*March* 11*th*, 1875.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Messrs. A. B. Kempe, B.A., and S. A. Renshaw were elected Members; Mr. J. H. Röhrs, M.A., was proposed for election; and Mr. Harry Hart, M.A., was admitted into the Society.

Mr. Roberts gave an account of his paper, "On a Simplified Method of obtaining the Order of Algebraical Conditions."

Mr. Sylvester, F.R.S., spoke on the subject of "An Orthogonal Web," pointing out a curious paradox when the reticulation was not all in the same plane.

Mr. Tucker read a portion of Mr. Darwin's paper, "On some proposed forms of Slide-rule."

The following presents were received:—

"Bulletin de la Société Mathématique de France," Tome ii., Fev. No. 5.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," Tome i., 2e Série, 1er Cahier. Paris, 1875.

"Bemerkungen zur Theorie der Ternären cubichen Formen von Axel Harnack." Erlangen, vom 8 Febr., 1875.

"Jahrbuch über die Fortschritte der Mathematik," viertes Band, Jahrgang 1872, Heft 3.

"Journal of the Institute of Actuaries," No. 97, Oct. 1874.

"Fifth Annual Report of the Association for the Improvement of Geometrical Teaching," January, 1875.

"Table des Fonctions Symétriques de Poids XI," dressée par le Chev. F. Faà de Bruno (extrait de la Théorie des Formes Binaires, du même auteur), Mars, 1875.

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## *On a Simplified Method of obtaining the Order of Algebraical Conditions.* By S. ROBERTS, M.A.

[*Read March* 11*th*, 1875.]

1. I propose to give some examples of a method of obtaining the order of the conditions for the co-existence of systems of equations. The method easily leads to the required expressions in the simpler cases, and shows the course of procedure where the actual expression is complicated and need not be evaluated.

One of the simplest examples will be given in detail. A few others have been chosen with a view to determining the characteristics of certain envelope surfaces to which the results will be applied.

2. Let it be required to determine the order of the conditions, two in number, that a pair of binary equations $U=0$, $V=0$, containing homogeneously the parameters $\alpha$, $\beta$, may have two common solutions. The coefficient of the highest power of $\alpha$ (say $\alpha^l$) in the first equation is supposed to be of the order $\lambda$ in uneliminated variables; that of $\alpha^{l-1}\beta$ is of the order $\lambda+a$; that of $\alpha^{l-2}\beta^2$ is of the order $\lambda+2a$, and so on. In the second equation, the coefficient of the highest power of $\alpha$ (say $\alpha^m$) is of the order $\mu$; that of $\alpha^{m-1}\beta$ is of the order $\mu+a$; that of $\alpha^{m-2}\beta^2$ is of the order $\mu+2a$, and so on. In other words, the equations are of the forms

$$U = \Sigma\, A_{\lambda+pa}\, \alpha^{l-p}\beta^p, \qquad p \not> l,$$
$$V = \Sigma\, B_{\mu+pa}\, \alpha^{m-p}\beta^p, \qquad p \not> m,$$

the suffixes denoting the order in which the uneliminated variables enter the coefficients.

Writing $N_m$ for the order required, so that $N_{m+1}$ denotes the corresponding order when $m$ is changed into $m+1$, we have first of all to determine $N_{m+1}-N_m$ or $\Delta N_m$. This difference being integrated, the result is to be made symmetrical with regard to $l$ and $m$, $\lambda$ and $\mu$. There will generally remain a constant to be determined by special values of $l$ and $m$. This is the general process which will be employed in all cases.

I use the notation ${}_i\left|\begin{matrix}U\\ V\\ \&c.\end{matrix}\right.$ to denote the order of the conditions that $U=0$, $V=0$, &c., may have $i$ common solutions.

In the case before us, suppose that $V$ is replaced by $V\,.\,X$, where $X$ is a linear factor $A\alpha+B\beta$. The coefficients $A$ and $B$ may be taken to be of the orders zero and $a$ respectively with regard to the uneliminated variables, so that in effect the orders of the coefficients of $V\,.\,X$ remain the same as those of $V$, while the degree of the equation is increased by unity. But we may, where it becomes expedient, simplify still further; for $X$ may be considered as identical with $\alpha$, one of the parameters, and the factor is then of the order zero as to the uneliminated variables. The effect of the breaking up of the equation into $V$ and a linear factor must be taken into consideration and allowed for. Thus we have

$$N_{m+1} = N_m + {}_2\left|\begin{matrix}U\\ X\end{matrix}\right. + {}_1\left|\begin{matrix}U\\ V\end{matrix}\right._1\left|\begin{matrix}U\\ X\end{matrix}\right. - {}_1\left|\begin{matrix}U\\ V\\ X\end{matrix}\right.$$

That is to say, the conditions of the case are fulfilled if $U$ and $V$ have two common solutions, or if $U$ and $X$ have two common solutions, or if U and V have a common solution, and U and X have another different common solution. We must consequently deduct the order of the conditions that U, V, and X may have a common solution.

The order ${}_{2}\left|\begin{matrix}U\\X\end{matrix}\right.$ is zero, so that

$$\Delta N_m = {}_{1}\left|\begin{matrix}U\\V\end{matrix}\right. {}_{1}\left|\begin{matrix}U\\X\end{matrix}\right. - {}_{1}\left|\begin{matrix}U\\V\\X\end{matrix}\right. = (\lambda m+\mu l+lma)(\lambda+la)-(\lambda+la)(\mu+ma)$$

and $N_m = \lambda^2\frac{m^2-m}{2} + \lambda\mu(l-1)m + \lambda a(2l-1)\frac{m^2-m}{2}$

$$+ \mu a(l^2-l)m + a^2(l^2-l)\frac{m^2-m}{2} + F.$$

To make this symmetrical in $l$ and $m$, $\lambda$ and $\mu$, we must take

$$F = \mu^2\frac{l^2-l}{2} - \lambda\mu l - \mu a\frac{l^2-l}{2} + C.$$

The expression must vanish for $l=1$, $m=1$, so that $C=\lambda\mu$ and the result is

$$\lambda^2\frac{m^2-m}{2} + \mu^2\frac{l^2-l}{2} + \lambda\mu(l-1)(m-1)$$

$$+ \frac{\lambda a}{2}m(m-1)(2l-1) + \frac{\mu a}{2}l(l-1)(2m-1) + \frac{a^2}{2}lm(l-1)(m-1),$$

and agrees with that arrived at in other ways.

3. To obtain the order of the conditions that two binary equations may have two common solutions, one of them a solution of a third equation, we may proceed in the same way.

Let the third equation $R=0$ be of the degree $n$, the coefficient being of the order $\gamma$ in uneliminated variables. For simplicity's sake we will here take $a=0$.

Then, if ${}^{1}\left|\begin{matrix}R\\ \overline{U}\end{matrix}\right.$ denote the order of the conditions that U, V may have two common solutions, one of them a solution of R,

$$N_{m+1} = {}^{1}_{2}\left|\begin{matrix}R\\ \overline{U}\\ V.X\end{matrix}\right. = {}^{1}_{2}\left|\begin{matrix}R\\ \overline{U}\\ V\end{matrix}\right. + {}^{1}_{2}\left|\begin{matrix}R\\ \overline{U}\\ X\end{matrix}\right. + {}_{1}\left|\begin{matrix}R\\U\\V\end{matrix}\right. {}_{1}\left|\begin{matrix}U\\X\end{matrix}\right. + {}_{1}\left|\begin{matrix}R\\U\\X\end{matrix}\right. {}_{1}\left|\begin{matrix}U\\V\end{matrix}\right. - 2\,{}_{1}\left|\begin{matrix}R\\U\\V\\X\end{matrix}\right.$$

But ${}^{1}_{2}\left|\begin{matrix}R\\ \overline{U}\\ X\end{matrix}\right. = 0$, and therefore

$$\Delta N_m = {}_{1}\left|\begin{matrix}R\\U\\V\end{matrix}\right. {}_{1}\left|\begin{matrix}U\\X\end{matrix}\right. + {}_{1}\left|\begin{matrix}R\\U\\X\end{matrix}\right. {}_{1}\left|\begin{matrix}U\\V\end{matrix}\right. - 2\,{}_{1}\left|\begin{matrix}R\\U\\V\\X\end{matrix}\right.$$

$$= \lambda(\mu\nu l + \lambda\nu m + \lambda\mu n) + \lambda\nu(\lambda m+\mu l) - 2\lambda\mu\nu,$$

and $N_m = \lambda^2\nu(m^2-m) + [\lambda^2\mu n + 2\lambda\mu\nu l - 2\lambda\mu\nu]m + F;$

and we must write for F,

$$\mu^2\nu\,(l^2-l)+\mu^2\lambda ln-2\lambda\mu\nu l+C.$$

The expression must vanish for $l=1$, $m=1$, giving

$$C+(\lambda^2\mu+\mu^2\lambda)\,n-2\lambda\mu\nu=0.$$

Finally,

$$N_n=\nu\,\{\lambda^2(m^2-m)+\mu^2(l^2-l)+2\lambda\mu\,(l-1)\,(m-1)\}\\ +n\lambda\mu\,\{\lambda\,(m-1)+\mu\,(l-1)\}.$$

4. Again, let it be required to find the order of the conditions that three ternary equations may have two common solutions, one of them a solution of a fourth equation.

The formation of the equations is supposed to be analogous to that of the previous binary equations, viz., they are of the form

$$\Sigma\,A_{\lambda+pa+qb}\,\alpha^{l-p-q}\,\beta^p\gamma^q,\qquad p+q \not< l.$$

We have, using an obvious extension of the notation,

$$N_n={}^1_2\left|\begin{matrix}\overline{R}\\U\\V\\W\,.\,X\end{matrix}\right.={}^1_2\left|\begin{matrix}\overline{R}\\U\\V\\W\end{matrix}\right.+{}^1_2\left|\begin{matrix}\overline{R}\\U\\V\\X\end{matrix}\right.+{}_1\left|\begin{matrix}R\\U\\V\\W\end{matrix}\right|^1\begin{matrix}U\\V\\X\end{matrix}+{}_1\left|\begin{matrix}R\\U\\V\\X\end{matrix}\right|^1\begin{matrix}U\\V\\W\end{matrix}-2\,{}_1\left|\begin{matrix}R\\U\\V\\W\\X\end{matrix}\right.$$

or

$$\Delta N_n={}^1_2\left|\begin{matrix}\overline{R}\\U\\V\\X\end{matrix}\right.+\left|\begin{matrix}R\\U\\V\\W\end{matrix}\right|^1\begin{matrix}U\\V\\X\end{matrix}+{}_1\left|\begin{matrix}R\\U\\V\\X\end{matrix}\right|^1\begin{matrix}U\\V\\W\end{matrix}-2\,{}_1\left|\begin{matrix}R\\U\\V\\W\\X\end{matrix}\right.$$

${}^1_2\left|\begin{matrix}\overline{R}\\U\\V\\X\end{matrix}\right.$ is known, since we have only to put $\alpha$ for X, and then make $\alpha=0$,

when the form is reduced to ${}^1_2\left|\begin{matrix}\overline{(R)}\\(U)\\(V)\end{matrix}\right.$ where (R), (U), (V) are binary.

It is unnecessary here to work out the general formula. We shall only require the case in which the coefficients of R are constant, and $a=0$, $b=0$. The degrees of R, U, V, W being $p$, $l$, $m$, $n$, and the orders of the coefficients 0, $\lambda$, $\mu$, $\nu$, we have

$$\Delta N_n=p\left\{\begin{matrix}\lambda\mu\,\{\lambda\,(m-1)+\mu\,(l-1)\}+(\lambda m+\mu l)(\mu\nu l+\lambda\nu m+\lambda\mu n)\\ +\lambda\mu\,(\lambda mn+\mu ln+\nu lm)-2\lambda\mu\nu\end{matrix}\right\},$$

and

$$N_n=p\left\{\begin{matrix}\lambda\mu\,(\lambda m+\mu l)\,(n^2-n)+\left\{\begin{matrix}\lambda\mu\,[\lambda(m-1)+\mu\,(l-1)]\\ +(\lambda m+\mu l)\,(\mu\nu l+\lambda\nu m)\\ +\lambda\mu\nu lm-2\lambda\mu\nu\end{matrix}\right\}n\end{matrix}\right\}\\ +F.$$

Making this symmetrical, and taking the constant so that the expression may vanish for $l=m=n=1$, we get

$$N_n = p\,\{\Sigma\lambda^2\mu n\,(mn-1) + \lambda\mu\nu\,(3lmn-2\Sigma l+3)\}.$$

Generally, for $k+1$ equations in all, we have

$$\Delta N = {}^{1}_{2}\left|\begin{array}{l} R \\ \hline U_1 \\ U_2 \\ \vdots \\ U_{k-1} \\ X \end{array}\right| + {}_{1}\left|\begin{array}{l} R \\ U_1 \\ U_2 \\ \vdots \\ U_k \end{array}\right| {}_{1}\left|\begin{array}{l} U_1 \\ U_2 \\ \vdots \\ U_{k-1} \\ X \end{array}\right| + {}_{1}\left|\begin{array}{l} R \\ U_1 \\ U_2 \\ \vdots \\ U_{k-1} \\ X \end{array}\right| \left|\begin{array}{l} U_1 \\ U_2 \\ \vdots \\ U_k \end{array}\right| - 2\; {}_{1}\left|\begin{array}{l} R \\ U_1 \\ U_2 \\ \vdots \\ U_k \\ X \end{array}\right|$$

the case of $k+1$ equations is thus reduced to that of $k$ equations and other known forms.

5. To find the order of the conditions that three ternary equations may have a pair of coincident solutions, or, what is the same thing, that three curves may touch at the same point. This is also a case which will be required subsequently. Using $t$ to denote the condition of touching, we have

$$N_{n+1} = {}_{t}\left|\begin{array}{l} U \\ V \\ W \end{array}\right| + {}_{t}\left|\begin{array}{l} U \\ V \\ X \end{array}\right| + 2\; {}^{t}_{1}\left|\begin{array}{l} U \\ V \\ \hline W \\ X \end{array}\right|$$

that is to say, we may have U, V, W touching at the same point, or U, V having a common tangent X, with the same point of contact; or we may have U, V touching at an intersection of W and X. This, we know, counts as two contacts.

Finally, then,

$$\Delta N_n = \left|\begin{array}{l} U \\ V \\ X \end{array}\right| + 2\; {}_{1}\left|\begin{array}{l} U \\ V \\ \hline W \\ X \end{array}\right| = 2\lambda\mu\{\lambda(\mu-1)+\mu(l-1)\} + 2\; {}_{2}\left|\begin{array}{l} U \\ V \\ \hline W \\ X \end{array}\right|$$

We will suppose here also that $a=0$, $b=0$, $a$ and $b$ being the increments of the orders of the coefficients, as we proceed from term to term. But

$${}^{t}_{1}\left|\begin{array}{l} U \\ V \\ \hline W \\ X \end{array}\right| = {}_{1}\left|\begin{array}{l} U_1V_2-U_2V_1 \\ U \\ V \\ W \\ X \end{array}\right| - {}_{1}\left|\begin{array}{l} U_1V_2-U_2V_1 \\ V \\ \gamma \\ W \\ X \end{array}\right| + {}_{1}\left|\begin{array}{l} V_1 \\ V_2 \\ \gamma \\ W \\ X \end{array}\right|$$

where $U_1$ is written for $\frac{dU}{da}$, $U_2$ for $\frac{dU}{d\beta}$ &c., and $\gamma$ is the third variable.

Consequently,

$$ {}^{1}_{1}\begin{vmatrix} U \\ V \\ \hline W \\ X \end{vmatrix} = (\lambda+\mu)(\lambda\mu n+\lambda\nu m+\mu\nu l) + \lambda\mu\nu(l+m-2)-(\lambda+\mu)\mu\nu+\mu^2\nu $$

and

$$ \Delta N_n = 2\lambda^2\mu(m+n-1) + 2\lambda\mu^2(l+n-1) + 2\lambda^2\nu m + 2\mu^2\nu l + 2\lambda\mu\nu(2l+2m-3). $$

Hence,

$$ N_n = \lambda^2\mu(n^2-3n+2mn) + \lambda\mu^2(n^2-3n+2ln) + 2\lambda^2\nu mn + 2\mu^2\nu ln + 2\lambda\mu\nu(2l+2m-3)n + F. $$

The expression must vanish for $l=m=n=1$, so that ultimately

$$ N_n = \Sigma\lambda^2\mu(n^2-3n+2mn) + 2\lambda\mu\nu\{2\Sigma lm - 3\Sigma l + 3\}. $$

6. I might proceed in the same manner to determine the order of the conditions that two binary equations may have three common solutions. The formula is, however, well known, and I only write it down for the sake of co venience, as it will be required presently. The order of the coefficients of the two equations being $\lambda$, $\mu$ respectively, and their degrees being $l$, $m$, we have (Salmon's "Lessons on Higher Algebra," p. 229)

$$ \frac{m(m-1)(m-2)}{2.3}\lambda^3 + \frac{l(l-1)(l-2)}{2.3}\mu^3 + \tfrac{1}{2}(m-1)(m-2)(l-2)\lambda^2\mu + \tfrac{1}{2}(l-1)(l-2)(m-2)\lambda\mu^2. $$

To obtain the order of the conditions that three ternary equations may have three common solutions, we set out with the following equality :—

$$ N_{n+1} = {}_{3}\begin{vmatrix} U \\ V \\ W.X \end{vmatrix} = {}_{3}\begin{vmatrix} U \\ V \\ W \end{vmatrix} + {}_{3}\begin{vmatrix} U \\ V \\ X \end{vmatrix} + {}_{2}\begin{vmatrix} U \\ V \\ W \end{vmatrix}\begin{vmatrix} U \\ \Lambda \\ X \end{vmatrix} + {}_{1}\begin{vmatrix} U \\ V \\ W \end{vmatrix}_{2}\begin{vmatrix} U \\ V \\ X \end{vmatrix} - {}^{1}_{2}\begin{vmatrix} W \\ \hline U \\ V \\ X \end{vmatrix} - {}^{1}_{2}\begin{vmatrix} X \\ \hline U \\ V \\ W \end{vmatrix} $$

or

$$ \Delta N_n = {}_{3}\begin{vmatrix} U \\ \Lambda \\ X \end{vmatrix} + {}_{2}\begin{vmatrix} U \\ V \\ W \end{vmatrix}_{1}\begin{vmatrix} U \\ V \\ X \end{vmatrix} + {}_{2}\begin{vmatrix} U \\ V \\ X \end{vmatrix}_{1}\begin{vmatrix} U \\ V \\ W \end{vmatrix} - {}^{1}_{2}\begin{vmatrix} W \\ \hline U \\ V \\ X \end{vmatrix} - {}^{1}_{2}\begin{vmatrix} X \\ \hline U \\ V \\ W \end{vmatrix} $$

I shall only require the case in which the coefficients of each equation are of the same order throughout.

We have for this case, using the preceding formula,

$$ \Delta N_n = \frac{ln(m-1)(m-2)}{2.3}\lambda^3 + \frac{l(l-1)(l-2)}{2.3}\mu^3 + \tfrac{1}{2}l(l-1)(m-2)\lambda\mu^2 + \tfrac{1}{2}m(m-1)(l-2)\lambda^2\mu $$

$$+(\lambda m+\mu l)\left\{\Sigma\frac{lm\,(lm-1)}{2}\nu^2+\Sigma\{(ln-1)(lm-1)-\tfrac{1}{2}(l-1)(l-2)\}\mu\nu\right\}$$
$$+\,(\lambda mn+\mu ln+\nu lm)\left(\frac{l(l-1)}{2}\mu^2+\frac{m\,(m-1)}{2}\lambda^2+(l-1)\,(m-1)\,\lambda\mu\right)$$
$$-\{\Sigma\mu\nu l\,(ln-1)+(3lmn-2\,\Sigma l+3)\,\lambda\mu\nu\}$$
$$-\{\nu\,[l^2-l)\mu^2+(m^2-m)\lambda^2+2\,(l-1)\,(m-1)\,\lambda\mu]$$
$$+n\lambda\mu\,[\mu\,(l-1)+\lambda\,(m-1)]\}.$$

The expression for $N_n$ must vanish for $l=m=n=1$, and must be made symmetrical.

Thus, for the coefficient of $\lambda^3$ in $N_n$, we have to integrate

$$\frac{m(m-1)(m-2)}{2\,.\,3}+\frac{m^3n^2-m^2n}{2}+\frac{m^3n-m^2n}{2},$$

and we get

$$\tfrac{1}{2}\left\{m^3\left(\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{2\,.\,3}\right)+(m^3-2m^2)\frac{n^2-n}{2}+\frac{m\,(m-1)\,(m-2)}{3}\,n\right\}$$
$$=\frac{1}{2\,.\,3}\,mn\,(mn-1)\,(mn-2).$$

In the same way the coefficients of $\lambda^2\mu$, &c. are obtained, and I find

$$N_n=\Sigma\,\frac{mn\,(mn-1)\,(mn-2)}{2\,.\,3}\,\lambda^3$$
$$+\,\Sigma\mu^2\nu\,\{(lm-2)\,(ln-1)\,(ln-2)-(ln-2)\,(l-1)\,(l-2)\}$$
$$+\,\frac{\lambda\mu\nu}{2}\left\{2\,l^2m^2n^2-5lmn\,\Sigma l+2\,\Sigma l^2+8\,\Sigma lm-12\,\Sigma l+9lmn+10\right\}$$

In a similar manner the value of $\Delta N_n$ can be reduced to known forms when there are $k$ equations homogeneous in $k$ variables. The form is

$$\Delta N_n={}_3\left|\begin{matrix}U_1\\U_2\\ \vdots\\U_{k-1}\\X\end{matrix}\right|+{}_2\left|\begin{matrix}U_1\\U_2\\ \vdots\\U_k\end{matrix}\right|\,{}_1\left|\begin{matrix}U\\U_2\\ \vdots\\U_{k-1}\\X\end{matrix}\right|+{}_2\left|\begin{matrix}U_1\\U_2\\ \vdots\\U_{k-1}\\X\end{matrix}\right|\,{}_1\left|\begin{matrix}U_1\\U_2\\ \vdots\\U_k\end{matrix}\right|-{}^1_2\left|\begin{matrix}\underline{U_k}\\U_1\\ \vdots\\U_{k-1}\\X\end{matrix}\right|-{}^1_2\left|\begin{matrix}\underline{X}\\U_1\\U_2\\ \vdots\\U_k\end{matrix}\right|$$

The determination of the order is thus reduced to the case of $k-1$ equations in $k-1$ variables and other lower forms.

My intention here is to indicate a method, and show how it may prove useful. I have, therefore, only taken a few cases which are applicable to the geometrical problems I now proceed to consider.

## *Geometrical Applications.*

7. In plane space the ordinary singularities of a curve ultimately depend on two independent conditions in the coordinates. For two

such conditions determine punctually or tangentially a definite number of singular elements. On the other hand, in space of three dimensions we have to take into consideration systems of three independent conditions which determine groups of singular elements, points, or planes, as well as systems of two conditions determining singular curves on the surface, or developables generated by singular tangent planes.

Thus, if we have two equations,

$$\left.\begin{array}{l} U = a\,t^{m} + b\,t^{m-1} + c\,t^{m-2} + \&c. = 0 \\ V = a'\,t^{m'} + b'\,t^{m'-1} + c'\,t^{m'-2} + \&c. = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (A),$$

$a$, $b$, &c. being homogeneous functions of the order $\mu$ in the plane coordinates $x$, $y$, $z$, and $a'$, $b'$, &c. being similar functions of the order $\mu'$, the resultant with regard to $t$ equated to zero represents a curve of the order $\mu m' + \mu' m$.

The conditions in $x$, $y$, $z$ that the system may have two common solutions give the number of double points on the curve, viz. :—

$$\frac{m^2-m}{2}\mu'^2 + \frac{m'^2-m'}{2}\mu^2 + (m-1)(m'-1)\,\mu\mu' \quad \ldots\ldots\ldots\ldots\ldots (a).$$

But if we consider $a$, $b$, &c., $a'$, $b'$, &c., as containing four homogeneous coordinates in the same orders as before, we get, in the same way, a surface whose order is $\mu m' + \mu' m$, and whose double curve is of the order $(a)$.

We must now, however, proceed further, and consider singularities of isolated elements. In the first place, the three conditions that (A) may have three common solutions determine a certain number of triple points on the nodal curve in number

$$\frac{m(m-1)(m-2)}{2\,.\,3}\mu'^3 + \frac{m'(m'-1)(m'-2)}{2\,.\,3}\mu^3$$
$$+\tfrac{1}{2}(m-1)(m-2)(m'-2)\,\mu'^2\mu + \tfrac{1}{2}(m'-1)(m'-2)(m-2)\,\mu^2\mu'.$$

In the next place, three conditions are to be fulfilled if (A) have in common pairs of equal roots. The order of these conditions is easily seen to be

$$2\mu\mu'\left[\mu(m'-1) + \mu'(m-1)\right],$$

and this is the number of cuspidal points of the nodal curve. The remaining singularities can now be determined by means of the known general formulæ.

8. In like manner, if the given equations are three in number, homogeneous in the parameters $\alpha$, $\beta$, $\gamma$, and in the coordinates $x$, $y$, $z$, $w$, and the orders with respect to the parameters are $m$, $m'$, $m''$ respectively, and with regard to the coordinates $\mu$, $\mu'$, $\mu''$, the order of the surface represented by the resultant equated to zero is $\Sigma\mu m m'$. There is a nodal curve whose order, by the formulæ of my paper on the Plückerian

characteristics, &c. ("Quarterly Journal of Mathematics, Vol. xii., p. 282*) is

$$\tfrac{1}{2}\{\Sigma\mu mm'(\Sigma\mu mm'-\Sigma\mu)-(\Sigma m-3)\Sigma\mu\mu'm''\}.$$

Further, we have to consider the three conditions that the given system may have three common solutions. We can at once apply the result of § 6 to this case, and thus have the number of triple points on the surface and the nodal curve.

But, again, we have to consider another case involving three conditions in the coordinates.

It will be observed that the three given equations may have in common two coincident solutions, or, as it may be expressed, the three curves in $\alpha$, $\beta$, $\gamma$ may have a common point of contact. There are thus a certain number of cuspidal points on the nodal curve, and we have for that number, by § 5,

$$\Sigma\mu^2\mu'(m''^2-3n''+2m'm'')+2\mu\mu'\mu''(2\Sigma mm'-3\Sigma m+3).$$

If there are more than three equations and more than three homogeneous parameters, and the number of equations is equal to the number of parameters, and if the coefficients contain four homogeneous space coordinates, we get, in like manner, a surface represented by the resultant equated to zero. Also, the order of the nodal curve and the number of triple points are determined by the formulæ of the paper before referred to, combined with the formula expressing the order of the conditions that such a system may have three common solutions. I have shown that the expression of this formula is reduced to the difficulty of somewhat lengthy calculations.

When, however, we consider whether there are cuspidal points in general on the nodal curve, it is seen that these will not exist. For the condition to be fulfilled is that there are to be two common coincident solutions of the system, and when the number of equations and parameters is greater than three, more than three conditions in the coordinates must be satisfied.

9. We will now consider the case of envelopes where there is one equation given containing several parameters.

If $U=0$ is the equation containing homogeneously the parameters $a_1, a_2, \ldots a_k$ and the space coordinates $x, y, z, w$, the envelope we are concerned with is obtained by eliminating the parameters from

$$\frac{dU}{da_1}=0,\quad \frac{dU}{da_2}=0 \;\ldots\ldots\ldots \frac{dU}{da_k}=0,$$

and equating the result to zero.

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* In that paper the applications refer to plane space. The formulæ are, however, applicable, as far as they extend, to space of three dimensions, and I am, therefore, obliged to refer to them several times.

The formulæ of my paper before mentioned (Quarterly Journal, vol. xii., pp. 289, 293, 297) give, therefore, the order of the surface, of its nodal and cuspidal curves. The points $j$ are absorbed in the cuspidal curve. Moreover, when we seek to determine the number of triple points we find included amongst them both the points $\beta$, intersections of the nodal and cuspidal curves, stationary points on the cuspidal curve, and the points $\gamma$, intersections of the nodal and cuspidal curves stationary points on the nodal curve. That is to say, we get $\beta+\gamma+t$ instead of $t$, as in the case of simple resultants.

Bearing in mind these considerations, we can now treat the envelope of the planes

$$U = A\alpha^m + B\beta^m + C\gamma^m + \&c. = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (B),$$

homogeneous and of the order $m$ in the parameters $\alpha$, $\beta$, $\gamma$, and as to the coefficients A, B, C, &c., of the first order in $x$, $y$, $z$, $w$. This envelope is referred to in Dr. Salmon's "Geometry of Three Dimensions," 3rd ed., p. 537, where he has determined the characteristics in the case of $m=3$.

The order of the surface is the order of the system

$$\frac{dU}{d\alpha} = 0, \quad \frac{dU}{d\beta} = 0, \quad \frac{dU}{d\gamma} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (B'),$$

and is therefore $3(m-1)^2$. The order of the nodal curve is the order of the conditions that the system (B′) may have two distinct solutions in common, and is $\frac{(m-1)(m-2)(9m^2-9m-33)}{2}$. Similarly, the order of the cuspidal curve is $12(m-1)(m-2)$.

Again, putting $\lambda=\mu=\nu=1$ and $(m-1)$ for $l$, $m$, $n$ in the formula of § 6, we get for $\beta+\gamma+t$

$$\tfrac{1}{2}\{9(m-1)^6-54(m-1)^4+27(m-1)^3+80(m-1)^2-72(m-1)+10\}.$$

For the order of the tangent cone, we condition (B) and

$$U' = A'\alpha^m + B'\beta^m + C'\gamma^m + \&c. = 0$$

(where the accents denote that $x$, $y$, $z$, $w$ are replaced by $x'$, $y'$, $z'$, $w'$) to have two coincident common solutions. The order is $3m(m-1)$. The class of the surface is evidently $m^2$.

The number of cuspidal edges of the tangent cone is the order of the conditions that $U=0$ and $U'=0$, considered as representing curves in $\alpha$, $\beta$, $\gamma$ coordinates, may have three-point contact. We have, then, to make $\mu'=0$, $m'=m$, $\epsilon=\eta=0$, in the formulæ given at p. 293 and at p. 297 (Quarterly Journal, vol xii.), by which means we get

$$\delta+\kappa = \tfrac{1}{2}(9m^4-18m^3-m^2+12m),$$
$$\kappa = 6m^2-9m.$$

These values, again, give $m^2$ for the class. They also show that the reciprocal surface has no cuspidal curve.

We can also determine $\sigma$, the number of intersections of the curve of contact of the tangent cone with the cuspidal curve. For this purpose we must combine the equation $U'=0$ with the system

$$\frac{d^2U}{d\alpha^2}\cdot\frac{d^2U}{d\beta^2}-\left(\frac{d^2U}{d\alpha\,d\beta}\right)^2=0,\ \frac{dU}{d\alpha}=0,\ \frac{dU}{d\beta}=0,\ \frac{dU}{d\gamma}=0,$$

and make the same reduction for a double solution satisfying $\gamma=0$ as at p. 285 of the paper before cited. Thus the order sought is the order of a system

$$\left|\begin{matrix}2, & 2m-4\\ 1, & m-1\\ 1, & m-1\\ 1, & m-1\\ 0, & m\end{matrix}\right| \text{ less twice the order of a system } \left|\begin{matrix}0, & 1\\ 1, & m-1\\ 1, & m-1\\ 1, & m-1\\ 0, & m\end{matrix}\right|$$

that is to say,

$$\sigma=2m(m-2)+6m(m-1)-2m=4m(2m-3).$$

The value for $\beta+\gamma+t$ is

$$\tfrac{1}{2}(9m^6-54m^5+81m^4+63m^3-190m^2+11m+90),$$

and we also have, by the general formulæ for reciprocal surfaces (Salmon's "Geometry of Three Dimensions," p. 543),

$$\begin{aligned}2\beta+3\gamma+3t &= b(m-2)-\rho,\\ 4\beta+\gamma &= c(m-2)-2\sigma.\end{aligned}$$

From this system we get the separate numbers $\beta$, $\gamma$, $t$. Also $j$ is found to be zero. The numbers $h$, $k$, $q$, $r$ can now be determined. It remains to consider the tangential singularities. We have already the class, the class of the plane section, and there is no cuspidal curve on the reciprocal. We get, therefore,

$$\kappa'=3a-3(m)+c,\ \text{ and }\ 2\delta'=a(a-1)-(n)-3\kappa,$$

where $(m)$, $(n)$ are written for the order and class of the surface in the general formulæ. The rest of the required numbers follow in like manner from those formulæ, and the results may be collected as follows:—

| | |
|---|---|
| Order of surface.......................... | $3(m-1)^2$ |
| Order of tangent cone .................... | $3m(m-1)$ |
| Number of its double edges............ | $\frac{1}{2}(9m^4-18m^3-13m^2+30m)$ |
| " " cuspidal edges ......... | $3m(2m-3)$ |
| Class of nodal torse ...................... | $9m^4-27m^3-m^2+30m$ |
| " cuspidal torse .................. | $4m(2m-3)$ |
| Order of nodal curve...................... | $\frac{m-1\,.\,m-2}{2}(9m^2-9m-33)$ |

Number of its pinch points ............ $0$

Order of cuspidal curve.................. $12\,(m-1)\,(m-2)$

Number of intersections of nodal and cuspidal curves, stationary points on cuspidal curve ..................... $50m^2-192m+168$

Number of intersections of nodal and cuspidal curves, stationary points on nodal curve ......................... $12\,(m-3)\,(3m^3-6m^2-11m+18)$

Class of surface ............................ $m^2$

Class of plane section .................. $3m\,(m-1)$

Number of its double tangents......... $\frac{1}{2}\,(m-1)\,(9m^3-9m^2-42m+48)$

" " inflexions ............... $3\,(m-1)\,(4m-5)$

Order of node couple curve ........... $3\,(m-1)^2\,(m^2+m-5)$

Class of node couple torse.............. $\frac{m\,(m-1)}{2}\,(m^2+m-3)$

Number of its pinch planes ........... $6\,(m-1)^2$.

11. In a similar way we can obtain most of the characteristics of the surface represented by equating to zero the result of eliminating $k$ homogeneously entering parameters $a_1$, $a_2$ ...... $a_k$ from the system

$$\frac{dU}{da_1}=0,\ \frac{dU}{da_2}=0\ \ldots\ldots\ \frac{dU}{da_k}=0,$$

where U is linear in $x$, $y$, $z$, $w$.

If the order of U in the parameters is $m$, we get

Order of surface ................................ $k\,(m-1)^{k-1}$

Order of tangent cone ........................... $\frac{k\,.\,k-1}{2}\,m\,(m-1)^{k-2}$

Class of cuspidal torse ... $\frac{k\,.\,k-1}{2\,.\,3}\,m\,(m-1)^{k-3}\,\{(k^2-1)\,m+k+3-2k^2\}$

Order of nodal curve ...

$$\tfrac{1}{2}k\,(m-1)^{k-2}\left(k(m-1)^k-\frac{3k^2+k-2}{2}(m-1)+\frac{3k^2-k-2}{2}\right)$$

Number of pinch points ................................ $0$

Order of cuspidal curve ........................ $\frac{k\,.\,k^2-1}{2}\,(m-1)^{k-2}\,(m-2)$

Class of surface .................................. $\frac{k\,.\,k-1\,.\,k-2}{1\,.\,2\,.\,3}\,m^2\,(m-1)^{k-3}$.

From these can be determined $\delta'$, $\kappa'$, $\delta$, $\kappa$, $c'$, $b$. To show how the above numbers may be established, I consider the class. The class is the order of the conditions for the coexistence of

$$\frac{dU}{da_1}=0\ \ldots\ldots\ \frac{dU}{da_k}=0,\quad U'=0,\ U''=0,$$

where the accents denote that $x$, $y$, $z$, $w$ are replaced by the same letters accented like the functions. The order of the above system, by the general formula for a system having one common solution, is the number above given.

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## *On some proposed forms of Slide-Rule. By* G. H. Darwin.

[*Read March 11th,* 1875.]

The object of the author was to devise a form of slide-rule which should be small enough for the pocket, and yet be a powerful instrument.

The first proposed form was to have a pair of watch-spring tapes graduated logarithmically, and coiled on spring bobbins side by side. There was to be an arrangement for clipping the tapes together and unwinding them simultaneously. Two modifications of this idea were given.

The second form was explained as the logarithmic graduation of several coils of a helix engraved on a brass cylinder. On the brass cylinder was to fit a glass one, similarly graduated.

To avoid the parallax due to the elevation of the glass above the other scale, the author proposed that the glass cylinder might be replaced by a metal corkscrew sliding in a deep worm, by which means the two scales might be brought flush with one another.

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## *April 8th,* 1875.

### Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Mr. J. H. Röhrs, M.A., was elected a Member, and Messrs. Nanson and Ritchie were admitted into the Society.

Mr. G. H. Darwin gave an account of two applications of Peaucellier's Cells: first, to "the Mechanical Description of Equipotential Lines"; and, secondly, to "a Mechanical Method of making a Force which varies inversely as the Square of the Distance from a Fixed Point." In this latter case, let O be the fixed pivot of a cell, and suppose the cell to be in equilibrium under the action of two forces P and P′ acting at B and D.