

ART. XVIII.—*On a method of swinging Pendulums for the determination of Gravity, proposed by M. Faye; by C. S. PEIRCE.*

[Read before the National Academy of Sciences, April 17th, 1879, with authority of the Superintendent of the U. S. Coast and Geodetic Survey.]

AT the Stuttgart, 1878, meeting of the International Geodetic Association, M. Faye suggested a method of avoiding the flexure of a pendulum-support which promises important advantages. The proposal was that two similar pendulums should be oscillated on the same support with equal amplitudes and opposite phases. If the pendulums could be made precisely alike, the amplitudes precisely equal, and the phases precisely opposite, it is obvious that the support would be continually solicited by two equal and opposite forces and would undergo no horizontal flexure, except from the distortion of the parts between the two edges. But since none of these three elements can be made equal, it is necessary to inquire what would be the effect of such slight imperfections in their equalization as would have to be expected in practice.

I had the advantage many years ago of learning the main characteristics of the mutual influence of pendulums from Professor Benjamin Peirce. As my father's studies of the subject were never, I believe, written out, I am unable to say definitely what I derive from that source. But the truth is the little knowledge I have of mathematics was learned from him, and from him I got a clear idea of the nature of this particular problem; so that acknowledgments of detail, even if I were able to make them, would be quite inadequate.

In M. Faye's proposed experiment, four finite forces would be in operation, namely: the weights of the two pendulums, the elastic force tending to restore the two knife-edge supports to their position of equilibrium when they are both displaced together, and the elastic force tending to restore them when their relative positions are displaced. The system has, also, four degrees of freedom corresponding to motions against each of the four finite forces. Accordingly there will be four differential equations of motion. By neglecting the terms of the second order, these equations are made linear, and by the general theory of such equations, they indicate that each of the four motions of the system (*viz.*, those of the pendulums and of the two knife-edges) is compounded of four simple harmonic motions. Two of these will have periods nearly equal to those of the pendulums; the other two will be mere tremors having periods nearly those of the natural elastic oscillations of the supports. These tremors will be so small that they may be neglected. In fact, if we simply suppose that the knife-edges are constantly in equilibrium under the various forces which solicit them (which is simply to neglect their living forces under their very small velocities) the tremors disappear, to the great simplification of the formulæ.

Putting, then,  $\varphi_1$  and  $\varphi_2$  for the momentary angles of displacement of the two pendulums,  $s_1$  and  $s_2$  for the momentary horizontal displacements of the two knife-edges,  $l_1$  and  $l_2$  for the lengths of the two equivalent simple pendulums (on an absolutely rigid support),  $g$  for the acceleration of gravity, and  $t$  for the time, we have

$$l_1 \frac{d^2 \varphi_1}{dt^2} + \frac{d^2 s_1}{dt^2} = -g \varphi_1,$$

$$l_2 \frac{d^2 \varphi_2}{dt^2} + \frac{d^2 s_2}{dt^2} = -g \varphi_2.$$

These equations are exactly like what we have in the case of a single pendulum on a flexible support; and I have shown their correctness in my paper on that subject.

There would be no difficulty in making the two pendulums so nearly alike that they might be regarded as entirely so in their actions on the stand, the whole amount of which is small.

We may also consider the parts of the stand on which the two knives rest as equally elastic. We may therefore take  $\frac{1}{2}(s_1 + s_2)$  as proportional to  $\frac{1}{2}(\varphi_1 + \varphi_2)$ , and  $\frac{1}{2}(s_1 - s_2)$  as proportional to  $\frac{1}{2}(\varphi_1 - \varphi_2)$ . Denoting, then, by  $x$  and  $y$  two constants whose values will be easily determinable by experiments we have

$$\begin{aligned} s_1 + s_2 &= (x + y) (\varphi_1 + \varphi_2) \\ s_1 - s_2 &= (x - y) (\varphi_1 - \varphi_2); \\ \text{or} \quad s_1 &= x\varphi_1 + y\varphi_2 \\ s_2 &= x\varphi_1 + y\varphi_2. \end{aligned}$$

Substituting these values of  $s_1$  and  $s_2$  in the differential equations, and also writing  $l + \delta l$  for  $l_1$  and  $l - \delta l$  for  $l_2$ , they become

$$\begin{aligned} (l + x + \delta l) \frac{d^2 \varphi_1}{dt^2} + y \frac{d^2 \varphi_2}{dt^2} &= -g \varphi_1 \\ (l + x - \delta l) \frac{d^2 \varphi_2}{dt^2} + y \frac{d^2 \varphi_1}{dt^2} &= -g \varphi_2. \end{aligned}$$

The solution of these equations is ( $A$ ,  $B$ ,  $t_1$ , and  $t_2$  being the arbitrary constants)

$$\begin{aligned} \phi_1 &= A \cos \left\{ \sqrt{\frac{g}{l+x-\sqrt{(\delta l)^2+y^2}}} \cdot (t-t_1) \right\} + B \cos \left\{ \sqrt{\frac{g}{l+x+\sqrt{(\delta l)^2+y^2}}} \cdot (t-t_2) \right\} \\ \phi_2 &= -A \left( \frac{\delta l}{y} + \sqrt{1 + \left( \frac{\delta l}{y} \right)^2} \right) \cos \left\{ \sqrt{\frac{g}{l+x-\sqrt{(\delta l)^2+y^2}}} \cdot (t-t_1) \right\} \\ &\quad - B \left( \frac{\delta l}{y} - \sqrt{1 + \left( \frac{\delta l}{y} \right)^2} \right) \cos \left\{ \sqrt{\frac{g}{l+x+\sqrt{(\delta l)^2+y^2}}} \cdot (t-t_2) \right\} \end{aligned}$$

The condition that the pendulums are started by drawing them away from their positions of equilibrium and then letting them escape nearly at the same instant makes  $t_1$  and  $t_2$  nearly equal. We may reckon the time from the mean instant of starting. Then at that instant we have very nearly

$$\begin{aligned} \varphi_1 &= A + B \\ \varphi_2 &= -A \left( \frac{\delta l}{y} + \sqrt{1 + \left( \frac{\delta l}{y} \right)^2} \right) - B \left( \frac{\delta l}{y} - \sqrt{1 + \left( \frac{\delta l}{y} \right)^2} \right); \end{aligned}$$

or if we write  $z$  for  $\frac{\delta l}{y}$ ,

$$\varphi_2 = -A (z + \sqrt{1+z^2}) - B (z - \sqrt{1+z^2}).$$

And since the amplitudes are nearly equal and the phases nearly opposite,

$$\varphi_1 = -\varphi_2,$$

$$\text{or } A + B = (\text{nearly}) A (z + \sqrt{1+z^2}) + B (z - \sqrt{1+z^2})$$

This gives

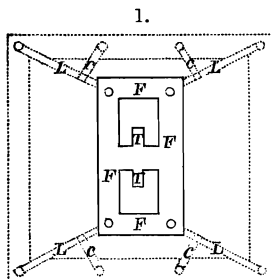
$$\frac{B}{A} = (\text{nearly}) \frac{\sqrt{1+z^2} - 1 + z}{\sqrt{1+z^2} + 1 - z}.$$

There would be no insuperable difficulty in making the pendulums so near alike that  $\delta l$  should be less than  $y$ , even if the latter quantity were smaller than it would be likely to be. But it will be seen presently that care must be taken in the construction not to make  $y$  too small.

We shall have then  $\delta l < y$  or  $z < 1$ ; whence  $B < A$ . Thus the amplitudes

of the first terms in the expressions for both  $\varphi_1$  and  $\varphi_2$  are greater than those of the second terms, while the period of the first terms is shorter than that of the second terms. From this it

can be shown to follow that the whole oscillations of the two pendulums have the same period, which is that of the harmonic motions represented by the first terms of their values. Thus, in the figure, the abscissas representing the time, we have a wave of short period and large amplitude placed in comparison with a wave of long period and small amplitude.



The phase of the short wave advances on the long one and goes over and over it. In each complete cycle of the curve representing the short wave, beginning and ending at  $y=0$ , it must cut the other curve twice unless the latter has mean time crossed the axis of abscissas once and not twice. When this happens there will be three intersections or only one, according to the direction of the crossing. Hence when the short curve has advanced over any even number of crossings by the long one of the axis of abscissas, the mean number of intersections per cycle of the short curve will be exactly two. Now let the short curve represent the first term in the expressions for  $\varphi_1$  or  $\varphi_2$  and let the long curve represent the second term *with its sign changed*; then, the intersections will represent passages of the pendulum over the vertical, and it will be seen that there are two for each complete period of the quicker harmonic component of the motion.

The mean period, then, of the oscillation of either pendulum will be

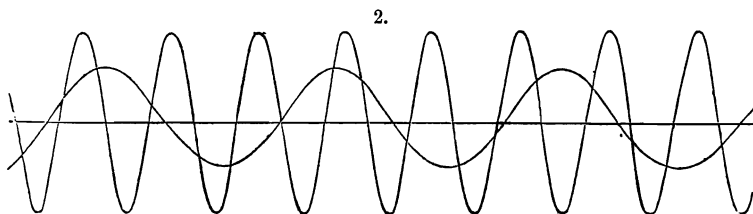
$$T = \pi \sqrt{\frac{l+x - \sqrt{(\delta l)^2 + y^2}}{g}}.$$

Now let us suppose that  $\delta l$  is so small that  $\frac{1}{2} \frac{(\delta l)^2}{ly}$  may be neglected, being less than one millionth. This would happen, for instance, if  $l$  were one meter,  $y$  a half a millimeter (so that the stand would be somewhat less stiff than the Repsold tripod), and  $\delta l$  were one twenty-fifth of a millimeter, so that the difference between the natural times of oscillation of the two pen-

dulums was not over four seconds a day, a perfectly attainable adjustment. Then the period would reduce to

$$T = \pi \sqrt{\frac{l+x-y}{g}}.$$

The terms  $x-y$  here indicate that the apparatus would still be subject to a correction for flexure: but it would be only for the relative flexure due to the distortion of the support between the two knife-edges. This could of course be made very small. It would still have to be measured: but it would be measured once for all, since it would be the same at all stations. At present, the measurement of the flexure at each station, involving as it does the erection of a separate pier, threatens to be one of the most troublesome and expensive parts of the whole work of determining gravity. This would be entirely obviated by M. Faye's plan, except that the small differential flexibility would have to be determined once for all. The proper way to make the stand so as to bind the two knives to their relative position as firmly as possible while allowing a moderately large flexibility to the whole stand, so that the two pendulums could freely influence one another, would easily be found out.



The accompanying figure, for instance, represents one such arrangement as viewed from above. T, T, are tongues upon which the pendulums would rest. These would be cast in one piece with the heavy frame F, F, F, F. This frame would rest on four legs L, L, L, L, which would spread at the bottom in the direction of the motion of the pendulum. At the bottom they would be bolted into another heavy frame. The cross braces C, C, C, C, would prevent twisting.

The average period of oscillation of either pendulum, after correction for flexure, would be that belonging to a simple pendulum having the length  $l$ , the mean of the lengths of the two simple pendulums whose natural periods of oscillation would be the same as those of the given pendulums. But although this would be the average time of oscillation of either pendulum, yet neither pendulum would have all its oscillations of the same duration. It is, therefore, necessary to inquire what error might arise owing to the observations not extending over any exact number of cycles of motion, so that the mean

of the observed periods would not be the same as the mean of the periods of a cycle.

The quickest oscillation of either pendulum would occur when the phases of the component harmonic motions were coincident, the slowest when these phases were opposed. The period of the slow harmonic component motion would be

$$\pi \sqrt{\frac{l+x+y}{g}}$$

or the mean of the periods of the two given pendulums oscillating on the given stand with coincident phases, so as to be affected by the flexibility of the whole stand but not by its liability to distortion. Suppose, then, that in the course of the experiment an instant comes at which the pendulums are vertical at once. Let us reckon the time from this instant, and put

$$I = \frac{A}{B} \cdot \frac{\sqrt{1+z^2}-1+z}{\sqrt{1+z^2}-1-z},$$

so that  $I$  is nearly unity. Then using the abbreviations

$$\begin{aligned}\sin_1 &= \sin \left\{ \sqrt{\frac{g}{l+x-\sqrt{(\delta l)^2+y^2}}} \cdot t \right\} \\ \sin_2 &= \sin \left\{ \sqrt{\frac{g}{l+x+\sqrt{(\delta l)^2+y^2}}} \cdot t \right\}\end{aligned}$$

we have

$$C \varphi_1 = (\sqrt{1+z^2}+1-z) \sin_1 \pm I (\sqrt{1+z^2}-1+z) \sin_2,$$

$$C \varphi_2 = (-\sqrt{1+z^2}-1-z) \sin_1 \pm I (-\sqrt{1+z^2}+1+z) \sin_2,$$

where the double sign distinguishes between coincidence and opposition of the phases of the harmonic constituents at the zero of  $t$ .

Then since the value of  $z$  is between 0 and unity, the values of these four coefficients lie

$\sqrt{1+z^2}+1-z$	between 2 and 1.414
$\sqrt{1+z^2}-1+z$	0      1.414
$-\sqrt{1+z^2}-1-z$	-2      -3.414
$-\sqrt{1+z^2}+1+z$	0      0.586

It follows that for one pendulum the phases of the harmonic constituents are coincident at the moment when they are for the other in exact opposition. Hence, one pendulum is making its slowest oscillation at the moment when the other is making its quickest, and *vice versa*. Then from the symmetrical character of harmonic motion it follows that if observations were taken of

both pendulums during any interval of time, then the mean of the average periods of the two during that interval, would give the mean period of either through a complete cycle of motion. A better method of observing, however, would be to set up a lens between the two pendulums, so as to bring the plane of oscillation of the one into focus on the plane of oscillation of the other. Then, by means of a reading telescope set up at a little distance, the oscillation at which both crossed on the vertical could be noted with some accuracy. It would then only be necessary to determine the mean period of oscillation of either from one such event to another. As the difference between the longest and shortest periods of oscillation would only amount to a few ten-thousandths of a second, it would not be necessary to be very exact in the time of beginning or ending the experiment. The number of oscillations between one coincidence at the vertical and another would afford a very accurate determination of  $y$ . For suppose  $n$  to be that number. Then

$$n \sqrt{\frac{l+x-y}{g}} = (n-1) \sqrt{\frac{l+x+y}{g}},$$

whence

$$l+x = \left( n - \frac{n-1}{2n-1} \right) y.$$

But as  $n$  is large (several thousand) we may take  $\frac{n-1}{2n-1} = \frac{1}{2}$ , and  $x$  as equal to  $y$ .

This gives  $y = \frac{l}{n-\frac{1}{2}}$ .

Then  $x-y$  having been determined, we ascertain the value of  $x$  also.

The greatest departure of the oscillations of the two pendulums from complete opposition of phase would occur when the phases of the harmonic components differed by a quadrant. In this case, the pendulums would cross at an angle equal to  $CI \frac{\delta l}{y}$  from the vertical. The difference in the time of their passage over the vertical could only amount to a minute fraction of a second.

If the pendulums should not be nearly enough adjusted to the same natural period, or if the stand should be too stiff, so that  $\delta l$  were greater than  $y$ , the slower harmonic component would have a greater amplitude than the quicker one. In this case, the pendulums would pass over all differences of phase, and whether the mean period of oscillation were that of the faster or of the slower component might depend upon the initial phases, or, if  $\delta l$  were still larger relatively to  $y$  it might be the same as if the pendulums were oscillating with coinci-

dent phases. Care would have to be taken to avoid such a state of things.

On the whole, it appears that the suggestion of M. Faye, though it was thrown out on the spur of the moment, and was not received with very warm approval on every hand, is as sound as it is brilliant, and offers some peculiar advantages over the existing method of swinging pendulums.

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