ON THE OUTER ISOMORPHISMS OF A GROUP

By W. BURNSIDE.

[Received December 11th, 1911.—Read December 14th, 1911.]

WHILE preparing the second edition of my *Theory of Groups* for the press I made many ineffectual attempts to determine whether an outer isomorphism of a group necessarily permutes some of its conjugate sets, or, in the alternative, if groups exist some outer isomorphisms of which change every operation into a conjugate operation.

I have since succeeded in constructing comparatively simple examples showing that of the two suppositions the latter is the correct one. One of the simplest of these is given below.

Taking p^3 symbols $x_{\alpha,\beta,\gamma}$, where the suffixes are reduced (mod p), the four substitutions

$$A_1 \quad \text{or} \quad x'_{\alpha,\beta,\gamma} = x_{\alpha+1,\beta,\gamma},$$

$$A_2 \quad \text{or} \quad x'_{\alpha,\beta,\gamma} = x_{\alpha,\beta+1,\gamma},$$

$$A_3 \quad \text{or} \quad x'_{\alpha,\beta,\gamma} = \omega^{\alpha+\beta} x_{\alpha,\beta,\gamma-\beta},$$

$$A_4 \quad \text{or} \quad x'_{\alpha,\beta,\gamma} = \omega^{\beta} x_{\alpha,\beta,\gamma+\beta-\alpha},$$

where ω is an assigned primitive p-th root of unity, generate a group of monomial substitutions of finite order.

The substitution that corresponds to $A_3^{-1}A_1^{-1}A_3A_1$ is determined by eliminating the intermediate symbols between

$$x'_{\alpha,\beta,\gamma} = \omega^{-\alpha-\beta} x_{\alpha,\beta,\gamma+\beta},$$

$$x''_{\alpha,\beta,\gamma} = x'_{\alpha-1,\beta,\gamma},$$

$$x'''_{\alpha,\beta,\gamma} = \omega^{\alpha+\beta} x''_{\alpha,\beta,\gamma-\beta},$$

$$x''''_{\alpha,\beta,\gamma} = x'''_{\alpha+1,\beta,\gamma}.$$

Hence $A_3^{-1}A_1^{-1}A_3A_1$ is $x'_{\alpha,\beta,\gamma} = \omega x_{\alpha,\beta,\gamma}$.

In the same way it is shewn that

$$A_4^{-1}A_1^{-1}A_4A_1 \quad \text{is} \quad x'_{a,\beta,\gamma} = x_{a,\beta,\gamma-1},$$
$$A_3^{-1}A_2^{-1}A_3A_2 \quad \text{is} \quad x'_{a,\beta,\gamma} = \omega x_{a,\beta,\gamma-1},$$
$$A_4^{-1}A_2^{-1}A_4A_2 \quad \text{is} \quad x'_{a,\beta,\gamma} = \omega x_{a,\beta,\gamma+1},$$

while both $A_1^{-1}A_2^{-1}A_1A_2$ and $A_3^{-1}A_4^{-1}A_3A_4$ are E, the identical substitution.

Denoting the two substitutions

$$x'_{a,\beta,\gamma} = \omega x_{a,\beta,\gamma},$$

and

$$x'_{a,\beta,\gamma} = x_{a,\beta,\gamma-1},$$

by P and Q, which are obviously independent substitutions of order p, it may be at once verified that both P and Q are permutable with A_1, A_2, A_3, A_4; while, on repeating them, it is found that A_1, A_2, A_3, A_4 are all of order p. It follows that every substitution of the group is expressible in the form

$$A_1^{x_1}A_2^{x_2}A_3^{x_3}A_4^{x_4}P^yQ^z,$$

where the indices take all values from 0 to $p-1$, and that the p^6 substitutions contained in this form are all distinct, so that the group is one of order p^6.

The existence of this group of linear substitutions proves that the relations

$$A_1^p = A_2^p = A_3^p = A_4^p = P^p = Q^p = E,$$
$$A_1^{-1}A_2^{-1}A_1A_2 = A_3^{-1}A_4^{-1}A_3A_4 = E,$$
$$A_3^{-1}A_1^{-1}A_3A_1 = P, \qquad A_4^{-1}A_1^{-1}A_4A_1 = Q,$$
$$A_3^{-1}A_2^{-1}A_3A_2 = PQ, \qquad A_4^{-1}A_2^{-1}A_4A_2 = PQ^{-1},$$

and relations expressing that both P and Q are permutable with A_1, A_2, A_3, A_4 are the complete defining relations of an abstract group G of order p^6.

Denoting $A_1^{x_1}A_2^{x_2}A_3^{x_3}A_4^{x_4}$ by R, the general inner isomorphism of G is given by

$$R^{-1}A_1R = A_1P^{-x_3}Q^{-x_4}, \qquad R^{-1}A_2R = A_2P^{-x_3-x_4}Q^{-x_3+x_4},$$
$$R^{-1}A_3R = A_3P^{x_1+x_2}Q^{x_2}, \qquad R^{-1}A_4R = A_4P^{x_2}Q^{x_1-x_2}.$$

This changes $A_1^{u_1}A_2^{u_2}A_3^{u_3}A_4^{u_4}$ into $A_1^{u_1}A_2^{u_2}A_3^{u_3}A_4^{u_4}P^yQ^z$, where

$$\left.\begin{aligned} y &\equiv -u_1x_3-u_2(x_3+x_4)+u_3(x_1+x_2)+u_4x_2 \\ z &\equiv -u_1x_4-u_2(x_3-x_4)+u_3x_2+u_4(x_1-x_2) \end{aligned}\right\} \pmod{p}.$$

Now $A_1^{u_1}A_2^{u_2}A_3^{u_3}A_4^{u_4}$ is obviously not a self-conjugate operation of G. If it is one of p conjugate operations, then y/z must be independent of x_1, x_2, x_3, x_4. Now, if y/z is λ, then

$$x_4[\lambda(u_1-u_2)-u_2]+x_3(\lambda u_2-u_1-u_2)+x_2[\lambda(u_4-u_3)+u_3+u_4] \\ +x_1(-\lambda u_4+u_3) \equiv 0 \pmod{p}.$$

Hence, if λ is independent of x_1, x_2, x_3, x_4, then

$$\begin{vmatrix} \lambda & -\lambda-1 & . & . \\ -1 & \lambda-1 & . & . \\ . & . & -\lambda+1 & \lambda+1 \\ . & . & 1 & -\lambda \end{vmatrix} \equiv 0 \pmod{p},$$

or

$$[(\lambda-1)^2-2]^2 \equiv 0 \pmod{p}.$$

It follows that if 2 is a quadratic non-residue (mod p), then $A_1^{u_1}A_2^{u_2}A_3^{u_3}A_4^{u_4}$ is one of a set of p^2 conjugate operations of G; or, in other words, that

$$A_1^{u_1}A_2^{u_2}A_3^{u_3}A_4^{u_4}P^iQ^j \quad (i, j = 0, 1, \dots, p-1)$$

constitute a conjugate set for G for all values of u_1, u_2, u_3, u_4, except

$$u_1 = u_2 = u_3 = u_4 = 0.$$

Now the defining relations of G are unaltered if $A_1P^{i_1}Q^{j_1}$, $A_2P^{i_2}Q^{j_2}$, $A_3P^{i_3}Q^{j_3}$, $A_4P^{i_4}Q^{j_4}$ are written for A_1, A_2, A_3, A_4, while P and Q are unchanged. Hence

$$\begin{pmatrix} A_1, & A_2, & A_3, & A_4 \\ A_1P^{i_1}Q^{j_1}, & A_2P^{i_2}Q^{j_2}, & A_3P^{i_3}Q^{j_3}, & A_4P^{i_4}Q^{j_4} \end{pmatrix}$$

gives an isomorphism of G, whatever the i's and j's may be. Moreover, any two of these isomorphisms carried out successively give a third isomorphism of the same form. There arises thus an Abelian group of isomorphisms of G of order p^8 and type $(1, 1, \dots, 1)$; and every isomorphism contained in this group changes any operation of G into a conjugate operation. The order of the group of inner isomorphisms of G is p^4. Hence G admits outer isomorphisms which change every operation into a conjugate operation.

The above group of isomorphisms of order p^8 is the greatest which changes every operation into a conjugate operation, since it changes the four generating operations into any four which are conjugate with each of them respectively. Every isomorphism of G not contained in this group must permute some of the conjugate sets.