

30.

Demonstrationes theorematum et solutiones problematum quorundam a celeb. Hill Vol. 7. p. 102. hujus operis propositorum.

(Auct. Th. Clausen. Mun.)

3. Theor.

$$\int_0^\pi du (x^2 + 2ax \cos u + a^2)^{\frac{n}{2}} \cos \left\{ ru - n \left(\arctan \frac{a \sin u}{x + a \cos u} \right) \right\} = \pi a^r x^{n-r} \frac{n \cdot n-1 \cdot \dots \cdot n-r+1}{1 \cdot 2 \cdot \dots \cdot r}$$
 pro $r =$ numero integro positivo valet, existente $x > a$.

Quarto hujus operis volumine p. 281. formulis (1.) et (2.) hasce serierum infinitarum summas erui:

$$(1+2z \cos \varphi + z^2)^{\frac{m}{2}} \cos m \cdot \arctan \frac{z \sin \varphi}{1+z \cos \varphi} = 1 + \frac{m}{1} z \cos \varphi + \frac{m \cdot m-1}{1 \cdot 2} z^2 \cos 2\varphi + \frac{m \cdot m-1 \cdot m-2}{1 \cdot 2 \cdot 3} z^3 \cos 3\varphi + \dots$$

$$(1+2z \cos \varphi + z^2)^{\frac{m}{2}} \sin m \cdot \arctan \frac{z \sin \varphi}{1+z \cos \varphi} = \frac{m}{1} z \sin \varphi + \frac{m \cdot m-1}{1 \cdot 2} z^2 \sin 2\varphi + \frac{m \cdot m-1 \cdot m-2}{1 \cdot 2 \cdot 3} z^3 \sin 3\varphi + \dots$$

unde statim sequitur:

$$(1+2z \cos \varphi + z^2)^{\frac{m}{2}} \cos \left(r\varphi - m \cdot \arctan \frac{z \sin \varphi}{1+z \cos \varphi} \right) = \cos r\varphi + \frac{m}{1} z \cos (r-1)\varphi + \frac{m \cdot m-1}{1 \cdot 2} z^2 \cos (r-2)\varphi + \dots$$

quae series in $d\varphi$ multiplicata, et a $\varphi = 0$ usque ad $\varphi = \pi$ (semiperipheriam circuli) integrata, statim supeditat:

$$\int_0^\pi (1+2z \cos \varphi + z^2)^{\frac{m}{2}} \cos \left(r\varphi - m \cdot \arctan \frac{z \sin \varphi}{1+z \cos \varphi} \right) d\varphi = \pi \frac{m \cdot m-1 \cdot \dots \cdot m-r+1}{1 \cdot 2 \cdot \dots \cdot r} z^r;$$

omnibus terminis praeter $(r+1)^{\text{tium}}$ evanescentibus, serieque desuper post quosdam terminos convergente. Quodsi itaque ponatur $z = \frac{a}{x}$; $\varphi = u$; $m = n$; theorema propositum sponte emergit.

10. Theor. $\int_0^{n\pi} x dx \log(\sin x)^2 = -n^2 \pi^2 \log 2^*$, designante n numerum integrum positivum vel negativum.

*) In expositione theorematum signum — omissum est.

Habetur loco s. m.

$$-\log \sin x = \log 2 + \cos 2x + \frac{1}{2} \cos 4x + \frac{1}{3} \cos 6x + \dots;$$

hinc sequitur

$$-\int dx \log \sin x = x \log 2 + \frac{1}{2} \left(\sin 2x + \frac{1}{2^2} \sin 4x + \frac{1}{3^2} \sin 6x + \dots \right)$$

$$-\int dx \int dx \log \sin x = \frac{1}{2} x^2 \log 2 - \frac{1}{4} \left(\cos 2x + \frac{1}{2^2} \cos 4x + \frac{1}{3^2} \cos 6x + \dots \right);$$

inter limites vero praescriptos $k=0$, $x=n\pi$, designante n numerum integrum

$$-\int_0^{n\pi} dx \log \sin x = n\pi \log 2$$

$$-\int_0^{n\pi} dx \int_0^x dx \log \sin x = \frac{1}{2} n^2 \pi^2 \log 2.$$

Cum vero sit:

$$\int 2x dx \log \sin x = 2x \int dx \log \sin x - 2 \int dx \int dx \log \sin x,$$

sponte sua valor integralis definiti

$$\int_0^{n\pi} x dx \log (\sin x)^2 = -n^2 \pi^2 \log 2$$

demumat.

16. Probl. Data functione f , invenire functionem ϕ ex aequatione:

$$\frac{\phi(fx)}{\phi x} = \frac{dfx}{dx}.$$

Ponatur fx e duobus constare partibus $\psi x + \psi' x$; quod si substituat in aequationem datam

$$\frac{\phi(\psi x + \psi' x)}{\phi x} = \frac{d\psi x}{dx} + \frac{d\psi' x}{dx}$$

suppeditat; inde facile concluditur:

$$\phi(\psi x + \psi' x) = \phi(\psi x) + \phi(\psi' x)$$

atque hinc generaliter:

$$\phi(\psi x + \psi' x + \psi'' x + \dots) = \phi(\psi x) + \phi(\psi' x) + \phi(\psi'' x) + \dots$$

Cum vero quaelibet functio generaliter loquendo, in seriem infinitam

$$a + bx + cx^2 + \dots$$

resolvi potest, nihil restat quam functionem ϕ pro valore $fx = x^m$ definire.

Hoc valore in aequationem datam substituto, prodit:

$$\frac{\phi x^m}{\phi x} = m x^{m-1}$$

cui aequationi per aequationem

$$\varphi x^m = x \frac{dx^m}{dx}$$

satisfit. Unde concludere possumus:

$$\varphi(fx) = x \frac{dfx}{dx}.$$

14. Probl. „Ex aequationibus linearibus:

- 1) $\frac{1}{2} c_2 = c_1 - c_{29};$
- 2) $\frac{1}{2} c_4 = c_2 - c_{28};$
- 3) $\frac{1}{2} c_6 = c_3 - c_{27};$
- 4) $\frac{1}{2} c_8 = c_4 - c_{26};$
- 5) $\frac{1}{2} c_{10} = c_5 - c_{25};$
- 6) $\frac{1}{2} c_{12} = c_6 - c_{24};$
- 7) $\frac{1}{2} c_{14} = c_7 - c_{23};$
- 8) $\frac{1}{2} c_{16} = c_8 - c_{22};$
- 9) $\frac{1}{2} c_{18} = c_9 - c_{21};$
- 10) $\frac{1}{2} c_{20} = c_{10} - c_{20};$
- 11) $\frac{1}{2} c_{22} = c_{11} - c_{19};$
- 12) $\frac{1}{2} c_{24} = c_{12} - c_{18};$
- 13) $\frac{1}{2} c_{26} = c_{13} - c_{17};$
- 14) $\frac{1}{2} c_{28} = c_{14} - c_{16};$
- 15) $\frac{1}{3} c_3 = c_1 - c_{19} + c_{21};$
- 16) $\frac{1}{3} c_6 = c_2 - c_{18} + c_{22};$
- 17) $\frac{1}{3} c_9 = c_3 - c_{17} + c_{23};$
- 18) $\frac{1}{3} c_{12} = c_4 - c_{16} + c_{24};$
- 19) $\frac{1}{3} c_{15} = c_5 - c_{15} + c_{25};$
- 20) $\frac{1}{3} c_{18} = c_6 - c_{14} + c_{26};$
- 21) $\frac{1}{3} c_{21} = c_7 - c_{13} + c_{27};$
- 22) $\frac{1}{3} c_{24} = c_8 - c_{12} + c_{28};$
- 23) $\frac{1}{3} c_{27} = c_9 - c_{11} + c_{29};$
- 24) $\frac{1}{3} c_5 = c_1 - c_{11} + c_{13} - c_{23} + c_{25};$
- 25) $\frac{1}{3} c_{10} = c_2 - c_{10} + c_{14} - c_{22} + c_{26};$
- 26) $\frac{1}{3} c_{15} = c_3 - c_9 + c_{15} - c_{21} + c_{27};$
- 27) $\frac{1}{3} c_{20} = c_4 - c_8 + c_{16} - c_{20} + c_{28};$
- 28) $\frac{1}{3} c_{25} = c_5 - c_7 + c_{17} - c_{19} + c_{29};$

quantitatum c tot, quot fieri potest per reliquas definire.”

Facile derivatur ex aequationibus 15) et 23), quatuordecimque prioribus aequatio 16), similique modo e 17) et 21) aequatio 20). Porro e 6) et 12) concluditur

$$29) \quad c_{12} = \frac{2}{3}c_6 + \frac{4}{3}c_{18}; \quad c_{24} = \frac{4}{3}c_6 - \frac{1}{3}c_{18}$$

in quas aequationes si substituantur 16) et 20), inveniuntur ope quatuordecim prioribus aequationibus 18) et 22). Simili omnino modo 25) e 24) et 28), atque 27) e 25) derivatur. Aequatio 26) denique per 15), 17), 19), 21), 23) atque 24) et 26) obtinetur.

Aequationes igitur hae sub reliquis contentae sunt: 16), 18), 20), 22), 25), 26), 27). Restant ideo viginti et una aequationes, quae si diversae essent, saltem viginti et unam quantitatum c determinare liceret, octo ad libitum assumtis. Jam vero observo in aequationibus 15), 17), 19), 21), 23), 24), 28), quindecim tantummodo quantitatum c occurrere, eas scilicet, quae indici impari gaudent. Quodsi itaque octo harum quantitatum ad libitum assumantur, ita tamen ut nulla harum aequationum assumptioni repugnet; septem reliquae per septem aequationes modo nominatas determinari possunt, si modo nulla sub reliquis contenta est. Assumantur jam quantitates $c_1, c_5, c_9, c_{13}, c_{17}, c_{21}, c_{25}, c_{29}$ tanquam cognitae; tunc per eliminationem prodit:

$$\begin{aligned} c_3 &= -\frac{3}{2}c_1 + \frac{3}{2}c_5 + \frac{3}{2}c_9 - \frac{3}{2}c_{13} + \frac{3}{2}c_{17} - \frac{1}{2}c_{21} - \frac{6}{5}c_{25} + \frac{3}{2}c_{29}; \\ c_7 &= -\frac{3}{2}c_1 + \frac{6}{5}c_5 + \frac{1}{2}c_9 - \frac{1}{2}c_{13} + \frac{3}{2}c_{17} - \frac{7}{5}c_{21} - \frac{3}{5}c_{25} + \frac{3}{2}c_{29}; \\ c_{11} &= -\frac{1}{2}c_1 + \frac{2}{3}c_5 + \frac{7}{6}c_9 - \frac{1}{2}c_{13} + \frac{1}{2}c_{17} - \frac{1}{2}c_{21} - \frac{1}{5}c_{25} + \frac{3}{2}c_{29}; \\ c_{15} &= \frac{3}{4}c_5 + \frac{3}{4}c_{25}; \\ c_{19} &= \frac{3}{2}c_1 - \frac{1}{5}c_5 - \frac{1}{2}c_9 + \frac{1}{2}c_{13} - \frac{1}{2}c_{17} + \frac{7}{5}c_{21} + \frac{2}{5}c_{25} - \frac{1}{2}c_{29}; \\ c_{23} &= \frac{3}{2}c_1 - \frac{3}{5}c_5 - \frac{7}{6}c_9 + \frac{3}{2}c_{13} - \frac{1}{2}c_{17} + \frac{1}{2}c_{21} + \frac{6}{5}c_{25} - \frac{3}{2}c_{29}; \\ c_{27} &= \frac{3}{2}c_1 - \frac{6}{5}c_5 - \frac{1}{2}c_9 + \frac{3}{2}c_{13} - \frac{3}{2}c_{17} + \frac{3}{2}c_{21} + \frac{3}{5}c_{25} - \frac{3}{2}c_{29}. \end{aligned}$$

Aequationes quatuordecim priores omnes quantitates c indice pari gaudentes suppeditant, simulac ceterae innotuere, scilicet nullo negotio $c_2, c_6, c_{10}, c_{14}, c_{16}, c_{22}, c_{26}$; deinde per aequationes 29) c_{12}, c_{24} ; porro ex aequationibus 2), 4), 8), 14) invenitur:

$$c_4 = -\frac{2}{15}c_2 + \frac{4}{15}c_{14} + \frac{8}{15}c_{22} + \frac{1}{15}c_{26};$$

cujus ope c_4 atque deinde facile $c_8, c_{10}, c_{20}, c_{28}$ determinantur. En hasce valores:

$$\begin{aligned} c_2 &= 2c_1 - 2c_{29}; \\ c_4 &= -4c_1 + \frac{8}{5}c_5 + \frac{6}{5}c_9 - \frac{8}{5}c_{21} - \frac{8}{5}c_{25} + 4c_{29}; \\ c_6 &= -6c_1 + \frac{18}{5}c_5 + 4c_9 - 6c_{13} + 6c_{17} - 4c_{21} - \frac{18}{5}c_{25} + 6c_{29}; \end{aligned}$$

$$\begin{aligned}
c_8 &= -8c_1 + \frac{16}{3}c_5 + \frac{16}{3}c_9 - 4c_{13} + 4c_{17} - \frac{16}{3}c_{21} - \frac{16}{3}c_{25} + 8c_{29}; \\
c_{10} &= 2c_5 - 2c_{25}; \\
c_{12} &= -\frac{12}{5}c_1 + \frac{36}{25}c_5 + \frac{16}{5}c_9 - \frac{12}{5}c_{13} + \frac{12}{5}c_{17} - \frac{16}{5}c_{21} - \frac{36}{25}c_{25} + \frac{12}{5}c_{29}; \\
c_{14} &= -6c_1 + \frac{18}{5}c_5 + \frac{10}{3}c_9 - 4c_{13} + 4c_{17} - \frac{10}{3}c_{21} - \frac{18}{5}c_{25} + 6c_{29}; \\
c_{16} &= -8c_1 + 4c_5 + 4c_9 - 4c_{13} + 4c_{17} - 4c_{21} - 4c_{25} + 8c_{29}; \\
c_{18} &= 2c_9 - 2c_{21}; \\
c_{20} &= \frac{4}{3}c_5 - \frac{4}{3}c_{25}; \\
c_{22} &= -4c_1 + \frac{6}{5}c_5 + \frac{10}{3}c_9 - 2c_{13} + 2c_{17} - \frac{10}{3}c_{21} - \frac{6}{5}c_{25} + 4c_{29}; \\
c_{24} &= -\frac{24}{5}c_1 + \frac{72}{25}c_5 + \frac{12}{5}c_9 - \frac{24}{5}c_{13} + \frac{24}{5}c_{17} - \frac{12}{5}c_{21} - \frac{72}{25}c_{25} + \frac{24}{5}c_{29}; \\
c_{26} &= 2c_{13} - 2c_{17}; \\
c_{28} &= 4c_1 - \frac{4}{5}c_5 - \frac{4}{3}c_9 + \frac{4}{3}c_{21} + \frac{4}{5}c_{25} - 4c_{29};
\end{aligned}$$

Hoc itaque modo viginti et una quantitatum per reliquas octo determinatae sunt; cum vero non plures quam viginti et una aequationes diversae datae sunt, plures determinari nequeunt.

München, 15. April 1831.