having its centre at the focus pole, is A.

The first polar of the origin is

$$2(\rho^2-2Bx+C^2)(C^2-Bx)-4A^2\rho^2=0.$$

Subtracting this from (A), we get

$$(\rho^2-2Bx+C^2)(\rho^2-C^2)=0.$$

Hence C is the radius of a circle whose centre is at the origin, and which passes through the points of contact of tangents drawn from the focus pole to the curve. The first factor relates to the tangents which make the focus.

The constants of (A) are now defined in relation to the curve.

* As to this, see "Higher Plane Curves," p. 126. Cartesians constitute a sub-class of a class of Bicircular Quartics. In the class, four concyclic foci have become linear. In the sub-class, since only three of these foci are finite, the remaining one is assumed to be infinitely distant. Infinitely distant points are, however, very often analytical conventions. In fact, all points which at infinity remain finitely distant from each other analytically coincide. Thus two parallel straight lines at a distance k from each other can only intersect the line at infinity in points at a distance k from each other; but by a convenient convention, the lines are said to intersect at infinity.

4. We proceed to identify (A) with the vectorial form $m\rho_1 - \rho = \kappa$; that is to say, with $m\sqrt{(\rho^2 - 2kx + k^2)} - \rho = \kappa$,

or $$\left(\rho^2 - \frac{2m^2k}{m^2-1}x + \frac{m^2k^2 - \kappa^2}{m^2-1}\right)^2 - 4\frac{\kappa^2}{(m^2-1)^2}\rho^2 = 0 \quad \text{.........(B).}$$

The equalities $A^2 = \frac{\kappa^2}{(m^2-1)^2}$, $B = \frac{m^2k}{m^2-1}$, $C^2 = \frac{m^2k^2 - \kappa^2}{m^2-1}$

give $$k^2 - \frac{B^2+C^2-A^2}{B}k + C^2 = 0 \quad \text{..................... (C),}$$

$$\kappa^2 + \frac{A^2+C^2-B^2}{A}\kappa + C^2 = 0 \quad \text{..................... (D),}$$

the roots of (C) representing, of course, the two axial foci not at the origin.

There is a noteworthy reciprocity between the forms

$$(\rho^2 - 2Bx + C^2)^2 - 4A^2\rho^2 = 0,$$
$$(\rho^2 - 2Ax + C^2)^2 - 4B^2\rho^2 = 0,$$

which continually reappears.

On the Polar Equation and its interpretation.

5. It is essential to distinguish between the ovals which are said to be conjugate to one another. For this purpose, we may advantageously use the equations

$$\rho^2 - 2A\rho - 2Bx + C^2 = 0 \quad \text{........................ (E),}$$
$$\rho^2 + 2A\rho - 2Bx + C^2 = 0 \quad \text{........................ (F),}$$

the left-hand members of which are the factors of (A). I find it less confusing to keep these forms distinct than to combine them in one sole equation.

If $\rho \cos w$ be written for x, the equations become polar, and the importance of distinctly determining their meaning makes it worth while to dwell on them at some length.

Negative values of ρ indicate that the corresponding lengths are measured on the vector or transversal produced below the axis. Positive values are taken above the axis.

The angle w being taken from 0° to 180° in both equations, we shall have the complete curve. The points determined by (F) are, however, the reflexions, relative to the axis, of the points determined by (E); and, generally speaking, we need only consider specially the latter form.

The portions of the ovals represented by (E) will reach the axis without passing it; they will consist, in fact, of half-ovals.

6. It may be remarked at once, that if C^2 be positive, the signs of the roots of (C) are alike. The origin, therefore, is an *extreme* focus.

On the other hand, if C^2 is negative, the roots of (C) are unlike in sign, and the origin is the *middle* focus.

We will, by convention, measure w from the side of the origin on which the triple focus lies; B will then be essentially positive.

Making $w=0$ and $=\pi$ successively in (E), we have for the axial points of the curve

$$\rho = A+B \pm \sqrt{\{(A+B)^2-C^2\}},$$
$$\rho = A-B \pm \sqrt{\{(A-B)^2-C^2\}}.$$

Consequently, if C^2 is negative, the four intersections are all real. If C^2 is positive, we must have $(A+B)^2 > C^2$ for a real curve. If $A \sim B$ is $>C$, there will be two real ovals; and if $A \sim B$ is $<C$, there will be one real oval only.

The discriminant of (C) is of the form

$$-\frac{(A+B+C)(A+B-C)(A-B+C)(-A+B+C)}{4B^2}.$$

It is easy to see that, $A+B$ being $>C$, the roots of (C) are imaginary if $A \sim B$ is $<C$, and *vice versâ*. Hence, *when there is only one real oval, there are two imaginary axial foci, and vice versâ.*

7. For an angle w, we have

$$\rho = A+B\cos w \pm \sqrt{\{(A+B\cos w)^2-C^2\}}.$$

If there be a passage from real to imaginary values of ρ as the vector moves about the focus pole, such passage will take place when there is a real tangent from the focus pole, and

$$\cos w = \frac{C-A}{B} \text{ or } -\frac{C+A}{B}.$$

There cannot be more than two breaks of this kind. The following schemes show the different cases for C^2 positive, and are obtained by reference to the limiting values of $\cos w$:—

$A<C$

(1)	No breaks	$C-A>B$,	$C+A>B$	(*a*),
	One break	$C-A>B$,	$C+A<B$	(*b*),
		$C-A<B$,	$C+A>B$	(*c*),
	Two breaks	$C-A<B$,	$C+A<B$	(*d*).

$A>C$

(2)	No breaks	$A-C>B$,	$C+A>B$	(*a′*),
	One break	$A-C>B$,	$C+A<B$	(*b′*),
		$A-C<B$,	$C+A>B$	(*c′*),
	Two breaks	$A-C<B$,	$C+A<B$	(*d′*).

As to (*a*) and (*b*), for a real Cartesian, we have seen that $(A+B)^2$ is $>C^2$; this disposes of these cases.

As to (c), when A is $<C$, we cannot have $A-B>C$ and $A>B$, nor can we have $B-A>C$, $B>A$, on account of $C+A>B$.

As to (*d*), we cannot have $A>B$, nor $C>B-A$, $B>A$.

As to (a'), we cannot have $B>A$.

As to (b'), the conditions are inconsistent.

As to (c'), we must have $B>A$, $C>B-A$.

And as to (d'), we cannot have $A>B$ nor $C>B-A$.

8. We have now a reduced scheme,

$$(3)\quad \begin{cases} \text{No breaks} & A>B,\ A-B>C, \\ \text{One break} & A \backsim B<C, \\ \text{Two breaks} & B>A,\ B-A>C. \end{cases}$$

In the first case, the equation (E) represents two half-ovals on the *same* side of the axis, and therefore the origin must be the *extreme* interior focus (§ 6).

In the second case, we have only one real oval, half of which is represented by (E).

In the third case, the equation (E) represents two half-ovals on *different* sides of the axis, and the origin must be the *external* focus (§ 6).

If C^2 be negative, there are evidently no breaks. The equation (E) represents two half-ovals, on *different* sides of the axis. The origin is the *middle* focus (§ 6).

Let the points in which a vector through the origin meets the part curve represented by (E) be called *corresponding* points. We shall see hereafter (§ 25) that these points are inverse with respect to a definite circle whose centre is the focus pole.

Then, in the first case, corresponding points lie on adjacent parts of the conjugate ovals. In the second case, they lie on the one real oval. In the third case, they lie on one oval. Lastly, when C^2 is negative, they lie on non-adjacent parts of the conjugate ovals.

In what precedes, I have, for simplicity's sake, not specially noticed the case of equalities. These indicate singularities, which do not affect general results. Thus, $C^2=(A-B)^2$ shows that the Cartesian is a Limaçon, and $C^2=(A+B)^2$ indicates a point. If $A=C$, the double tangent passes through the focus pole.

When we have recognized the position of corresponding points for a given focus as pole, the interpretation of the analytical results becomes easy. The conclusions just arrived at are therefore important in many applications.

On the Description of the Curves by the Transformation of a Circle.

9. The equation of a circle

$$\rho^2-2Bx+B^2-R^2=0$$

can be transformed into (E) and (F) by very simple substitutions.

Put $\rho \mp A$ for ρ, and $x-a$ for x. Then we have

$$(\rho \mp A)^2-2Bx+B^2+2Ba-R^2=0,$$

which coincides with (E) and (F) if

$$A^2+B^2+2Ba-R^2=C^2.$$

For brevity, I call a straight line parallel to the axis of y a y-line. Let ρ', x' be the mixed coordinates of a point in (E) and its reflection, and let ρ_1, x_1 be the mixed coordinates of the corresponding point and its reflection on the circle. Then a corresponding point and its reflection on the Cartesian are determined by taking $\rho'=\rho_1 \pm A$, $x'=x+a$; that is to say, by the intersection of the vector $\rho_1 \pm A$ with a y-line separated from the corresponding y-line relative to the circle by a distance a.

We may state the matter thus:—Given a straight line, a circle whose centre is upon the line, and a Cartesian having the line for its axis; we can always determine a point upon the line as pole of the circle, so that to a point of the circle whose coordinates are (ρ, x) shall correspond a point of the Cartesian whose coordinates are $(\rho \pm A, x+a)$ relative to an axial focus as origin.

The distance of the pole of the circle from its centre is evidently equal to the distance of the triple focus from the corresponding focus pole, and A is, as we have seen, the distance of the points of contact of the double tangent from the focus pole, if those points are real.

It is convenient to make corresponding y-lines coincide. This will be the case when the distance between the focus pole and the pole of the circle is a.

The mode of description now given is evidently a modified extension to Cartesian generally of the well known method of describing a Limaçon, the equation of which being written in the form $\rho = \mp A + B \cos w$, suggests the transformation $\rho \pm A$ for ρ in the equation $\rho = B \cos w$. In this case, however, the angles correspond; and in the general case it seems necessary to make the x coordinate correspond.

The auxiliary circle may even degenerate to a point; in which case, however, the construction depends on what is stated in the next article.

10. In the transformation now treated of, a real point on the Cartesian may correspond to an imaginary point on the auxiliary circle. We must take account therefore of the real coordinates (ρ, x) of such a point on the circle. It is evident that a constant length added to ρ may convert an imaginary point into a real one. If it were not easy to find the values of ρ in such cases, this mode of constructing the Cartesian would practically fail. The following methods of finding these values present themselves.

Let O be the pole, C the circle, L the foot of the y-line corresponding to $x = OL$.

Construction.—Through the pole draw a y-line, and take OM upon it equal to the radius of the circle C. Also through C, the centre of the circle, draw a y-line. Then, with centre L and radius LM, describe a circle meeting the y-line through C in N. CN is the length of the ρ corresponding to $x = OL$.

For we have, r being the radius of the circle,

$$OL^2+r^2 = CN^2+(OL-OC)^2,$$

or

$$CN^2-2OC\,.\,OL+OC^2-r^2 = 0.$$

Or we may, with L as centre, describe a circle cutting the given circle C orthogonally. A tangent from the pole to this circle will give the required vector length.

11. I find it best to take a=A, so that we have the forms

$$\rho^2-2A\rho-2Bx+(A+B)^2-R^2 = 0 \text{ .... (G)},$$

$$\rho^2+2A\rho-2Bx+(A+B)^2-R^2 = 0 \text{ .... (H)};$$

the pole of the auxiliary circle is here supposed to be intermediate between the focus pole and the centre. The circle is doubly tangential to the curve on the axis, and may be called a diametral circle.

Putting $x=\rho$, we have

$$\rho = A+B \pm R,$$

$$\rho = B-A \pm \sqrt{R^2-4AB}.$$

If R′ be the radius of the diametral circle corresponding to $x = \rho$ in (H), we have $R'^2 = R^2-4AB.$

The condition for imaginary axial foci is therefore

$$R'^2 = R^2-4AB < 0,$$

which, when $(A+B)^2-C^2$ is put for R^2, agrees with the condition previously given.

As a matter of construction, it is better to interpret (G) and (H) so that each may represent an oval. To this end, we must include some negative values in (G). For, to complete the outer oval, it will often be necessary to take the difference of the ρ of the circle and A, instead of the sum. This happens, for instance, when the outer oval is indented.

12. If the pole of the auxiliary circle is further from its centre than the focus pole, and on the same side, it is evident geometrically that we have to write $(B-A)^2-R^2$ instead of $(A+B)^2-R^2$. If the focus pole and the pole of the circle lie on different sides of the centre, we have B negative.

Supposing $R^2-4AB > 0$, so that there are two real ovals, there are two diametral circles in each case, by aid of which the Cartesian may be constructed when a focus is given. For if ρ_1, ρ_2 be the axial values for (G), and ρ', ρ'' those for (H), when $x=\rho$, we may take the diametral circle through (ρ_1, ρ_2), or the diametral circle through (ρ', ρ''), as the auxiliary circle. But it is to be observed, that we cannot for a given focus take at will any one of the six diametral circles as the auxiliary circle.

The matter will be made more clear by a figure (Fig. 1). Let F be the focus pole of the Cartesian, APB the given auxiliary circle whose centre is C. On FC as diameter draw the circle FPC.

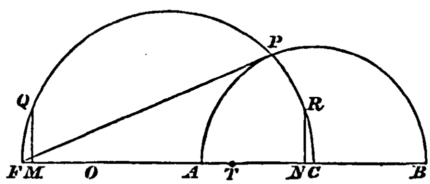


FIG. 1.

Draw the y ordinates of the circle FPC, $QM = RN = \frac{AC}{2}$.

Then if O, the pole of the circle, be anywhere between M, N, we have a Cartesian with a pair of imaginary axial foci. If O be on the left of F, F is the external focus, and the inner oval touches at A, B. If O lie between F and M, there are two ovals, and the circle touches the outer oval. If O be between N and C, there are two ovals, and F is the innermost focus. The circle APB lies between the ovals. If O lie beyond C, the quantity B becomes negative, or the position of the triple focus changes sign relative to FC. This indicates a change of front, so to speak, of the curve. The exterior focus changes sides with respect to the given circle.

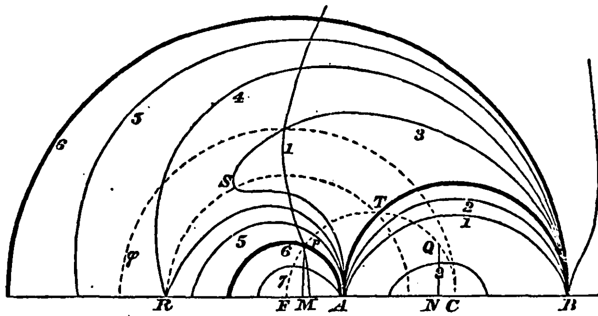
If F is situated within the circle APB, the focus F is the middle one. Taking FT = OC, T is the triple focus. Also FP is the radius C. The circle whose radius is FP also passes through the antifoci of the axial foci not at the origin. This is true also of a circle whose centre is the triple focus and whose radius is FO.

On Certain Systems of Cartesians.

13. In the series of curves which can be thus described, by the change of position of O, we have a focus, a diametral circle, and the axis given. Referring to (E), we have $A \pm B$ given and C^2 given. It follows that the apices relative to the given focus lie on the circle $\rho^2 = C^2$, and the vertices relative to the axis of the antifoci of the axial foci not at the origin are on the circle $\rho^2 = (A \pm B)^2$. The system shows how we pass from a Limaçon having a conjugate point through a series of Cartesians of the second kind to a Limaçon having a node.

The figure (2) will illustrate such a system. The figure is drawn, however, on too small a scale for minute accuracy.

FIG. 2.



The focus pole is F. The auxiliary circle is ATB, whose centre is C. The apices lie on the circle RST. The extra-axial vertices lie on the circle ϕC. The ovals and their conjugates are numbered in the order of the positions of the corresponding pole of the auxiliary circle. Thus, for (1), the pole is to the left of F; for (2), it lies between F and M $\left(\text{MP} = \frac{\text{AC}}{2}\right)$; for (3), between M and N $\left(\text{NQ} = \frac{\text{AC}}{2}\right)$, (3) therefore has no real conjugate oval; for (4), which is a Limaçon, the pole is N; for (5), the pole lies between N and C; and for (6), (concentric circles,) it is at C. The curve (7) shows the effect of taking the pole on the right of C. For the pole M, we should have a Limaçon with a conjugate point, which curve, however, is not drawn.

When two axial foci are imaginary, it cannot be properly said that they are within or without the curve; but it will be observed that their mean, which is real, may lie within or without the oval. The real antifoci are within the oval.

In a similar manner, we can readily construct a diagram to illustrate the system of Cartesians when A and B are given; for in this case we have F, O, C given, and the system of auxiliary circles will be concentric. We shall see that the apices relative to F will lie on a Limaçon, and a set of extra-axial vertices will lie on a circle.

It will be observed that two concentric circles constitute a complete Cartesian. The triple focus and two axial single foci are united at the common centre, and the third axial focus has gone off to infinity.

FIG. 3.

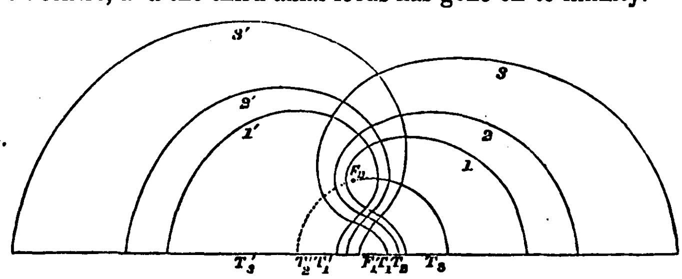


Figure (3) illustrates a system of tri-confocal Cartesians, two of the common axial foci being imaginary. F_1 is the real axial focus, F_2 is one of the real antifoci of the imaginary axial foci. There are then two sub-systems of curves, with respect to one of which systems F_1 lies between the axis of F_2 and the curve, and with respect to the other F_1 lies outside the axis of F_2. These systems cut each other orthogonally, but the curves of one and the same sub-system do not cut each other. The circle described, with F_1 as centre and F_1F_2 as radius, forms a limiting curve, or rather two limiting curves, of the series. In fact, the doubled arcs terminated at the real extra-axial foci constitute degenerate Cartesians. The letters T_1, T_2, T_3, $T_{1'}$, $T_{2'}$, $T_{3'}$ indicate the positions of the triple foci of the curves 1, 2, 3, 1′, 2′, 3′. For the circle, the triple focus unites with F_1. In the case of these Cartesians, the whole series of curves may have real contact with their double tangents. In order that there may be imaginary contact, the angle which F_1F_2 makes with the axis must be smaller than in the figure. For real contact, we must have generally, by (A′),

$$B^2+2A^2-C^2 > \frac{(A^2+2B^2-C^2)^2}{4B^2},$$

or

$$2AB > A^2 \sim C^2.$$

Now, if A is $> C$, we have, considering the triangle formed by F_1, F_2, and T, $C^2 = A^2+B^2-2AB\cos\theta$, or $A^2-C^2 = 2AB\cos\theta - B^2$. Hence, in this case, the contacts are real. But if C is $> A$, we have

$$A^2 = C^2+B^2-2BC\cos\phi, \quad \text{or} \quad C^2-A^2 = 2BC\cos\phi - B^2;$$

and for imaginary contacts we must have

$$C\cos\phi > A + \frac{B}{2},$$

which is not the case in the figure. The curves complete themselves symmetrically on the other side of the axis.

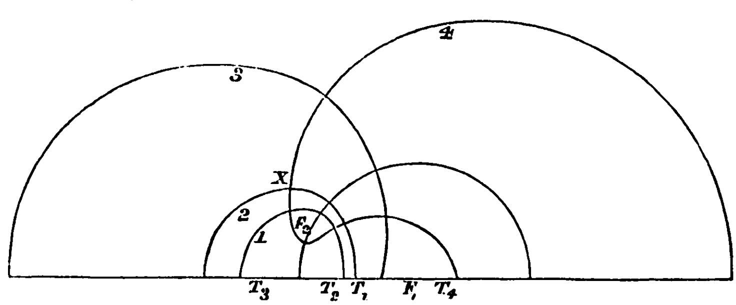


FIG. 4.

Figure (4) relates to a similar triconfocal system, in which, however, curves which have imaginary contact with their double tangents occur.

14. A system of Cartesians represented by $m\rho' - \rho = \kappa$, where ρ', ρ are vectors from two given axial foci, m is constant, and κ variable, may be simply defined as a system having in common two axial foci and the triple focus.

For, comparing (A) and (B), we see that B is constant if m and k are given.

The equation $m\rho' - \rho = 0$ represents a circle having its centre at the triple focus, passing through the antifoci of the given axial foci, and having the latter foci for inverse points. This circle is, in fact, one of the circles connected with the two axial foci, the triple focus, and the focus at infinity (which are four foci on a straight line), in the same manner that the confocal circles are connected with the three finite axial foci and the focus at infinity. In fact, all the circles so depending on the axial foci and the triple focus, taken in fours, stand in a more or less special relation to the Cartesian.

From the equation of the system we have, by differentiation,

$$\frac{m\,(xdx - kdx + ydy)}{\rho'} - \frac{xdx + ydy}{\rho} = 0.$$

The points of contact of tangents from the external focus constitute real apices of the curve. There are also imaginary apices relative to the internal axial foci, and real apices relative to the infinitely distant focus.

Making $\frac{dx}{dy} = 0$, we find that the locus of the points of contact of the double tangents of the system is the circle $m\rho - \rho' = 0$. Also, making $\frac{dy}{dx} = 0$, we find that the locus of the apices relative to the infinitely distant focus is $m^2(x-k)\,\rho^2 - x^2\rho'^2 = 0$. This is the locus of the vertex of a triangle, when the base and the ratio of the cosines of the angles at the base are given. The curve is mentioned by Mr. Crofton as the locus of the intersections of pairs of equal tangents to a Cartesian.

This quartic is circular and unicursal, having a double point at infinity. It is, in fact, one of those which I have shown to be traceable by means of a conchoidal motion (Proceedings, Vol. II., p. 125).

If a line RQ moves with its extremity Q on a given line, and is always a tangent to a given circle, the extremity of a line PQ perpendicular to RQ at Q, and equal to half the radius of the circle, will describe the locus in question, P being taken on the same side of RQ as the centre.

The curve consists of two infinite branches which cross at a node, or of two branches which turn back so as not to meet, in which case both the finite double points are conjugate points.

To find the locus of the apices relative to one of the given axial foci, we must eliminate κ between

$$m\rho' - \rho = \kappa, \quad \rho^2 = \frac{m^2k^2 - \kappa^2}{m^2 - 1}.$$

The locus consists of two circles through the focus

$$\rho^2 - kx \pm \frac{k}{m^2-1}\,y = 0.$$

15. The system of Cartesians given by

$$\rho^2 \mp 2A\rho - 2Bx + C^2 = 0,$$

where C^2 is alone variable, may be defined as a system having a common triple focus and a common axial focus, and such that the points of contact of its double tangents lie on a given circle having its centre at the given axial focus.

We get from the general equation, by differentiation,

$$2\left(x\,dx + y\,dy \mp A\,\frac{x\,dx + y\,dy}{\rho} - B\,dx\right) = 0.$$

Hence putting $\frac{dx}{dy} = 0$, we have for the points of contact of double tangents and the axial points $(\rho \mp A)\,y = 0$;
and for the apices relative to the infinitely distant focus, the locus is

$$(x - B)^2\rho^2 = A^2x^2,$$

which represents the conchoid of Nicomedes.

In order to obtain the locus of the apices relative to the common focus, we have to put ρ^2 for C^2 in the equation, and we obtain

$$\rho^2 = Bx \pm A\rho,$$

the well-known equation of a Limaçon of Pascal.

16. The conditions that the two Cartesians

$$\rho^2 \mp 2A\rho - 2Bx + C^2 = 0,$$
$$\rho^2 \mp 2A'\rho - 2B'x + C'^2 = 0,$$

may have the same axial foci are, by (C),

$$C^2 = C'^2, \quad \frac{B^2 + C^2 - A^2}{B} = \frac{B'^2 + C'^2 - A'^2}{B'}.$$

These two conditions express that the apices relative to a focus shall lie on a circle having that focus as centre, and the mean of the two other finite foci shall be the same in all the curves of the system.

On the Normals.

17. The condition that an axial circle may pass through the antifoci of the axial foci not at the origin, is

$$\frac{C^2 + K^2 - R^2}{K} = \frac{C^2 + B^2 - A^2}{B},$$

the equation of the circle being

$$\rho^2 - 2Kx + K^2 - R^2 = 0 \quad \text{(K)}.$$

If (K) intersect (E) in ρ', we have

$$\rho' = \frac{KA \pm BR}{K - B}.$$

For, eliminating x,

$$(K-B)\rho^2 - 2AK\rho + KC^2 - B(K^2 - R^2) = 0;$$

or, by the condition

$$(K-B)\rho^2 - 2AK\rho + \frac{K^2A^2 - B^2R^2}{K-B} = 0.$$

18. Following the analogy of the theory of Conics, we may call the part of the normal line intercepted between the point of the curve through which it is drawn and the axis, the *Normal.**

It is therefore the radius of the axial tangential circle whose centre is its foot. The equation of the circle being

$$\rho^2 - 2Kx + L^2 = 0,$$

we have, as we have seen, eliminating x between this and (E),

$$\rho^2 - \frac{2AK}{K-B}\rho + \frac{C^2K - BL^2}{K-B} = 0.$$

If the circle touches, we have the condition

$$(A^2 - C^2)K^2 + B(C^2 + L^2)K - B^2L^2 = 0.$$

And to determine ρ when K is given, or *vice versâ*, we have

$$\rho K - B\rho - AK = 0.$$

Writing $\rho = K$, we have $K = 0$, which relates to the focus, and $K = A + B$; in like manner, writing $\rho = -K$, we have $K = 0$ and $K = B - A$. Hence, for the vertices relative to the axis of the antifoci of the axial foci not at the origin, we have the vector and the axial coordinate of the foot of the normal equal. Hence if, with the focus pole as centre, we describe circles whose radii are $A + B$, $B - A$, the chords joining the points where the circles meet the corresponding ovals to the corresponding points where they meet the axis, are normals at the extra-axial vertices. It is here supposed that the focus is external, and therefore B is > A. The other cases can be similarly determined.

We have also

$$\text{Subnormal} = K - x = \frac{\rho^3 - 3A\rho^2 + (C^2 + 2A^2 - 2B^2)\rho - AC^2}{2B(\rho - A)}.$$

If N be the length of the normal, we get, writing $L^2 = K^2 - N^2$,

$$K^3 - \frac{C^2 + B^2 - A^2}{B}K^2 + (C^2 - N)K + BN = 0.$$

* It is evident that real normal distances may correspond analytically to imaginary normal lines.

Making $N=0$, we again obtain the equation (C) which determines the axial foci; and, generally, it appears that *there are three normals of given length, and the mean of the positions of their feet is the mean of the positions of the axial foci.*

If K′, K″, K‴ are the coordinates of the feet of such a set of normals, we have

$$N = -\frac{K'K''K'''}{B}.$$

For points whose vectors are ρ, we have

$$N = \frac{\rho^3 - \dfrac{C^2+A^2-B^2}{A}\rho^2 + C^2\rho}{\dfrac{(\rho-A)^2}{A}}.$$

When $N=0$, the equation (D) is satisfied. Hence $-\kappa$ is the value of ρ corresponding to the evanescent normals whose feet are the axial foci not at the origin.

19. The circles which have their centres on the axis are, as is well known, essentially connected in several ways with the Cartesians possessing that axis.

To exemplify this, we may make use of Professor Sylvester's remarkable theorem, that if a Cartesian passes through four concyclic points, these points are concyclic foci of a cubic passing through the foci of the Cartesian. Now since this Cartesian is symmetrical with regard to the axis, it follows that it passes through the four concyclic reflections of the given set of points. Hence the foci are determined as the linear intersections of two circular cubics which are the reflections of each other relative to the axis. These cubics intersect besides in six points on an axial circle. Hence the axis and the system of axial circles constitute the intersections of all such cubics corresponding to Cartesians having that axis.

20. If (E) and (K) intersect in ρ', we have seen (§17) that

$$\rho' = \frac{KA \pm BR}{K-B} \quad \text{..............................(L).}$$

But if K′ be the distance from the origin of the foot of the corresponding normal, we have $K' = \dfrac{B\rho'}{\rho'-A}$, and (L) is the condition that the normal may bisect the angle between the vector ρ' and the radius R, which meet on the curve. A circle passing through the axial foci and through the point (ρ') will meet the circle (K) orthogonally.

Hence Mr. Crofton's theorems,—The arc of the curve is equally inclined to the focal vector and the circle passing through a point on the curve and the other axial foci; and, Tri-confocal Cartesians cut at right angles.

21. If the equation of a Cartesian be written in the form

$$(\rho^2-k^2)^2+A_1x+B_1=0 \text{ ....} (M),$$

the radius of curvature R is given by means of

$$R^2=\frac{[16y^2(\rho^2-k^2)^2+\{A_1+4x(\rho^2-k^2)\}^2]^3}{16\{(\rho^2-k^2)[16y^2(\rho^2-k^2)^2+\{A_1+4x(\rho^2-k^2)\}^2]+2A_1^2y^2\}^2}.$$

This can be further reduced by means of the equation of the curve.

In this way the square root of the denominator becomes

$$4[\{16x^2(A_1x+B_1)-A_1^2\}(\rho^2-k^2)-8(A_1x+B_1)(A_1x+2B_1)-2A_1^2y^2];$$

and, equated to cipher, this function represents a circular cubic. Hence *the eight points of inflection of a Cartesian lie on a circular cubic.*

To find the radius of curvature at the points of contact of the double tangent.

Putting $\rho^2-k^2=0,\quad y^2=k^2-x^2=\dfrac{A_1^2k^2-B_1^2}{A_1^2},$

we have $R^2=\dfrac{A_1^6}{16.4.A_1^4y^4}=\dfrac{A_1^6}{64(A_1^2k^2-B_1^2)^2}.$

Comparing the forms (M) and (A′), we find finally

$$R^2=\frac{16B^6A^4}{\{4A^2B^2-(A^2-C^2)^2\}^2},\quad\text{or}\quad R=\frac{4B^3A^2}{4A^2B^2-(A^2-C^2)^2}.$$

We have $R=\infty$ if $A^2 \backsim C^2=2AB$. In this case the points of contact unite at the vertex of the outer oval.

22. If we are given an axial focus, the triple focus, and a point on the curve, as well as the circle having its centre at the axial focus and passing through the points of contact of the double tangent, we have a very simple construction for the normal at the given point. Let fP (Fig. 5) be the vertex of the given point from the given axial focus. Take $PS=A$ the radius of the given circle; then, joining S to the triple focus T, the normal PK is parallel to ST; for we have $\dfrac{fK}{fP}=\dfrac{fT}{fP-PS}$ (§18).

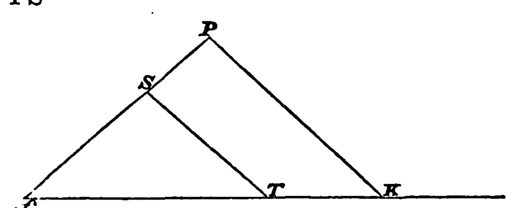


FIG. 5.

23. Denoting by w, w', the angles which the tangents from the external focus make with the axis, we have

$$\cos w = \frac{C-A}{B}, \quad \cos w' = -\frac{C+A}{B}.$$

Hence $$B \sin w \sin w' = \frac{\sqrt{(A^4+B^4+C^4-2A^2B^2-2A^2C^2-2B^2C^2)}}{B}.$$

And we have, by (C),

$$k = B\cos^2\frac{w+w'}{2}, \quad \text{or} \quad B\cos^2\frac{w-w'}{2}.$$

This gives an easy construction for the remaining foci when the external focus and the triple focus are given. We must draw vectors bisecting the sum and the difference of the angles made with the axis by the tangents to the two ovals from the given focus. Then, projecting the triple focus on these vectors and again on the axis, we have the foci.

Properties deducible from the general interpretation of the Polar Equation.

24. I propose to give some examples by way of illustrating the conclusions of §§ 5—8. It is known that if a transversal meet a complete Cartesian, the sum of the vectors to the points of intersection is algebraically constant. This result must be interpreted differently according to the focus which is taken as pole. Let the pole focus be external, and let the transverse meet both ovals.

To eliminate w between

$$\rho^2 - 2A\rho - 2B\cos w\,\rho + C^2 = 0,$$
$$(a\cos w + b\sin w)\,\rho + c = 0,$$

we have to put the last equation in the form

$$(a\cos w\rho + c)^2 - b^2\sin^2 w\,\rho^2 = 0;$$

that is to say, in effect, we take two lines equally inclined to the axis. Since the lines are reflexions of one another relative to the axis, we can refer these intersections to one of them. Remembering this and the nature of corresponding points, we see that, in the case supposed, the sum is $\rho_1+\rho_2-\rho'-\rho''$, the vectors ρ_1, ρ_2 belonging to the intersections on the outer oval, and ρ', ρ'' being the corresponding vectors relative to the inner oval.

But suppose that the transversal meets the same indented oval in four points, then the sum is $\rho_1+\rho_2+\rho_3+\rho_4$. For the equally inclined lines will intersect the oval at points whose vectors are positive.

Again, suppose that the focus pole is the extreme inner one. In this case the vectors are positive; and the sum is $\rho_1+\rho_2+\rho'+\rho''$. And if the pole is the middle focus, we have again the difference

$$\rho_1+\rho_2-\rho'-\rho''.$$

The algebraic sums in question do not therefore comprise all those obtained by alteration of signs.

25. Since the absolute term is the same in (C), (E), and (F), it follows that the product of the vectors of *corresponding points* is equal to the product of the distances of the other two foci from the focus pole.

The interpretation of this for a particular focus as pole, depends immediately upon what has been stated relative to corresponding points.

There are two real circles, and one imaginary circle, having their centres at a focus, and the two other foci for inverse points.

It follows that corresponding points are inverse to each other, with respect to the confocal circle which has its centre at the focus pole.

An oval is therefore inverse to itself relative to the confocal circle whose centre is the external focus. The adjacent parts of conjugate ovals are inverse with respect to the confocal circle whose centre is the extreme inner focus. And the non-adjacent parts of the conjugate ovals are inverse with respect to the imaginary confocal circle.

In the same manner, by reference to corresponding points, we can completely interpret M. Quetelet's theorems, where his statement is adapted to the case of a particular focus as pole.

26. Suppose we have a polar equation of the form

$$\rho^n + (\mathrm{A}\cos w + \mathrm{B}\sin w + \mathrm{C})\,\rho^{n-1}$$
$$+ (a\cos 2w + b\sin 2w + c\cos w + d\sin w + e)\,\rho^{n-2} + \&\mathrm{c}. = 0 \;\ldots\; (\mathrm{N}),$$

then the sum of the squares of the roots relative to ρ

$$\Sigma\rho^2 = (\mathrm{A}\cos w + \mathrm{B}\sin w + \mathrm{C})^2 - 2\,(a\cos 2w + b\sin 2w + \&\mathrm{c.})$$

Consequently $\Sigma\int_{w_1}^{w_2}\frac{\rho^2}{2}\,dw$ is integrable in finite terms.

If, therefore, the vector roots represent real points on the branches of the curve, the algebraic sum of the areas intercepted by the different vectors is capable of quadrature when there is no break of continuity.

The origin is a focus, for if odd powers of ρ are present in (N), the complete curve will be represented by a form $\phi^2 - \rho^2\psi = 0$.

The ovals of Cassini, and bicircular quartics generally, present a case in which only even powers of ρ are present.

In the case of a Cartesian represented by (E), we have

$$\frac{\rho'^2+\rho''^2}{2} = 2\,(\mathrm{A}+\mathrm{B}\cos w)^2 - \mathrm{C}^2$$
$$= (\mathrm{B}^2 + 2\mathrm{A}^2 - \mathrm{C}^2) + 4\mathrm{AB}\cos w + \mathrm{B}^2\cos 2w.$$

Therefore

$$\int\frac{\rho'^2+\rho''^2}{2}\,dw = (\mathrm{B}^2+2\mathrm{A}^2-\mathrm{C}^2)\,w + 4\mathrm{AB}\sin w + \frac{\mathrm{B}^2}{2}\sin 2w.$$

If the limits are 0, π, the origin being the extreme inner focus, we find that *the sum of the areas of the ovals is equal to twice the area of the circle* *whose centre is at the triple focus, and which passes through the points of contact of the double tangent.*

Hence, if we consider the ovals of the locus as forming a ring, a circle, whose radius is determinate, described about the inner oval, will divide the ring into equal portions.

The doctrine of corresponding points on a vector enables us to see that if the origin is external, the areas are intercepted by the same oval; if the origin is the middle focus, the areas are intercepted by non-adjacent parts of the conjugate ovals; if the origin is the innermost focus, they are intercepted by adjacent parts of the ovals.

We may vary the expression of the theorems thence arising in several ways. Thus, let the origin be the external focus, and let two vectors meet the outer oval in (P, Q), (R, S) and the inner oval in (p, q), (r, s), then Area PQpq — Area RSrs can be expressed in finite terms.

27. The equation of a Cartesian being

$$\rho^2-2A\rho-2B\cos w\rho+C^2=0,$$

Mr. William Roberts has inferred that the difference of the arcs of the conjugate ovals corresponding to the same position of the radius vector is expressible by an elliptic arc. (Liouville XI., p. 195, quoted by Salmon "Higher Plane Curves," p. 268.)

In fact, the difference of the elements of the arcs is represented by

$$2\sqrt{A^2+B^2-C^2+2AB\cos w}\,.\,dw.$$

Referring again, to what has been said of corresponding points, we are enabled to state the meaning of the analytical result more completely.

For (1), when the focus is external, the corresponding arcs are on the same oval,—Mr. Crofton has remarked this case from a different point of view; (2) if the interior extreme focus is the origin, the corresponding arcs are adjacent portions of the conjugate ovals; (3) if the middle focus is the origin, the corresponding arcs are non-adjacent portions of the conjugate ovals; (4) if two of the axial foci are imaginary, the corresponding arcs lie on the real oval of course, but the difference in question is expressible by two elliptic integrals of the first and second order. For in this case

$$(A \backsim B)^2<C^2; \quad \text{or,} \quad A^2+B^2-C^2<2AB.$$

Also, with respect to case (3), it will be observed that it is not really the geometrical difference, but the sum of the corresponding arcs, which is expressed by an elliptic arc. For the corresponding elements are on non-adjacent parts of the conjugate ovals, and one becomes negative to the other.

28. Since, when the foci are real, there are two interior foci, and in

order to obtain the whole length of a semi-oval, we must integrate from $w=0$ to $w=\pi$, we are enabled to draw a remarkable conclusion.

For let the length of the outer oval be $2L$, that of the inner $2l$. Then denoting a complete elliptic integral of the second kind by $E_{\frac{1}{2}\pi}$, we have, the extreme inner focus being the origin,

$$L-l = KE_{\frac{1}{2}\pi}.$$

But again, taking the middle forms as origin, we have

$$L+l = K'E'_{\frac{1}{2}\pi}.$$

Hence $\quad 2L = KE_{\frac{1}{2}\pi}+K'E'_{\frac{1}{2}\pi}, \quad 2l = K'E'_{\frac{1}{2}\pi}-KE_{\frac{1}{2}\pi}.$

That is to say, the lengths of the ovals of a Cartesian are expressed by syzygetic relations between two elliptic quadrants. I confess, however, that a verification of this is desirable.

29. In the case of a nodal Limaçon, the external and middle focus coincide in the node, and we have to modify our integrations. The arc of this curve can be expressed by an elliptic arc in virtue of its character as an Epitrochoid.

Referring the curve to its single focus as origin, we get the equation

$$\rho^2-2A\rho-2B\cos w\rho+(A-B)^2 = 0.$$

Hence the difference between the outer semi-loop and the inner semi-loop is given by

$$2\int_0^{\pi}\sqrt{2AB+2AB\cos w}\,.\,dw = 8\sqrt{AB}\int_0^{\frac{1}{2}\pi}\cos\frac{w}{2}\,.\,d\,.\,\frac{w}{2} = 8\sqrt{AB}.$$

Now, if the equation above given be compared with the form derived from the common equation $\rho = \pm a+b\cos w$, which is transformed into

$$\left(\rho^2-\frac{a^2}{b}x-\frac{b^4-a^4}{4b^2}\right)^2-a^2\left\{\rho^2+\frac{b^2-a^2}{b}x+\frac{(b^2-a^2)^2}{4b^2}\right\} = 0,$$

we see that $B = \dfrac{a^2}{2b}$, $A = \dfrac{b}{2}$, so that we have the theorem,

The difference of the lengths of the loops of a nodal Limaçon is four times the distance between the vertices.

It is remarkable that the difference is independent of the parameter b; if we take the node as pole, the difference in question comes out by means of a formula in the theory of elliptic functions. In consequence of the node, the integrations are not from 0 to π, but from 0 to w_1, and from 0 to w_2, when $\cos w_1 = -\dfrac{a}{b}$, and $w_2+w_1 = \pi$.

We then get a formula

$$L-l = 2(a+b)(2E_{\frac{1}{2}w_1}-E_{\frac{1}{2}\pi}),$$

and the equation of amplitudes for the comparison of elliptic arcs, sometimes written

$$\cos\sigma = \cos\phi\cos\psi-\sin\phi\sin\psi\sqrt{1-e^2\sin^2\sigma},$$

is satisfied. It follows that

$$2E_{\frac{1}{2}\omega_1} - E_{\frac{1}{2}\pi} = 1 - \sqrt{1-e^2} = \frac{2a}{a+b},$$

and thus the previous result is verified.

30. I may remark that the transformation given in § 9 is a particular case of a more general one. For it is evident that the equation of a Cartesian in the form $\rho^2 - 2A\rho - 2Bx + C^2 = 0$ is transformed into another of similar form by the substitutions $\rho - A'$, $x - a$, for ρ and x respectively. The circle employed in § 9 is itself a Cartesian, and any point may be considered as an axial focus of it, relative to the diameter through the point as axis. In the general case, the pole of the auxiliary equation will be one of the three axial foci. By means of these substitutions, we may derive a Cartesian of the second kind from one of the first kind, and *vice versâ*. It will be observed, however, that a real point on the derived Cartesian may correspond to an imaginary point on the auxiliary Cartesian, as in the particular case when this curve is a circle.

If we are given a parabola, we can also derive Cartesians of both kinds by an obvious and simple transformation.

The equation of the parabola being written in the form

$$y^2 - 2Ay - 2Bx + C^2 = 0,$$

we only have to substitute ρ for y to obtain the equation of a Cartesian. Hence, if the feet of the y ordinates be gathered at the origin, while the extremities slide on perpendiculars to the diameter which is the axis of x, the locus of the extremities will be a Cartesian. The limiting axial points correspond to $y = \pm x$ in the parabola. On account of the identity of the coefficients in the two forms of equation, the theory of Cartesians might be discussed with some advantage by means of the auxiliary parabola.

A slight discussion followed upon the reading of the paper. Dr. Henrici exhibited a plaster cast of the surface

$$xyz - \left(\frac{3}{7}\right)^3 (x+y+z-1)^3 = 0.$$

A cardboard model of this surface was shown to the Society at its meeting in the previous November.

The following presents had been received:—

"Sulle ventisette rette di una Superficie del terzo ordine, nota del Prof. Luigi Cremona;" from the Author.

"Monatsbericht;" März, 1870.