

# DIOPTRIC FORMULÆ FOR CYLINDRICAL LENSES COMBINED AT OBLIQUE AXES.

BY CHARLES SHEARD.

THE solution of the problem of transposing double cylinders at oblique angles into their equivalents with axes at right angles (the so-called cross-cylinder combination) was first worked out, as far as the writer is aware, by C. F. Prentice and is given in his book on Ophthalmic Lenses, pages 53-83. His method is somewhat involved and lengthy and the final equations given for obtaining the values of the dioptric powers of the cylinders and their axial angles involve several terms and algebraic processes. The writer of this paper will present in outline a much simpler and clearer solution of the problem. The method of curvatures has doubtless been previously applied to cylinders, but no such solutions as are given in the succeeding paragraphs are known to me. This paper also calls attention to the superiority of the method of curvatures in the solution of a large number of problems in geometrical optics.

In Fig. 1, let  $abmcd$  represent a cylinder with  $op$ , its axis, making angles  $\alpha$  and  $\beta$  respectively with the  $Y$  and  $X$  ( $90^\circ$  and  $0^\circ$ ) axes. The dioptric power in a direction at right angles to the axis  $op$  and the  $X, Y$  plane is determined by the curvature of the cylindrical surface; *i. e.*, by  $amb$ . Since the dioptric power is the reciprocal of the focal length, and

$$d = \text{dioptric power} = 1/f = (n - 1)1/r,$$

then  $\overline{d} \propto 1/r$ . By geometry,  $\overline{ap^2} = 2r \cdot mp - \overline{mp^2}$  and neglecting the term  $\overline{mp^2}$ ,

$$r = \frac{\overline{ap^2}}{2mp},$$

or

$$(1) \quad d \propto \frac{1}{r} \propto \frac{1}{\overline{ap^2}},$$

since  $mp$ , the thickness of the cylinder, is a constant for any particular case.<sup>1</sup> Fig. 1 represents a cylinder of definite thickness,  $mp$ , and definite

<sup>1</sup> The foregoing solution assumes a cylindrical element in which the thickness is very small in comparison to the radius of curvature of the curved surface, which is the case in actual practice. Therefore the arcs obtained by the intersections of the cylinder by planes while actually portions of ellipses may be taken as portions of circles.

semi-chord,  $ap$ . If the thickness of the cylinder is kept constant, it will be apparent to the reader that the value of the line  $ap$  will be altered when the curvature of the arc  $amb$  is changed. Since the width,  $ab$ , and the thickness,  $mp$ , are constants of a specified cylinder, the chords  $ab$ ,  $sb$  and  $ak$  may be used in the measurement of the curvatures of the arcs  $amb$ ,  $sfb$  and  $afk$  respectively in the manner indicated in equation (1).

The dioptric power of the cylinder in directions parallel to the  $X$  and  $Y$  axes may be obtained by passing planes through the cylinder intersecting its cylindrical surface in arcs  $afk$  and  $sfb$ .

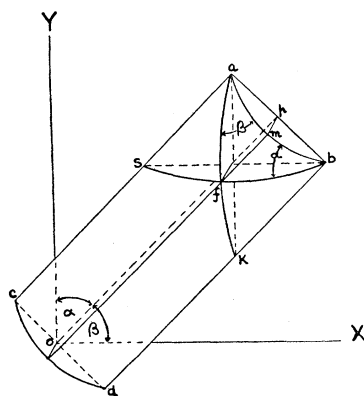


Fig. 1.

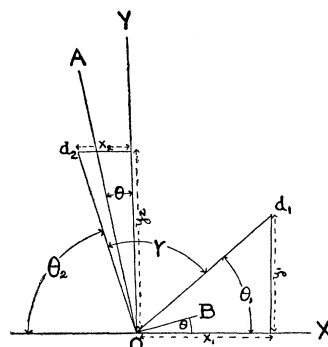


Fig. 2.

Since  $ak = ab/\cos \beta$  and  $sb = ab/\cos \alpha = ab/\sin \beta$ , by substitution in (1) we obtain

$$(2) \quad \begin{aligned} d_y &= \text{vertical dioptric power} = d \cos^2 \alpha = d \sin^2 \beta, \\ d_x &= \text{horizontal dioptric power} = d \cos^2 \beta. \end{aligned}$$

Hence the dioptric powers of any cylinder taken in right-angled directions are proportional respectively to the square of the cosine and sine of the angle which the axis of the cylinder makes with one of the coördinate axes.

In Fig. 2,  $Od_1$  and  $Od_2$  represent graphically the powers,  $d_1$  and  $d_2$ , of two cylinders with axes at the oblique angle  $\gamma$  and making angles  $\theta_1$  and  $\theta_2$  with the  $X$  axis. Let  $OA$  and  $OB$  represent the right-angled equivalents of  $Od_1$  and  $Od_2$ . The angle  $BOA$  is  $90^\circ$ ;  $\theta$  represents the angle which the axes of the equivalent cylinders make with the  $X$  and  $Y$  axes. Let the dioptric values of  $OA$  and  $OB$  be indicated by the symbols  $A$  and  $B$ .

The total dioptric power of the two cylinders  $Od_1$  and  $Od_2$  is (see Fig. 2)

$$(a) \quad \text{along } X \text{ axis, } x_1 + x_2 = d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2,$$

(b) along  $Y$  axis,  $y_1 + y_2 = d_1 \sin^2 \theta_1 + d_2 \sin^2 \theta_2$ ,

or

$$(3) \quad \Sigma x + \Sigma y = d_1 + d_2 = A + B.$$

$d_1 + d_2$  must be equal to  $A + B$  since the sum of the dioptric powers before and after transposition must remain the same.

Treating the cross cylinders  $A$  and  $B$ , which are to be equivalent to the original oblique-angled combination, in a similar manner it will be seen that

$$(4) \quad A \sin^2 \theta + B \cos^2 \theta = \Sigma x_{A, B} = X,$$

$$(5) \quad A \cos^2 \theta + B \sin^2 \theta = \Sigma y_{A, B} = Y.$$

Combining by subtraction (4) and (5) one obtains

$$(A - B) \sin^2 \theta - (B - A) \cos^2 \theta = \Sigma x_{A, B} - \Sigma y_{A, B}$$

or

$$(6) \quad (B - A) \cos 2\theta = \Sigma y_{A, B} - \Sigma x_{A, B} = Y - X$$

and by addition of equations (4) and (5) one obtains

$$(7) \quad A + B = Y + X = d_1 + d_2.$$

Therefore  $\Sigma x + \Sigma y$  for cylinders of powers  $d_1$  and  $d_2$  is equal to a similar quantity for cylinders of powers  $A$  and  $B$ .

If the  $X$ ,  $Y$  axes be rotated until the  $Y$  axis coincides in turn with  $Od_2$  and  $OA$  it will be seen that

$$(8) \quad \begin{aligned} A^2 + B^2 &= d_1^2 + d_2^2 + 2d_1d_2 \cos^2 \gamma \\ &= (d_1 + d_2)^2 - 2d_1d_2 \sin^2 \gamma, \end{aligned}$$

from which it follows that

$$(9) \quad A \cdot B = d_1d_2 \sin^2 \gamma.$$

Equations (6), (7) and (9) give the complete solution of the problem.

*Example.*—To transpose 1.50 cyl. ax.  $120^\circ + 1.00$  cyl. ax.  $80^\circ$  into the equivalent cross cylinder. We have

$$(7) \quad A + B = 2.50,$$

$$(9) \quad A \cdot B = 0.6195,$$

$$(6) \quad (B - A) \cos 2\theta = \Sigma y - \Sigma x = 1.6896.$$

Solving equations (7) and (9),  $A = 2.21$  diopters and  $B = 0.29$  diopter. Equation (6) gives  $\theta = 14^\circ 12'$ . Hence the solution,

$$2.21 \text{ cyl. ax. } 104^\circ 12' + 0.29 \text{ cyl. ax. } 14^\circ 12'.$$

By neutralization there was obtained

$$2.25 \text{ cyl. ax. } 104^\circ + 0.25 \text{ cyl. ax. } 14^\circ$$

and by Prentice's formulæ

$$2.22 \text{ cyl. ax. } 104^{\circ} 47' + 0.28 \text{ cyl. ax. } 14^{\circ} 47'.$$

Equations (6), (7) and (9) are applicable to the solution of the cross cylinder equivalents of any combination of converging or diverging cylinders if due regard is had for the signs intrinsically associated with the dioptric powers of the cylinders involved. Furthermore, the algebraic sign of the quantity  $(Y - X)$  determines the quadrant in which the axis of  $A$  or  $B$  lies; that is to say, whether  $A$  has its axis at an angle  $\theta$  or  $\theta + 90^{\circ}$ .

Tables have been prepared by the writer to facilitate in the calculation of such transpositions.

PHYSICAL LABORATORY,  
OHIO STATE UNIVERSITY.