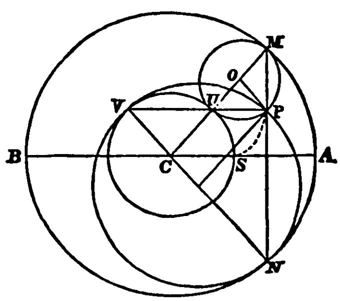
On Epicycloids and Hypocycloids. *By* Prof. WOLSTENHOLME.

[*Read April* 10*th*, 1873.]

The following method of considering epicycloids (and of course hypocycloids) seems to show more directly the relation between their twin modes of generation than the ordinary one of considering them primarily as loci. It also directly determines the class number of any such curve; and, from the examples given, it would seem also to lend itself to the discussion of the more important properties at least as easily.

Let the two points M, N describe the same circle, whose centre is O, with velocities in the ratio $m:n$, m and n being whole numbers prime to each other; AB a diameter of the circle such that when M is at A, N is at B; $\angle\,\text{ACM} = m\theta$, $\angle\,\text{BCN} = n\theta$. Then, if M′N′ be the consecutive position of MN, and P the point of ultimate intersection of MN, M′N′,



$$\text{MM}' : \text{NN}' = \text{MP} : \text{PN};$$

but $\text{MM}' : \text{NN}' = m : n,$

therefore $\dfrac{\text{MP}}{\text{PN}} = \dfrac{m}{n}.$

Describe a circle touching the prime circle at M and passing through P, take O its centre, and let it meet CM in U; then

$$\frac{\text{MO}}{\text{MC}} = \frac{\text{MP}}{\text{MN}} = \frac{m}{m+n}, \quad \text{or} \quad \frac{\text{MU}}{\text{MC}} = \frac{2m}{m+n};$$

therefore $\dfrac{\text{CU}}{\text{CM}} = \dfrac{n-m}{,+m}.$

(The figure has assumed n to be $>m$.) Describe another circle with centre C and radius CU, which is a fixed quantity, and let this circle meet CA in S; then

$$\angle\,\text{MCN} = \pi - n\theta + m\theta,$$

therefore $\angle\,\text{CMN} = \dfrac{n-m}{2}\,\theta,$

therefore $\angle\,\text{UOP} = (n-m)\,\theta$, and $\angle\,\text{UCS} = m\theta$,

therefore $\angle\,\text{UOP} : \angle\,\text{UCS} = n-m : m = \text{CU} : \text{UO};$

whence the arc PU is equal to the arc SU, and the locus of P is an epicycloid to which MN is a tangent and PU a normal at P. In exactly the same way, if we describe a circle through P touching the prime circle at N, this circle will be of fixed magnitude, its radius being to CA as $n : m+n$, and it will touch the same fixed circle in V, the point of concourse of PU, NC; the arc SV will be equal to the arc PV, and we get the second mode of generation of the epicycloid.

Moreover, since the angular velocity of U is that of M, and the angular velocity of V is that of N, we see that UV will touch a similar epicycloid, whose dimensions are to that of the original as $n-m : n+m$; or since UV is the normal at P, the evolute of any epicycloid is a similar epicycloid, whose linear dimensions are to that of the original as $n-m : n+m$.

If M and N move in opposite senses, we must put $-m$ for m, P will lie in NM produced, and we shall have the case of the hypocycloid, which does not need any separate discussion.

To find the *class* of the curve, consider any point L of the prime circle, and when M first arrives at L, let N_1, P_1 be the positions of N and P; then when M next arrives at L, N will have described an arc measured by $\frac{n}{m} \cdot 2\pi$, and is at N_2 suppose, P being at P_2, and so on, until M is at L for the mth time, when N will have described an arc $\frac{m-1}{m} \cdot 2n\pi$, and N_m, P_m are the positions of N and P; hence there will be m tangents drawn from L as a position of M, the m points $N_1, N_2, \ldots N_m$ will be the corners of a regular m-gon inscribed in the prime circle, and therefore $P_1, P_2, \ldots P_m$ will be the corners of a regular m-gon inscribed in the first moving circle when L is its point of contact.

In exactly the same way, we shall have n tangents drawn from L as an N point, and their points of contact will be the corners of a regular n-gon inscribed in the second moving circle when its point of contact with the prime circle is L.

There can be no other tangents from L, and the class number of the curve is thus $m+n$ both for the epicycloids and hypocycloids.

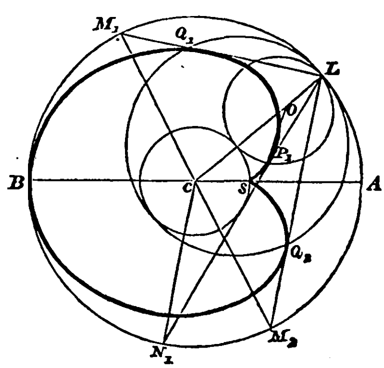
The prime circle is the circle through the vertices of the epicycloid; and we see that, if at any point of this circle we draw the two circles touching this circle and the concentric circle through the cusps of the epicycloid, these two circles will pass through the points of contact of all the tangents which can be drawn from that point, and these points of contact will be the corners of two regular polygons.

For a cardioid $n : m = 2 : 1$, and for a three-cusped hypocycloid $n : m = 2 : -1$, explaining somewhat the close relationship between these curves, each of which is of the fourth degree and third class, and each a projection of the other.

It follows also from the properties already proved, that if an epicycloid be generated by a circle of radius ma rolling without one of radius $(n-m)a$, m, n being whole numbers prime to each other, and if in the moving circle we inscribe a regular m-gon one of whose corners is the describing point, all the other corners will move in the same epicycloid, and the whole epicycloid will be completely generated by these different points in one complete revolution about the fixed circle. The same epicycloid may also be generated by the n corners of a regular n-gon inscribed in a circle of radius na rolling on the same fixed circle of radius $(n-m)a$, the contact being in this case internal.

So for hypocycloids, if a circle of radius ma, or of radius na, roll within a fixed circle of radius $(m+n)a$, the whole hypocycloid will be traced out in going once round the fixed circle by the corners of a regular m-gon, or a regular n-gon, inscribed in the moving circle. A singular case is $m=1$, $n=2$, where a circle of radius 2 rolls within one of radius 3; for if AB be a diameter of the moving circle, A, B trace out the same three-cusped hypocycloid to which AB is also a tangent throughout the motion.

It will, I think, be found in all cases quite as easy to prove any properties of special curves from this mode of generation as from the usual one. I append two or three examples.



For a cardioid, $n=2m$; and when M is at L, N is at N_1, such that $\angle BCN_1 = 2\angle ACL$, therefore

$$\angle ACL = \angle BLN_1;$$

and when N is at L, we shall have M at either M_1 or M_2, where

$$\angle ACM_1 = \tfrac{1}{2}(\pi + ACL),$$

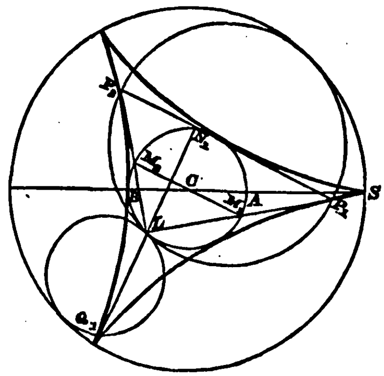
and M_2 is at the other end of the diameter through M_1, since M describes a half circumference while N describes the whole. Hence, if

$$LP_1 = \tfrac{1}{3}LN_1, \text{ and } \frac{LQ_1}{LM_1} = \frac{LQ_2}{LM_2} = \frac{2}{3},$$

P_1, Q_1, Q_2 will be the points of contact of the tangents from L. Since $\angle ACM_1 = \frac{1}{2}(\pi + ACL)$, CM_1 is at right angles to BL, and therefore parallel to AL; and since $\dfrac{AS}{AC} = \dfrac{LQ_1}{LM_1}$, SQ_1 is parallel to AL or CM_1, so also SQ_2 is parallel to CM_2; whence Q_1SQ_2 is one straight line, and the tangents drawn at the ends of a chord through the cusp are at right angles to each other, and intersect on a fixed circle.

Since $\angle BCN_1 = 2\angle ACL = 2\angle BAN_1$, $\angle ACL = \angle CAN_1$, or AN_1 is parallel to CL; whence SP_1 is parallel to either, since $\dfrac{CS}{CA} = \dfrac{LP_1}{LN_1}$. (This is really only the same theorem as proved before, viz., SQ_1 parallel to CM_1.) Hence, if the tangent at P_1 meet the axis in T, we have

$$\angle P_1TA = 3\angle ABL = \tfrac{3}{2}\angle TSP_1, \quad \text{or} \quad \angle SP_1T = \tfrac{1}{2}\angle P_1ST.$$



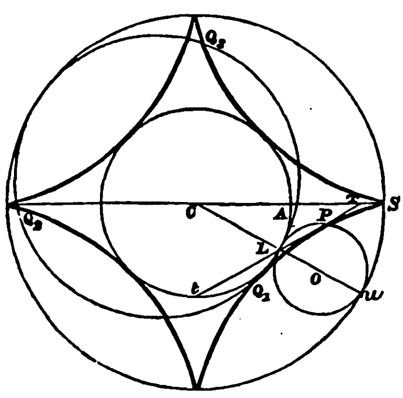
For the three-cusped hypocycloid, $n=-2m$; and M starting from A, and N starting from B, as before, S will be the cusp, where $BS=2AS$; and if $\angle BCL = 2\angle ACM_1$, then when N first gets to L, M will be at M_1; and when N next gets to L, M will be at M_2, the other end of the diameter M_1CM_2; and LM_1, LM_2 will be two tangents at right angles to each other, the points of contact being P_1, P_2, where $LP_1=2LM_1$, $LP_2=2LM_2$, therefore $P_1P_2=2M_1M_2$, or is a constant quantity. When M arrives at L, N will be at N_1, such that

$$\angle BCN_1 = 2\angle BCL = 4\angle ACM_1,$$

therefore $$\angle M_2CN_1 = 3\angle ACM_1 = \angle M_2CL,$$

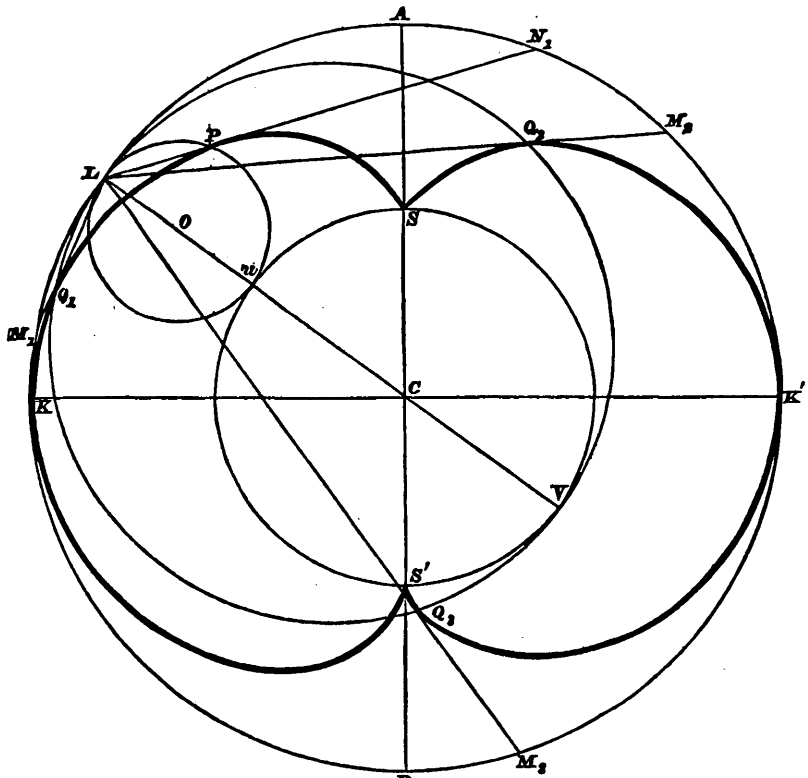
or LN_1 is perpendicular to M_1M_2; hence P_2, N_1, P_1 lie in one straight line at right angles to LN_1, which is therefore itself a tangent to the hypocycloid. Hence, if two tangents to this hypocycloid be at right angles, the straight line joining their points of contact is also a tangent to the hypocycloid. Also, of the three tangents drawn to the hypocycloid from any point on the inscribed circle, two are at right angles and the third is perpendicular to the chord of contact of the other two. These are the best known properties peculiar to this hypocycloid.

Similarly, the properties of the four-cusped hypocycloid, for which $n=3$, $m=-1$, may be discussed.



The radius of the prime circle being 2, that of the fixed circle (through the cusps) will be 4, and of the two moving circles 1 and 3. If L be a point on the prime circle, LP, LQ_1, LQ_2, LQ_3 the four tangents; Q_1, Q_2, Q_3 are corners of an equilateral triangle. In this case $\angle UOP = 4\angle UCS = 2\angle OLP$, therefore $\angle OLP = 2\angle LCS$; hence CL, LP are equally inclined to CS; and if LP meet CS and the diameter at right angles to it in

T, t, Tt = 2CL = CS, a well known property. This would follow at once from our fundamental idea; for if AB be a diameter, and M, N be originally at A, B, $\angle$BCN = 3 $\angle$ACM, therefore each of the angles CMN, CNM = 2MCA, the same result as above.



In a two-cusped hypocycloid, $m=1$, $n=3$; S, S' the two cusps; K, K' the two vertices; LN_1, LM_1, LM_2, LM_3 the four tangents from a point on the prime circle; P, Q_1, Q_2, Q_3 the points of contact; then Q_1, Q_2, Q_3 are the corners of an equilateral triangle.

If KCL$=\theta$,

$$KCM_1 = \frac{\theta}{3},\quad KCM_2 = \frac{2\pi+\theta}{3},\quad KCM_3 = \frac{4\pi+\theta}{3},\quad KCN_1 = 3\theta;$$

and if ρ, ρ_1, ρ_2, ρ_3 be the radii of curvature at P, Q_1, Q_2, Q_3,

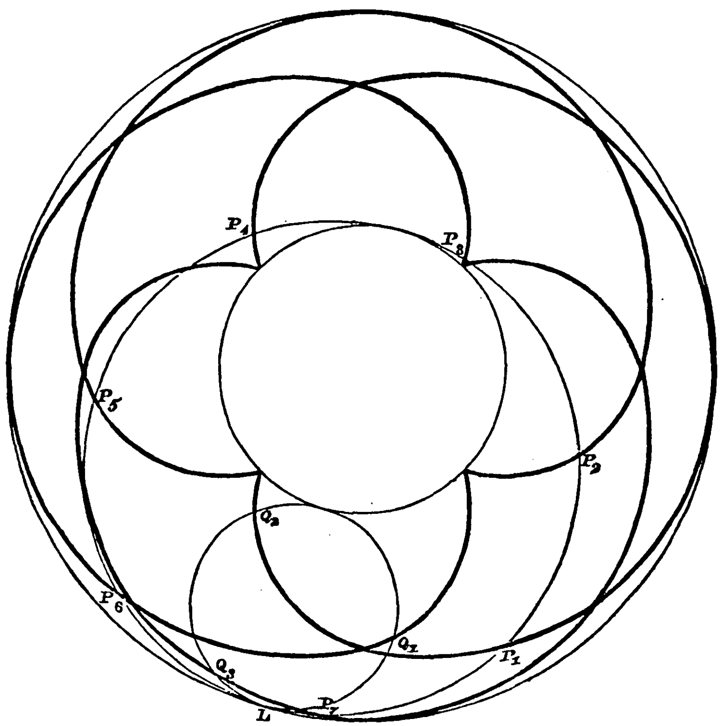
$$\rho = \tfrac{3}{2}PU,\quad \rho_1 = \tfrac{1}{2}VQ_1,\quad \&c.;$$

and $\angle CLP = \frac{\pi}{2}-\theta$, therefore $PU = UL\cos\theta = CU\cos\theta$;

$$\angle \mathrm{CLQ}_1 = \frac{\pi}{2} - \frac{\theta}{3}, \quad \text{therefore} \quad \mathrm{VQ}_1 = \mathrm{VL}\cos\frac{\theta}{3} = 3\,\mathrm{CU}\cos\frac{\theta}{3}$$

therefore $\rho = \frac{3}{2}\,\mathrm{CU}\cos\theta, \quad \rho_1 = \frac{3}{2}\,\mathrm{CU}\cos\frac{\theta}{3},$ &c.,

the four radii of curvature being as $\cos\theta : \cos\frac{\theta}{3} : \cos\frac{2\pi+\theta}{3} : \cos\frac{4\pi+\theta}{3}$.
Hence, of the radii of curvature at Q_1, Q_2, Q_3, the greatest is equal to the sum of the other two, and their product varies as the radius of curvature at P.



If $n=7$, $m=3$, the radii of the two circles by which the epicycloid is described will be 4, 3, or 4, 7; and if L be any point on the prime circle (radius 10), the tangents will be LP_1, LP_2, LP_3 ... LP_7, where P_1, P_2 ... P_7 are corners of a regular heptagon in a circle of radius 7; and LQ_1, LQ_2, LQ_3 where Q_1, Q_2, Q_3 are corners of an equilateral triangle inscribed in a circle of radius 3.

Hence we see that, if a circle of radius 3 roll on a circle of radius 4, the corners of an equilateral triangle inscribed in the moving circle will trace out the same epicycloid, and in one complete revolution round the fixed curve, the three points will between them have traced out the complete epicycloid.

Since any point on the general epicycloid is determined by equations of the form $x = m \cos n\theta + n \cos m\theta$, $y = m \sin n\theta + n \sin m\theta$; we get the order of the curve at once by considering the number of points lying on any line $px + qy = 1$; and if we put $\cos\theta = \frac{1-z^2}{1+z^2}$, $\sin\theta = \frac{2z}{1+z^2}$, the resulting equation in z is of the degree $2n$, $(n > m)$. The curves being obviously unicursal, the deficiency is 0; and to account for this it becomes necessary to ascertain the nature of the circular points.

By taking $x + y\sqrt{-1} \equiv X$, $x - y\sqrt{-1} = Y$, $1 = Z$, I find that at the point (0, 1, 0) the form of the curve is $Z^{m+n} = Y^m X^n$, $(n > m)$; and that for the hypocycloids we get the correct result by putting $-m$ for m, so that the line at ∞ is a double tangent, n-fold at each point. The circular points are therefore multiple points of the nth degree, but their effect in reducing the class is still greater on account of the tangents all coinciding. There are, in general, also impossible ordinary double points, and I have not yet completed the theory of these. The class and order of the hypocycloids were determined by Mr. Cotterill in the "Educational Times," for Nov. 1865.

On the Locus of the Point of Concourse of Perpendicular Tangents to a Cardioid. By Prof. Wolstenholme, M.A.

[*Read April 10th*, 1873.]

If P be a point generating a cardioid by the rolling of a circle on an equal circle, O the centre of the fixed circle, U the point of contact, S the cusp, then PU is the normal at P; and if $\angle \text{SOU} = \theta$ and a be the radius of either circle, the equation of the tangent at P is

$$x \sin\frac{3\theta}{2} - y \cos\frac{3\theta}{2} = 3a \sin\frac{\theta}{2},$$

or, writing $\pi - 2\phi$ for θ, and b for $3a$,

$$-x \cos 3\phi + y \sin 3\phi = b \cos\phi.$$

In order to get a tangent at right angles to this, we must put $\frac{\pi}{6} + \phi$, $\frac{\pi}{2} + \phi$, or $\frac{5\pi}{6} + \phi$ for ϕ, the subsequent values giving no new tangent, the cardioid being of the third class.

The second of these gives the corresponding tangent

$$x \sin 3\phi + y \cos 3\phi = b \sin\phi,$$