

The Primitive Ray Chamber Lift: Lattice Visibility via Tolled Conformal Scale

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Abstract

We give a red-team-safe coordinate atlas for primitive lattice visibility. Every nonzero lattice point

$$\mathbf{p} = (x, y) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$$

decomposes uniquely as

$$\mathbf{p} = g\mathbf{q}, \quad g = \gcd(|x|, |y|), \quad \gcd(|q_x|, |q_y|) = 1.$$

The primitive point \mathbf{q} is visible from the origin, while the integer g records the radial obstruction layer. In complex notation,

$$z_p = gz_q = e^u z_q, \quad u = \log g.$$

Multiplication by g is a conformal dilation of the plane: it preserves angle and scales length by g . Thus $\log g$ is treated as a discrete additive scale-coordinate, not as new smooth conformal geometry.

We define a tolled radial length functional

$$L_{\text{tolled}}(\mathbf{p}) = r(\mathbf{q}) + \lambda \log g,$$

where $r(\mathbf{q}) = \|\mathbf{q}\|_2$ and $\lambda > 0$ is a free exchange-rate parameter converting logarithmic scale-cost into length-units. The central object is not a universal identity but a break-even locus:

$$(g - 1)r(\mathbf{q}) = \lambda \log g, \quad r^*(g) = \frac{\lambda \log g}{g - 1}.$$

Equivalently, in the additive scale coordinate $u = \log g$,

$$r^*(u) = \frac{\lambda u}{e^u - 1}.$$

We also record the classical density bridge

$$\prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

which describes both visible lattice points and square-free integers. The square-free analogue is written in the form

$$n = ms^2, \quad m \text{ square-free}, \quad \log n = \log m + 2 \log s.$$

This note proposes an atlas and bookkeeping framework only. It does not claim a new theorem, a new conformal geometry, or a two-point metric without an explicit path-metric closure.

1 Purpose and nonclaim

This note packages a small but coherent structure:

primitive ray + integer dilation + logarithmic scale-cost
+break-even locus + $\zeta(2)$ density + square-free mirror.

The purpose is descriptive. The construction is intended as a coordinate atlas for known arithmetic geometry around primitive lattice points and square-free obstruction.

Nonclaim 1 (Scope). *No claim is made that the tolled radial length is a new metric on \mathbb{Z}^2 , that the break-even locus proves any visibility theorem, or that the conformal-scale wording defines new conformal geometry. The classical results used here are the primitive lattice decomposition and the classical densities of visible lattice points and square-free integers. The new content is the organization of these ingredients into a single descriptive coordinate frame.*

2 Primitive rays and lattice visibility

Let

$$X = \mathbb{Z}^2 \setminus \{(0, 0)\}.$$

For

$$\mathbf{p} = (x, y) \in X,$$

define

$$g(\mathbf{p}) = \gcd(|x|, |y|).$$

When the meaning is clear, write simply g .

Definition 1 (Primitive visible point). *A point $\mathbf{q} = (a, b) \in X$ is called primitive, or visible from the origin, when*

$$\gcd(|a|, |b|) = 1.$$

Lemma 1 (Primitive ray decomposition). *Every point $\mathbf{p} = (x, y) \in X$ has a unique decomposition*

$$\mathbf{p} = g\mathbf{q},$$

where

$$g = \gcd(|x|, |y|) \geq 1$$

and \mathbf{q} is primitive.

Proof. Set

$$\mathbf{q} = \frac{1}{g}\mathbf{p} = \left(\frac{x}{g}, \frac{y}{g}\right).$$

Then $\mathbf{q} \in \mathbb{Z}^2$, and by construction its coordinates are coprime. Conversely, if $\mathbf{p} = h\mathbf{q}'$ with \mathbf{q}' primitive, then

$$\gcd(|x|, |y|) = \gcd(|hq'_x|, |hq'_y|) = h \gcd(|q'_x|, |q'_y|) = h.$$

Thus $h = g$, and consequently $\mathbf{q}' = \mathbf{q}$. Hence the decomposition is unique. \square

The Euclidean radial length

$$r(\mathbf{p}) = \|\mathbf{p}\|_2 = \sqrt{x^2 + y^2}$$

is homogeneous, so

$$r(\mathbf{p}) = g r(\mathbf{q}).$$

The radial distance lying behind the primitive visible point is therefore

$$r(\mathbf{p}) - r(\mathbf{q}) = (g - 1)r(\mathbf{q}).$$

This is the first chamber wall: geometry already knows the obstruction multiplier.

3 Discrete conformal scale

Identify \mathbb{R}^2 with \mathbb{C} by

$$z_p = x + iy.$$

The primitive decomposition becomes

$$z_p = g z_q.$$

Since $g > 0$, multiplication by g preserves argument:

$$\arg(z_p) = \arg(z_q),$$

and multiplies modulus:

$$|z_p| = g |z_q|.$$

The map

$$D_g(z) = gz$$

is a conformal dilation of the complex plane. It preserves angles and scales all lengths by g ; this is the standard complex-analytic sense of conformality [3].

Write

$$g = e^u, \quad u = \log g.$$

Then

$$z_p = e^u z_q.$$

The coordinate $u = \log g$ is additive under composed dilations:

$$g_1 g_2 = e^{u_1} e^{u_2} = e^{u_1 + u_2}, \quad \log(g_1 g_2) = \log g_1 + \log g_2.$$

This is the reason $\log g$ is the natural scale-cost in this atlas. Only this elementary additivity is needed below. Classical characterization results also single out logarithmic behavior among additive arithmetic functions under monotonicity-type hypotheses [4, 5, 6].

Remark 1 (Terminology). *The phrase discrete conformal scale means only that each obstruction layer is represented by an integer conformal dilation along a fixed ray, with additive scale-coordinate $\log g$. This is not a claim of a smooth conformal factor on a surface.*

A red-team-safe summary is:

Lattice obstruction is radial dilation along a preserved conformal direction.

The multiplier g is the dilation factor. The coordinate $\log g$ is its additive scale parameter.

4 Tolled radial length

Definition 2 (Tolled radial length). *Fix $\lambda > 0$. For $\mathbf{p} = g\mathbf{q}$, with \mathbf{q} primitive, define*

$$L_{\text{tolled}}(\mathbf{p}) = r(\mathbf{q}) + \lambda \log g.$$

The parameter λ has units of length per logarithmic cost-unit. It is not determined by the construction.

The first term

$$r(\mathbf{q})$$

is the visible primitive radial length. The second term

$$\lambda \log g$$

is a scale-cost toll expressed in length-units. It is not the Euclidean distance from \mathbf{q} to \mathbf{p} . The Euclidean hidden radial distance is

$$(g - 1)r(\mathbf{q}).$$

The point of the break-even construction is to compare these two quantities.

Nonclaim 2 (Metric status). *The functional L_{tolled} is a cost-from-origin or point-length functional. It is not automatically a two-point metric. A genuine path metric could be built by assigning edge or path costs and taking shortest paths, but that is a separate construction not carried out here.*

5 The break-even locus

The full radial distance to \mathbf{p} is

$$r(\mathbf{p}) = gr(\mathbf{q}).$$

The tolled visible-plus-scale path equals the full radial distance exactly when

$$r(\mathbf{q}) + \lambda \log g = gr(\mathbf{q}).$$

Equivalently,

$$(g - 1)r(\mathbf{q}) = \lambda \log g.$$

This is not a universal identity. For fixed λ , it is a locus in the (g, r) -plane.

Definition 3 (Break-even radius). *For $g > 1$, define*

$$r^*(g) = \frac{\lambda \log g}{g - 1}.$$

The superscript star marks the selected break-even value of the primitive radius for a given scale layer g .

Thus the break-even condition is

$$r(\mathbf{q}) = r^*(g).$$

At this boundary, logarithmic scale-cost pays exactly the hidden radial toll:

$$\boxed{\lambda \log g = (g - 1)r(\mathbf{q})}$$

Proposition 1 (Monotone decay of the break-even radius). *For $g > 1$, the function*

$$r^*(g) = \frac{\lambda \log g}{g - 1}$$

is strictly decreasing. Moreover,

$$\lim_{g \rightarrow 1^+} r^*(g) = \lambda$$

as a smooth analytic extension, and

$$r^*(g) \sim \lambda \frac{\log g}{g}$$

as $g \rightarrow \infty$.

Proof. It is enough to study

$$f(g) = \frac{\log g}{g - 1}.$$

Then

$$f'(g) = \frac{(g - 1)/g - \log g}{(g - 1)^2}.$$

Let

$$h(g) = 1 - \frac{1}{g} - \log g.$$

We have $h(1) = 0$, and for $g > 1$,

$$h'(g) = \frac{1}{g^2} - \frac{1}{g} = \frac{1 - g}{g^2} < 0.$$

Hence $h(g) < 0$ for $g > 1$, so $f'(g) < 0$. The limit at $g = 1$ follows from

$$\lim_{g \rightarrow 1^+} \frac{\log g}{g - 1} = 1.$$

The asymptotic relation follows because $g - 1 \sim g$. □

Since g is an integer in the lattice, $g = 1$ is the primitive layer and not an obstruction. The first genuine obstructed layer is

$$g = 2, \quad r^*(2) = \lambda \log 2.$$

The value $r^*(1) = \lambda$ is a smooth extension, useful for reading the geometry of the curve.

6 The additive scale form

Set

$$g = e^u, \quad u = \log g.$$

Then the break-even locus becomes

$$(e^u - 1)r = \lambda u,$$

and

$$r^*(u) = \frac{\lambda u}{e^u - 1}.$$

This coordinate makes the conformal-scale meaning visible. For small u ,

$$e^u - 1 = u + \frac{u^2}{2} + \frac{u^3}{6} + \dots,$$

so

$$r^*(u) = \lambda \left(1 - \frac{u}{2} + \frac{u^2}{12} + O(u^4) \right).$$

For large u ,

$$r^*(u) \sim \lambda u e^{-u}.$$

Thus logarithmic scale grows additively, while the hidden Euclidean radial distance grows exponentially in that scale.

7 The zeta layer count

Classically, primitive lattice points have natural density

$$\frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

There are two equivalent ways to see why this factor appears.

7.1 Prime obstruction survival

A prime p blocks visibility precisely when

$$p \mid x \quad \text{and} \quad p \mid y.$$

For a random lattice point, this has local density

$$\frac{1}{p^2}.$$

Therefore the survival factor for that prime is

$$1 - \frac{1}{p^2}.$$

Multiplying over all primes gives the Euler product

$$\prod_p \left(1 - \frac{1}{p^2} \right) = \frac{1}{\zeta(2)}.$$

7.2 Discrete radial dilation layers

Every nonzero lattice point lies on exactly one primitive ray layer:

$$\mathbf{p} = g\mathbf{q}.$$

In two dimensions, scaling by g multiplies area by g^2 . Thus the density weight of the g -layer is proportional to

$$\frac{1}{g^2}.$$

Summing all layers gives

$$\sum_{g=1}^{\infty} \frac{1}{g^2} = \zeta(2).$$

Hence the primitive layer has reciprocal density

$$\frac{1}{\zeta(2)}.$$

Remark 2 (Zeta wording). *A safe interpretation is:*

$\zeta(2)$ counts the density-weighted stack of radial dilation layers,
 $1/\zeta(2)$ is the primitive visible layer.

This is a classical arithmetic density statement, not a new consequence of conformal geometry.

8 Square-free mirror in valuation space

The square-free analogue lives not in Euclidean space but in log-height and valuation space. It is written here in the form

$$n = ms^2$$

so that the variable core m appears first and the square-scale layer s^2 appears second. Here m denotes a square-free integer core. No physical interpretation is attached to this notation in the present paper.

Every positive integer n decomposes uniquely as

$$n = ms^2,$$

where m is square-free and s^2 is the largest square dividing n . Taking logs gives

$$\log n = \log m + 2 \log s.$$

This mirrors the lattice log-scale decomposition

$$\log r(\mathbf{p}) = \log r(\mathbf{q}) + \log g.$$

The square-free condition is

$$\mu^2(n) = 1,$$

which is equivalent to

$$s = 1.$$

The obstruction is the repeated-square layer s^2 . A natural height-cost is

$$C_{\text{sf}}(n) = \log s,$$

or, if one wants to count repeated prime-factor depth more finely,

$$C_{\text{excess}}(n) = \sum_p \max(0, v_p(n) - 1) \log p.$$

The first cost measures the square-root of the largest square divisor. The second measures excess valuation beyond the first copy of each prime.

The same $1/\zeta(2)$ density appears. A prime p obstructs square-freeness when

$$p^2 \mid n,$$

which has local density p^{-2} . Therefore

$$\prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)}$$

is also the density of square-free integers.

Thus visible lattice points and square-free integers share the same per-prime survival sieve:

$$\boxed{p^{-2} \text{ obstruction at each prime.}}$$

The ambient geometries differ. Lattice obstruction is radial dilation in the plane. Square-free obstruction is repeated valuation depth in log-height space.

9 Why the arc comparison is cut

An earlier possible comparison was

$$r(\mathbf{q}) + \lambda C(g) = R\theta,$$

where

$$\theta = \arg(z_q) = \arg(z_p).$$

This is not used here. The reason is structural: dilation by g preserves angle. Thus θ carries no information about the radial obstruction layer. The conformal-scale interpretation strengthens this deletion rather than repairing it.

Angle may become relevant in a separate study of primitive directions, Farey sequences, Stern-Brocot structure, or angular equidistribution of visible lattice points. That is an orthogonal project, not part of the radial toll model.

10 Red-team ledger

R1. The construction rests on the exact identity $\mathbf{p} = g\mathbf{q}$, not on a new theorem.

R2. The phrase *discrete conformal scale* means integer dilation along rays, not new smooth conformal geometry.

R3. The parameter λ is free. It is an exchange rate, not a derived constant.

R4. The break-even relation is a locus:

$$(g - 1)r = \lambda \log g.$$

It is not a universal identity.

R5. The functional L_{tolled} is a point-length or cost-from-origin functional. It is not asserted to be a two-point metric.

R6. The factor $1/\zeta(2)$ is classical density and sieve structure.

R7. The square-free analogue lives in log-height and valuation space, not Euclidean conformal space.

R8. The angular comparison is cut because angle is invariant under radial dilation.

R9. Physical analogies are deliberately excluded from the present mathematical note. Any such bridge belongs in a separate, independently grounded companion work.

11 Conclusion

The primitive ray chamber is small but stable. Each nonzero lattice point decomposes into a visible primitive point and a discrete radial dilation:

$$\mathbf{p} = g\mathbf{q}.$$

The primitive point gives the ray. The multiplier gives the obstruction layer. In complex form,

$$z_p = e^u z_q, \quad u = \log g,$$

so $\log g$ becomes the natural additive coordinate of scale.

The tolled length

$$L_{\text{tolled}}(\mathbf{p}) = r(\mathbf{q}) + \lambda \log g$$

records visible distance plus logarithmic scale-cost. Its break-even boundary

$$(g - 1)r(\mathbf{q}) = \lambda \log g$$

marks where the cost of describing hidden scale equals the radial toll of obstruction.

The same prime-square survival factor

$$\prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)}$$

links primitive lattice visibility with square-free integers. The shared structure is not accidental: in both settings, each prime contributes a local obstruction of size p^{-2} .

In short:

Primitive visibility is the unobstructed ray. Logarithmic scale is the toll of hidden dilation.

12 Closing summary

The construction above is a mathematical atlas only. Its objects are primitive rays, integer dilation layers, logarithmic scale-cost, zeta density, and square-free valuation structure. Any physical analogy involving scale, curvature, or information should be developed separately, with independent physical grounding and without altering the arithmetic claims of this note.

References

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