

The Radical Chamber Lift abc, Valuation Excess, and Additive Closure

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Abstract

This expository note organizes the classical theory surrounding the abc conjecture around two arithmetic measurements of a coprime triple $a + b = c$. The radical $\text{rad}(n) = \prod_{p|n} p$ is the square-free skeleton of n , while the valuation excess

$$E(n) = \log n - \log \text{rad}(n) = \sum_{p|n} (v_p(n) - 1) \log p$$

records the prime-power mass discarded by the radical. In these terms the quality

$$q(a, b, c) = \frac{\log c}{\log \text{rad}(abc)}$$

compares additive height against multiplicative footprint, and the threshold surplus

$$\Delta_\varepsilon = \log c - (1 + \varepsilon) \log \text{rad}(abc)$$

satisfies $\Delta_\varepsilon > 0$ if and only if $q > 1 + \varepsilon$. The abc conjecture of Masser and Oesterle asserts that, for every $\varepsilon > 0$, only finitely many coprime triples satisfy $q > 1 + \varepsilon$. We record the elementary prime-allocation lemma for coprime sums, the proven function-field analogue of Stothers and Mason, the relation with modified Szpiro, and a bounded computational grid that visualizes the radical chamber. No proof of abc is claimed.

Nonclaim 1 (Scope). This paper is an atlas, not a proof. The terms chamber lift, radical footprint, square-free skeleton, valuation excess, and additive closure are organizational vocabulary for known arithmetic objects. The abc conjecture remains open in the mainstream mathematical literature. We do not endorse Inter-universal Teichmüller theory as an accepted proof, and we do not claim that the Chamber Lift framework supplies a proof method. The finite computation in Section 9 is a visualization only.

1 Introduction

The abc conjecture is usually stated with three positive integers

$$a + b = c,$$

subject to a coprimality condition, and with the radical

$$\text{rad}(n) = \prod_{p|n} p,$$

the product of the distinct prime divisors of n . The conjecture says that c cannot be much larger than the radical footprint of the whole product abc except for finitely many triples above any fixed threshold.

This makes abc a natural continuation of the preceding Chamber Lift papers. The primitive-ray and dimensional-visibility notes separated common-divisor stripping from primitive residue structure. The Zeta-Exponent note distinguished coordinate obstruction exponents, valuation-depth exponents, and averaging-frequency exponents. The present paper moves to a global additive equation. The radical remembers which primes occur. The valuation excess remembers how deeply those primes are stacked. The equation $a + b = c$ then forces the multiplicative ledger into a rigid additive closure.

The core thesis is therefore:

abc compares additive height with square-free multiplicative footprint.

The paper makes this comparison precise through $q(a, b, c)$, $E(n)$, and Δ_ε .

2 Radical, quality, and valuation excess

Definition 1 (Radical). For $n \geq 1$, define

$$\text{rad}(n) = \prod_{p|n} p,$$

with the empty product convention $\text{rad}(1) = 1$. Thus $\text{rad}(n)$ discards all repeated prime powers and keeps only the square-free prime support of n .

Definition 2 (abc quality). For a positive triple $a + b = c$ with $\text{rad}(abc) > 1$, the quality is

$$q(a, b, c) = \frac{\log c}{\log \text{rad}(abc)}.$$

The base of the logarithm is irrelevant.

A triple has $q > 1$ precisely when

$$c > \text{rad}(abc).$$

Such triples are called abc triples in de Smit's tables [4].

Definition 3 (Valuation excess). For $n \geq 1$, define

$$E(n) = \log n - \log \text{rad}(n).$$

Equivalently, if $n = \prod_p p^{v_p(n)}$, then

$$E(n) = \sum_{p|n} (v_p(n) - 1) \log p.$$

Proposition 1 (The radical gap). For every $n \geq 1$,

$$E(n) \geq 0,$$

with equality if and only if n is square-free. If $\gcd(m, n) = 1$, then

$$E(mn) = E(m) + E(n).$$

Proof. The identities

$$\log n = \sum_{p|n} v_p(n) \log p, \quad \log \operatorname{rad}(n) = \sum_{p|n} \log p$$

give the stated formula after subtraction. Each coefficient $v_p(n) - 1$ is nonnegative, and all are zero precisely when no prime exponent exceeds one. Additivity for coprime m, n follows from disjoint prime support. \square

In Chamber Lift language, $\operatorname{rad}(n)$ is the footprint and $E(n)$ is the hidden valuation depth above that footprint. This is the same valuation-depth ledger that appears in the k -free decomposition

$$n = ms^k, \quad m \text{ is } k\text{-free,}$$

but here it is read logarithmically rather than as a density.

3 Prime allocation under additive closure

The abc conjecture is usually stated for coprime triples. In the equation $a + b = c$, this has a simple but important consequence.

Lemma 1 (Pairwise coprimality). *Let a, b, c be positive integers satisfying $a + b = c$. If $\gcd(a, b) = 1$, then a, b, c are pairwise coprime. Equivalently, for such triples, the condition $\gcd(a, b, c) = 1$ is the same as pairwise coprimality.*

Proof. The pair a, b is coprime by hypothesis. If a prime p divides both a and c , then p divides $c - a = b$, contradicting $\gcd(a, b) = 1$. Hence $\gcd(a, c) = 1$. Similarly, if p divides both b and c , then p divides $c - b = a$, again a contradiction. Thus the three pairs are coprime. \square

Corollary 1 (Disjoint radical branches). *For a coprime triple $a + b = c$,*

$$\operatorname{rad}(abc) = \operatorname{rad}(a) \operatorname{rad}(b) \operatorname{rad}(c)$$

and

$$\log \operatorname{rad}(abc) = \log \operatorname{rad}(a) + \log \operatorname{rad}(b) + \log \operatorname{rad}(c).$$

Proof. The preceding lemma says the prime supports of a, b, c are disjoint. The radical is multiplicative on coprime arguments. \square

Thus a prime belongs to at most one branch of the triple. This is the precise sense in which a coprime abc triple has a three-branch prime allocation. The phrase is descriptive, not a mechanism.

4 The abc conjecture

Theorem 1 (abc conjecture, Masser-Oesterle). *For every $\varepsilon > 0$, there are only finitely many coprime triples of positive integers $a + b = c$ such that*

$$c > \operatorname{rad}(abc)^{1+\varepsilon}.$$

This is a conjecture, not a theorem proved here. An equivalent constant form says that for every $\varepsilon > 0$ there is a constant K_ε such that every coprime triple $a + b = c$ satisfies

$$c \leq K_\varepsilon \operatorname{rad}(abc)^{1+\varepsilon}.$$

Definition 4 (Threshold surplus). For a coprime triple and $\varepsilon \geq 0$, define

$$\Delta_\varepsilon(a, b, c) = \log c - (1 + \varepsilon) \log \operatorname{rad}(abc).$$

Proposition 2 (Quality and surplus). For every coprime triple $a + b = c$ with $\operatorname{rad}(abc) > 1$,

$$\Delta_\varepsilon(a, b, c) = \log \operatorname{rad}(abc)(q(a, b, c) - (1 + \varepsilon)).$$

Consequently,

$$\Delta_\varepsilon > 0 \iff q(a, b, c) > 1 + \varepsilon.$$

Proof. Substitute the definition of q :

$$\log c = q(a, b, c) \log \operatorname{rad}(abc).$$

Then subtract $(1 + \varepsilon) \log \operatorname{rad}(abc)$. □

The conjecture can therefore be read as follows: for every fixed positive threshold ε , only finitely many coprime triples have positive surplus Δ_ε . This is the paper's central gauge.

Remark 1 (The essential role of $\varepsilon > 0$). The statement is not true with $\varepsilon = 0$. There are infinitely many triples with $q > 1$, although abc predicts that no fixed margin above 1 persists infinitely often. Thus one should never say that abc predicts $\sup q = 1$. Known individual triples exceed 1. The correct asymptotic language is the finite-exception statement above.

5 Valuation excess under a coprime sum

For a coprime triple $a + b = c$, the valuation excess of the whole product decomposes as

$$E(abc) = E(a) + E(b) + E(c),$$

because the three prime supports are disjoint. Expanding gives

$$E(abc) = \sum_{p|abc} (v_p(abc) - 1) \log p.$$

The quality, however, is not determined by $E(abc)$ alone. It compares $\log c$ with $\log \operatorname{rad}(abc)$:

$$q(a, b, c) = \frac{\log c}{\log \operatorname{rad}(abc)}.$$

Thus the radical-compression phrase should be read carefully. High quality means that the additive height $\log c$ is large relative to the square-free footprint $\log \operatorname{rad}(abc)$, not merely that the product abc has large repeated exponents.

A safe chamber sentence is:

abc bounds persistent additive height surplus over the radical footprint.

A less safe sentence is: “valuation excess explains abc .” It does not. The equation $a + b = c$ is the global constraint, and the radical is the multiplicative footprint being compared with c .

6 The function-field contrast: Mason-Stothers

The closest proven analogue of abc is the polynomial theorem of Stothers and Mason.

Theorem 2 (Mason-Stothers, cited). *Let $A(t), B(t), C(t)$ be relatively prime polynomials over a field, satisfying*

$$A + B = C,$$

and suppose they do not all have vanishing derivative. Then

$$\max\{\deg A, \deg B, \deg C\} \leq \deg \text{rad}(ABC) - 1.$$

Here $\text{rad}(ABC)$ denotes the product of the distinct irreducible factors of ABC .

This theorem was proved by Stothers [5] and independently by Mason [6]. The elementary proof uses the Wronskian

$$W(A, B) = AB' - A'B.$$

The key multiplicity-stripping fact is that

$$\frac{F}{\text{rad}(F)} \mid F'$$

for a polynomial F in characteristic zero, because differentiation lowers the multiplicity of a repeated root by one. This derivative mechanism has no direct integer analogue. The polynomial theorem therefore motivates abc but does not prove it.

7 The Szpiro relation and the IUT status

The abc conjecture arose from work of Masser and Oesterle related to Szpiro's conjecture on elliptic curves [1, 2]. In one standard form, the modified Szpiro conjecture asserts that for every $\varepsilon > 0$ there is a constant C_ε such that every elliptic curve over \mathbb{Q} satisfies

$$\max\{|c_4|^3, c_6^2\} \leq C_\varepsilon N^{6+\varepsilon},$$

where c_4, c_6 are the usual invariants and N is the conductor. This modified form is equivalent to abc up to ineffective constants, through the work surrounding Oesterle, Nitaj, and the Frey-Hellegouarch construction.

Mochizuki released four papers developing Inter-universal Teichmüller theory and claiming a proof of abc; these papers were published in *Publications of the Research Institute for Mathematical Sciences* in 2021 [12]. The claimed proof has not been accepted by the broader mathematical community as settling abc. Scholze and Stix circulated a critique focused on Corollary 3.12 of the third IUT paper [13], and Mochizuki disputed their assessment. This paper treats abc as open and uses none of IUT.

8 Examples and high-quality triples

The following table gives several illustrative triples. The first four appear in the finite grid used below. The last line is the highest-quality triple listed in de Smit's high-quality table, due to Eric Reyssat [4].

Table 1: Selected abc triples and qualities. The record line is cited, not rediscovered by the grid.

a	b	c	$\text{rad}(abc)$	q
1	8	9	6	1.22629
3	125	128	30	1.42657
1	2400	2401	210	1.45567
1	4374	4375	210	1.56789
2	$3^{10} \cdot 109$	23^5	15042	1.62991

The record should be stated empirically: it is the highest quality currently listed in the standard table, not a proven upper bound. The abc conjecture predicts finite exceptions above each threshold $1 + \varepsilon$, not a particular numerical ceiling for the largest known examples.

9 A bounded computational grid

To visualize the radical chamber, we enumerated coprime triples

$$a + b = c, \quad c \leq 20000, \quad 1 \leq a \leq b, \quad \gcd(a, b) = 1.$$

This gave 60,795,197 coprime triples. For each one, we computed $\text{rad}(abc)$, q , $E(abc)$, and $\Delta_{0.1}$. The top-quality triple in this grid was

$$(1, 4374, 4375),$$

with

$$\text{rad}(abc) = 210, \quad q \approx 1.56789.$$

In the same grid, the counts above selected thresholds were:

$$\#\{q > 1\} = 204, \quad \#\{q > 1.1\} = 82, \quad \#\{q > 1.3\} = 8.$$

As proportions of the whole grid, these are approximately

$$0.000336\%, \quad 0.000135\%, \quad 0.000013\%.$$

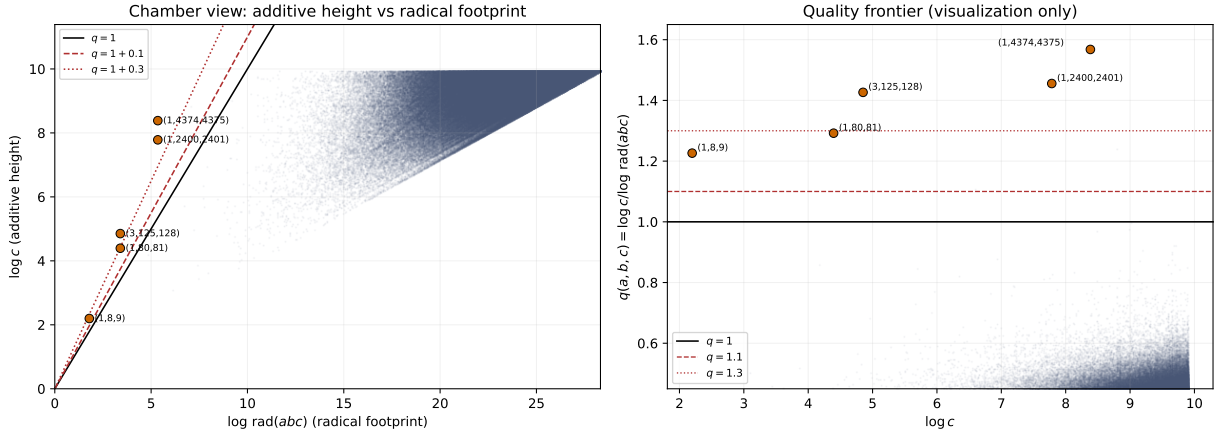


Figure 1: Coprime triples $a + b = c$ with $c \leq 20000$, plotted as radical footprint $\log \text{rad}(abc)$ against additive height $\log c$ and as quality against $\log c$. The solid line is $q = 1$; the dashed and dotted lines mark $q = 1.1$ and $q = 1.3$. The threshold lines are steeper than the $q = 1$ diagonal because $q = 1 + \varepsilon$ means $\log c = (1 + \varepsilon) \log \text{rad}(abc)$. The figure is a visualization only; it is not evidence for abc.

Remark 2 (Computation is not confirmation). A finite census cannot support a universal finiteness conjecture. The grid is useful because it makes the definitions visible: high-quality triples are sparse in this bounded sample, and they occupy the region above the $q = 1$ line. It is not a proof, and it cannot detect the asymptotic behavior asserted by abc.

10 Relation to square-free and k -free ledgers

The radical is the square-free skeleton of an integer. It is related to, but distinct from, the k -free decomposition. For $k \geq 2$, every positive integer can be written uniquely as

$$n = ms^k$$

where m is k -free after taking residues of all valuations modulo k . The density of k -free integers is

$$\prod_p (1 - p^{-k}) = \frac{1}{\zeta(k)}.$$

In the present paper, however, no Euler product proves abc. The radical and valuation excess supply descriptive coordinates, but the equation $a + b = c$ is a global additive condition. This is the important break from the earlier local-density chambers.

11 Consequences and limitations

The abc conjecture is famous partly because many deep results would follow from it. A few representative consequences are:

- Asymptotic Fermat-type results follow by comparing $x^n + y^n = z^n$ with the radical of xyz ; this is a direct elementary consequence in standard surveys [3].

- The Fermat-Catalan conjecture follows from abc by more technical Diophantine arguments; this is a standard conditional consequence.
- Roth-type results follow from abc through work of Bombieri and van Frankenhuysen [10].
- Elkies showed that abc implies an effective form of Mordell’s conjecture [8]. Mordell itself is unconditionally known through Faltings’ theorem.
- Silverman showed that abc implies infinitude results for non-Wieferich primes in fixed bases [9].

Catalan’s conjecture should not be listed as an abc consequence in the same sense. It was proved unconditionally by Mihailescu [11]. abc can imply broader finiteness statements for differences of perfect powers, but the theorem itself is already established.

12 Conclusion

The Radical Chamber Lift isolates three quantities:

$$\text{rad}(abc), \quad E(abc), \quad \Delta_\varepsilon.$$

Together they express the abc conjecture as a comparison between additive height and square-free multiplicative footprint. The pairwise-coprime lemma makes the prime support split cleanly across the three branches of a coprime sum. Mason-Stothers shows that a closely related function-field statement is true because differentiation strips multiplicity. But no such derivative exists for integers, and the Chamber Lift language does not fill that gap.

The result is an atlas of the problem rather than a tunnel through it. It shows where the pressure sits: additive closure asks three disjoint prime branches to meet exactly, while the radical records only the square-free footprint left behind.

References

- [1] D. W. Masser, “Open problems,” in *Proceedings of the Symposium on Analytic Number Theory*, W. W. L. Chen, ed., Imperial College, London, 1985.
- [2] J. Oesterle, “Nouvelles approches du theoreme de Fermat,” *Seminaire Bourbaki*, exp. 694, *Asterisque* 161-162 (1988), 165-186.
- [3] A. Granville and T. J. Tucker, “It’s as easy as abc,” *Notices of the AMS* 49 (2002), no. 10, 1224-1231.
- [4] B. de Smit, “ABC triples,” Mathematisch Instituut, Universiteit Leiden. Available at <https://pub.math.leidenuniv.nl/~smitbde/abc/>.
- [5] W. W. Stothers, “Polynomial identities and hauptmoduln,” *Quarterly Journal of Mathematics* 32 (1981), 349-370.
- [6] R. C. Mason, *Diophantine Equations over Function Fields*, London Mathematical Society Lecture Note Series 96, Cambridge University Press, 1984.

- [7] N. Snyder, “An alternate proof of Mason’s theorem,” *Elemente der Mathematik* 55 (2000), 93-94.
- [8] N. D. Elkies, “ABC implies Mordell,” *International Mathematics Research Notices* 1991, no. 7, 99-109.
- [9] J. H. Silverman, “Wieferich’s criterion and the abc-conjecture,” *Journal of Number Theory* 30 (1988), no. 2, 226-237.
- [10] M. van Frankenhuysen, “The abc conjecture implies Roth’s theorem and Mordell’s conjecture,” *Matematica Contemporanea* 16 (1999), 45-72.
- [11] P. Mihalescu, “Primary cyclotomic units and a proof of Catalan’s conjecture,” *Journal fuer die reine und angewandte Mathematik* 572 (2004), 167-195.
- [12] S. Mochizuki, “Inter-universal Teichmuller Theory I-IV,” *Publications of the Research Institute for Mathematical Sciences* 57 (2021), no. 1/2.
- [13] P. Scholze and J. Stix, “Why abc is still a conjecture,” manuscript, 2018.