Note on the simple group of order 504.

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The simple group of order 504 is completely defined by the relations

(1) $A^7 = 1, \quad B^2 = 1, \quad (AB)^3 = 1, \quad (A^3BA^5BA^3B)^2 = 1.$

To verify this statement, let

$$\tau = A^3BA^5BA^3B,$$

and

$$\sigma = ABA^4B.$$

By the defining relations τ is an operation of order 2; and since

$$\begin{aligned}\sigma &= B.BABA.A^3B,\\ &= B.A^6B.A^3B,\\ &= BA^5.ABA.A^2B,\\ &= BA^5B.A^6.BA^2B;\end{aligned}$$

σ is an operation of order 7.

Now

$$\begin{aligned}\sigma^2 &= ABA^3.ABABA.A^3B\\ &= ABA^3BA^3B,\end{aligned}$$

and

$$\sigma^{-2} = BA^4BA^4BA^6.$$

Hence

$$\begin{aligned}\sigma^2B\sigma^{-2} &= ABA^3BA^3BA^4BA^4BA^6,\\ &= ABA^3BA.A^2BA^4BA^4B.A^6.\\ &= ABA^3BA.BA^3BA^3BA^5.A^6,\end{aligned}$$

(since $A^2BA^4BA^4B$, being conjugate to τ, is an operation of order 2)

$$\begin{aligned}&= ABA^2.ABABA.A^2BA^3BA^4,\\ &= ABA^2BA^2BA^3BA^4;\end{aligned}$$

and

$$\begin{aligned}\sigma^2B\sigma^{-2}\tau &= ABA^2BA^2BABA^3B,\\ &= ABA^2BABA^2B,\\ &= (AB)^3 = 1.\end{aligned}$$

Hence
$$\tau = \sigma^2 B \sigma^{-2}.$$
But
$$B\sigma^{-1}B\sigma = A^3BA^6BABA^4B,$$
$$= A^3BA^5BA^3B,$$
$$= \tau;$$
and therefore

(2) $$B\sigma^{-1}B\sigma = \sigma^2 B\sigma^{-2}.$$

I have shewn (Messenger of Mathematics, Vol. XXV, pp. 187—189) that (2) is the sufficient condition that σ and B, of orders 2 and 7, should generate a group of order 56, which contains 7 permutable operations of order 2 and 8 subgroups of order 7. Next let
$$\varrho = BA^3;$$
then
$$\varrho^3 = BA^3BA^3BA^5.A^5,$$
$$= A^2BA^4BA^4BA^5,$$
(since $BA^3BA^2BA^5$ is of order 2)
$$= A^2BA^3.AB.A^4BA^5.$$
Now AB is an operation of order 3, and therefore ϱ is an operation of order 9. Moreover ϱ and B generate the group, since
$$(B\varrho)^5 = A.$$
Hence, if it be shewn that the cyclical group generated by ϱ and the group of order 56 generated by σ and B are permutable with each other, the relations (1) must define a group of finite order. Now $\sigma^{-2}B\sigma^2$, or B', transforms ϱ into ϱ^{-1}. For
$$B'\varrho = BA^4BA^4BA^6.B.ABA^3BA^3B.BA^3$$
$$= BA^4BA^4BA^5BA^2BA^6.$$
Now
$$BA^5B = ABA^2BA.$$
Hence
$$B'\varrho = BA^4BA^5BA^2BA^3BA^6,$$
$$= BA^5BA^2BA^3BA^3BA^6,$$
$$= ABA^2BA^3BA^3BA^3BA^6,$$
$$= ABA^4.A^5BA^3BA^3B.A^3BA^6.$$
Hence $B'\varrho$ is conjugate to $A^5BA^3BA^3B$, which is an operation of order 2. Therefore
$$(B'\varrho)^2 = 1,$$
or

(3) $$\varrho^n B' \varrho^n = B',$$

for all values of n.

Moreover it may be easily verified that

$$(4)\qquad \begin{cases} \rho\sigma\rho^2 = B, \\ \rho^2\sigma\rho^4 = \sigma B, \\ \rho^3\sigma\rho^3 = \sigma^6, \\ \rho^4\sigma\rho^7 = B\sigma^4 B, \\ \rho^5\sigma\rho^8 = B\sigma^3 B, \\ \rho^6\sigma\rho^5 = (B\sigma)^3, \\ \rho^7\sigma\rho^6 = (B\sigma)^4, \\ \rho^8\sigma\rho = B\sigma^6. \end{cases}$$

For instance,

$$\begin{aligned} \rho^4\sigma\rho^7 &= (BA^3)^4 ABA^4B(BA^3)^7 = (BA^3)^3A^3(BA^3)^4 \\ &= (BA^3)^2BA^6BA^3(BA^3)^3 = (BA^3)^2ABA^4(BA^3)^3 \\ &= BA.A^2BA^4BA^4B.A^3(BA^3)^2 \\ &= BABA^3BA^3BABA^3BA^3 \\ &= A^6BA^2BA^2BA^2BA^3 \\ &= (A^6BA^3)^4 = (BABA^4)^4 \\ &= (B\sigma B)^4 \quad = B\sigma^4 B. \end{aligned}$$

But (3) and (4) are sufficient conditions to ensure that the cyclical group $\{\rho\}$, generated by ρ, and the group $\{\sigma, B\}$, generated by σ and B' or B, should be permutable with each other. Hence the group $\{A, B\}$, where A and B satisfy the relations (1), is a group of finite order.

The simple group of order 504 is given by the totality of the linear substitutions

$$z' \equiv \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \not\equiv 0, \quad (\text{mod. } 2),$$

where $\alpha, \beta, \gamma, \delta$ are powers of a root of the irreducible congruence

$$x^3 + x + 1 \equiv 0 \quad (\text{mod. } 2).$$

This group is generated by the substitutions

$$a \quad \text{or} \quad z' \equiv xz$$

and

$$b \quad \text{or} \quad z' \equiv \frac{z + x^2}{x^2 z + 1};$$

and it is easy to verify that a and b satisfy the relations (1). Hence $\{a, b\}$ and $\{A, B\}$ are isomorphic; and since the order of $\{A, B\}$ cannot exceed 504, the isomorphism is simple. Finally therefore the group $\{A, B\}$, where A and B satisfy the relations (1), is simply isomorphic with the simple group of order 504.