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XIII. *On the Remainder of the Series in the development of $(1+x)^{-n}$, and on a Theorem respecting the products of Squares.* By J. R. YOUNG, Professor of Mathematics, Belfast*.

IN the last Number of the Philosophical Magazine, there is a very interesting paper, by Professor Graves, On the Calculus of Operations, in which he has communicated a valuable theorem in that important department of analysis, which I believe has not hitherto appeared in a complete form.

Professor Graves has been enabled to deduce this theorem from the previous development of $(1+x)^{-n}$; which, by means of the differential calculus, he has exhibited in connexion with the remainder of the series.

This completed form of the expansion may be readily obtained by a process imitative of that employed in my paper published in the November Number of this Journal, and without involving any operation of a more advanced character than that of common algebraical division. It is as follows:—

As in Professor Graves's notation, let

$$A_n = \frac{n(n+1) \dots (n+m-2)}{1.2 \dots (m-1)} \\ = \frac{m(m+1) \dots m+n-2}{1.2 \dots (n-1)}.$$

Put also

$$(1+x)^{-1}(-x)^m = R_1, \quad (1+x)^{-2}(-x)^m = R_2,$$

$$(1+x)^{-3}(-x)^m = R_3, \text{ \&c. ;}$$

then, since

$$(1+x)^{-1} = 1 - x + x^2 - \text{\&c.} \dots (-x)^{m-1} + R_1,$$

we shall have, by dividing the terms on the right severally by $(1+x)$, the following rows of results, namely,

$$\begin{aligned} (1+x)^{-2} &= 1 - x + x^2 - x^3 + \dots (-x)^{m-1} + R_1 \\ &\quad - x + x^2 - x^3 + \dots (-x)^{m-1} + R_1 \\ &\quad + x^2 - x^3 + \dots (-x)^{m-1} + R_1 \\ &\quad - x^3 + \dots (-x)^{m-1} + R_1 \\ &\quad \text{\&c.} \qquad \text{\&c.} \qquad + R_2; \end{aligned}$$

that is,

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots A_2(-x)^{m-1} + A_2R_1 + R_2.$$

Similarly,

* Communicated by the Author.

$$\begin{aligned}
(1+x)^{-3} &= 1 - 3x + 6x^2 - 10x^3 + \dots A_3(-x)^{m-1} + A_3R_1 \\
&\vdots \\
&\quad + A_2R_2 + R_3 \\
&\vdots \\
(1+x)^{-n} &= 1 - nx + \frac{n(n+1)}{1 \cdot 2} x^2 - \&c \dots A_n(-x)^{m-1} + A_nR_1 \\
&\quad + A_{n-1}R_2 + A_{n-2}R_3 + \dots + A_1R_n,
\end{aligned}$$

the A_1 being introduced before R_n for the sake of uniformity of notation: its value is evidently unit. When R_r is replaced by $(1+x)^{-r}(-x)^m$, this result becomes the same as that in Professor Graves's paper; and it must certainly strike a reader as a circumstance worthy of notice, that an expression thus obtained by aid of only the first principles of algebra, should virtually involve a theorem of such interest in the higher researches of analysis as that given in the paper alluded to.

I fear, from the remark in the first paragraph of that paper, that I must have expressed myself somewhat obscurely in reference to the Calculus of Operations making "no provision for the correction" I had adverted to as necessary. I think I ought to have added, that this provision should always be furnished by the theorem for quantity whence that for operations is derived.

I take this opportunity of mentioning that the general form of the theorem respecting squares, namely,

$$\sum_{16} q' \sum_{16} q'' = \sum_{32} q',$$

which the Rev. Mr. Kirkman has done me the favour to insert at page 500 of the last volume of this Journal, I should prefer to have appeared in the following more comprehensive shape:

$$\sum_{8m} q' \sum_{8n} q'' = \sum_{8mn} q',$$

to which may be added the analogous theorems

$$\sum_{4m} q' \sum_{4n} q'' = \sum_{4mn} q'$$

$$\sum_{2m} q' \sum_{2n} q'' = \sum_{2mn} q',$$

where m and n are any positive whole numbers whatever.

Belfast, Jan. 13, 1849.

Note. I submitted the substance of the foregoing investigation to Professor Graves, who, in reply, did me the favour to communicate to me a sketch of two other methods of arriving, algebraically, at the same result: this I here give in his own words:—

* * * "I indicated the method of obtaining the remainder by differentiation, because that process admits of being described in the fewest words, though it is far from being the simplest. I know of two algebraical methods by which the result is obtained more easily.

“One follows the ordinary track; showing that, if the theorem holds for $(1+x)^{-n+1}$, it will hold likewise for $(1+x)^{-n}$. And this is readily proved by means of the fundamental property of the binomial coefficients; viz. that the algebraical sum of the coefficients of x^{r-1} and x^r in the development of $(1+x)^p$ is equal to the coefficient of x^r in the development of $(1+x)^{p+1}$.

“My other method having something peculiar in it, I shall give it in full. Using S to denote the sum of the first m terms in the development of $(1-x)^{-n}$, and R the remainder after S , we shall have

$$R = \frac{1 - (1-x)^n \cdot S}{(1-x)^n}.$$

Now, when we come to examine the numerator in this value of R , we find that it contains only powers of x , from x^m up to x^{m+n-1} . R may therefore be put into the form $x^m \cdot f(1-x)^{-1}$, $f(x)$ being used to denote a series of n terms proceeding according to positive integer powers of x , from x up to x^n .

“We have now ascertained the *form* of the remainder about which we are inquiring, and it will be easy to determine the coefficients in f . For this purpose let us take the equation

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1 \cdot 2} x^2 + \dots + A_n x^{m-1} + x^m \cdot f(1-x)^{-1},$$

and in it interchange x with $1-x$, and m with n . Then we shall have

$$x^{-m} = 1 + m(1-x) + \frac{m(m+1)}{1 \cdot 2} (1-x)^2 + \dots + A_n (1-x)^{n-1} + (1-x)^n f'x^{-1};$$

where $f'x^{-1}$ stands for a series of powers of x^{-1} , from x^{-1} up to x^{-m} . Multiply the last equation by $x^m(1-x)^{-n}$, and it will become

$$(1-x)^{-n} = x^m \left\{ (1-x)^{-n} + m(1-x)^{-n+1} + \frac{m(m+1)}{1 \cdot 2} (1-x)^{-n+2} + \dots + A_n (1-x)^{-1} \right\} + x^m f'x^{-1}.$$

On comparing the several terms of the two finite developments thus given for $(1-x)^{-n}$, it is obvious that we must have

$$x^m f'x^{-1} = S,$$

and

$$R = x^m \left\{ (1-x)^{-n} + m(1-x)^{-n+1} + \frac{m(m+1)}{1 \cdot 2} (1-x)^{-n+2} + \dots + A_n (1-x)^{-1} \right\}."$$