Divisor-Filtered Farey Chambers and Localized Discrepancy

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Abstract

For each integer $n \geq 1$, the elementary chamber grid

$$a(n,k) = n - 1 + \frac{k}{n}, \qquad 1 \leq k \leq n,$$

becomes, after translation to the unit interval and reduction to lowest terms, a divisor-filtered family of reduced fractions: a reduced denominator q appears in chamber n if and only if $q \mid n$, and the number of entries of reduced denominator q is $\varphi(q)$. This note studies the discrepancy of such reduced fractions when the allowed denominators are restricted to a divisor window $D \subseteq \mathrm{Div}(n)$. For $0 \leq x \leq 1$, define

$$A_{n,D}(x) = \sum_{q \in D} \#\{\, 1 \leq a \leq qx : (a,q) = 1 \,\}, \qquad \Phi_D = \sum_{q \in D} \varphi(q),$$

and

$$\Delta_{n,D}(x) = A_{n,D}(x) - x\,\Phi_D.$$

We give a Möbius-inversion formula for $\Delta_{n,D}$, prove the uniform bound

$$\sup_{0 \leq x \leq 1} |\Delta_{n,D}(x)| \leq \sum_{q \in D} 2^{\omega(q)},$$

and identify several instructive special cases. The full divisor window $D = \mathrm{Div}(n)$ collapses exactly to the regular grid, $A_{n,\mathrm{Div}(n)}(x) = \lfloor nx \rfloor$. The exact shell $D = \{n\}$ recovers the classical reduced-residue discrepancy, while a prime-power shell $D = \{p^a\}$ has the sharper exact supremum $1 - 1/p$, with a telescoping refinement for prime-power ladders. We also describe a *chamber blow-up* coordinate that renormalizes each chamber's cells to unit width, separating continuous position from arithmetic resolution and showing that discrepancy extrema occur at, or as one-sided limits at, cell boundaries. These results are finite, elementary, and local. They are not global Farey-discrepancy estimates, and they carry no implication for the Riemann hypothesis.

Contents

1 Introduction

Classical number theory organizes reduced rational numbers in many ways. Farey sequences arrange all reduced fractions with bounded denominator; the Stern–Brocot and Calkin–Wilf trees enumerate the positive rationals through recursive constructions; reduced residue systems organize coprime numerators modulo a fixed denominator.

The triangular-fractional chamber grid is a deliberately elementary alternative. For each $n \geq 1$, define

$$a(n,k) = n - 1 + \frac{k}{n}, \qquad 1 \leq k \leq n.$$

Before reduction, chamber n is just the regular grid $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right\}$ translated into the interval $(n-1, n]$. After reduction, however, the same points are partitioned by their reduced denominators. If $\frac{k}{n} = \frac{a}{q}$ in lowest terms, then

$$q = \frac{n}{(k,n)}.$$

Thus every reduced denominator in chamber n divides n. Conversely, every reduced fraction $a/q \in (0,1]$ with $q \mid n$ appears uniquely in chamber n, namely as $\frac{k}{n} = \frac{a}{q}$ with $k = a\,\frac{n}{q}$. This observation gives a chamber-level form of Gauss's identity:

$$\sum_{q \mid n} \varphi(q) = n.$$

The whole chamber is therefore not a mysterious or irregular point set. As an unlabeled set of points, it is exactly the regular grid. The arithmetic structure appears only when the chamber is resolved by reduced denominator.

The purpose of this note is to study the discrepancy of such denominator-resolved windows. Instead of taking all denominators $q \mid n$, we choose a divisor window $D \subseteq \mathrm{Div}(n)$ and count only reduced fractions with denominators in D. This produces a finite, localized discrepancy functional

$$\Delta_{n,D}(x) = A_{n,D}(x) - x\,\Phi_D.$$

The main theorem expresses this discrepancy by Möbius inversion and bounds it by a divisor-window sum of $2^{\omega(q)}$. We then record what happens for the full divisor window, for the exact shell $D = \{n\}$, and for structured windows such as prime-power ladders and high-denominator bands. A short companion section introduces a *chamber blow-up* coordinate, a coordinate-normalization device that rescales each chamber's cells to unit width and pins down where the discrepancy extrema occur.

The results here should be read at the correct scale. The global Farey discrepancy studied by Franel, Landau, Mikolás, Niederreiter, Dress, and many later authors is a different object. It aggregates all denominators $q \le Q$, and certain discrepancy rates are equivalent to the Riemann hypothesis. The present note changes the object from global Farey discrepancy to a finite divisor-window discrepancy problem. It is a local arithmetic restriction, not progress toward the Riemann hypothesis.

2 Notation and conventions

Throughout,

$$\mathrm{Div}(n) = \{\, q \in \mathbb{N} : q \mid n \,\}$$

denotes the divisor set of n. We write μ for the Möbius function, φ for Euler's totient function, and $\omega(q)$ for the number of distinct prime divisors of q. We use the convention $\omega(1) = 0$, so that $2^{\omega(1)} = 1$.

For real $y \ge 0$, define the coprime-counting function

$$C_q(y) = \#\{\, a \in \mathbb{Z} : 1 \le a \le y,\ (a, q) = 1 \,\}.$$

Equivalently, the upper limit is $\lfloor y \rfloor$.

Convention 2.1 (Endpoint convention)**.** All counts use the closed upper range $1 \le a \le y$. Thus for $x \in [0, 1]$,

$$C_q(qx) = \#\{\, 1 \le a \le qx : (a, q) = 1 \,\}.$$

The point $1 = 1/1$ is carried by the degenerate denominator $q = 1$. In particular,

$$C_1(x) = \lfloor x \rfloor,$$

so $C_1(x) = 0$ for $0 \le x < 1$, and $C_1(1) = 1$, matching $\varphi(1) = 1$.

Definition 2.2 (Divisor-window counting function)**.** Let $n \ge 1$, let $D \subseteq \mathrm{Div}(n)$, and let $0 \le x \le 1$. Define

$$A_{n,D}(x) = \sum_{q \in D} C_q(qx) = \sum_{q \in D} \#\{\, 1 \le a \le qx : (a, q) = 1 \,\}.$$

Also set

$$\Phi_D = \sum_{q \in D} \varphi(q),$$

and define the divisor-window discrepancy

$$\Delta_{n,D}(x) = A_{n,D}(x) - x\,\Phi_D.$$

Remark 2.3 (Multiplicity and reduced denominators)*.* The sum defining $A_{n,D}$ is a sum over reduced-denominator classes. A reduced fraction a/q has a unique reduced denominator q, so different q-classes do not double-count reduced fractions in $(0, 1)$. The endpoint 1 is handled by the convention that $1 = 1/1$ is carried by $q = 1$. Thus on $(0, 1)$ the windowed count $A_{n,D}(x)$ coincides with the cardinality of the union of the per-denominator reduced point sets, and only the single value 1 requires separate bookkeeping.

3 The chamber dictionary

We first record the precise relationship between a chamber and reduced denominators.

Proposition 3.1 (Divisor-filtered chamber dictionary)**.** *Fix $n \geq 1$. Translating chamber n back to the unit interval gives the grid $\left\{\frac{k}{n} : 1 \leq k \leq n\right\}$. If $\frac{k}{n} = \frac{a}{q}$ in lowest terms, then $q \mid n$. Conversely, if $q \mid n$ and $1 \leq a \leq q$ with $(a, q) = 1$, then $\frac{a}{q} = \frac{k}{n}$ for $k = a\,\frac{n}{q}$. Hence chamber n, after reduction and translation to $(0, 1]$, is the divisor-filtered Farey family*

$$\mathcal{D}_n = \left\{ \frac{a}{q} \in (0,1] : q \mid n,\ 1 \leq a \leq q,\ (a,q) = 1 \right\}.$$

For each $q \mid n$, the number of chamber entries with reduced denominator q is $\varphi(q)$, and therefore $\sum_{q \mid n} \varphi(q) = n$.

Proof. Write $g = (k, n)$. Then $\frac{k}{n} = \frac{k/g}{n/g}$, so the reduced denominator is $q = n/g$, which divides n. Conversely, suppose $q \mid n$ and $(a, q) = 1$. Let $k = a\,\frac{n}{q}$. Since $1 \leq a \leq q$, we have $1 \leq k \leq n$, and

$$\frac{k}{n} = \frac{a(n/q)}{n} = \frac{a}{q}.$$

Uniqueness follows from uniqueness of reduced form. For each fixed q, the allowed numerators a are precisely the $\varphi(q)$ integers in $1 \leq a \leq q$ coprime to q. Summing over $q \mid n$ gives n, since every $k \in \{1, \ldots, n\}$ appears exactly once. $\square$

Remark 3.2. This proposition is the reason the full chamber has trivial point-set discrepancy: before denominator labels are remembered, chamber n is simply the regular grid $\{k/n\}_{k=1}^{n}$. The nontrivial finite arithmetic lives in the denominator-resolved windows $D \subseteq \mathrm{Div}(n)$.

4 Chamber blow-up and unit-cell normalization

This section introduces a refined coordinate system for the chamber grid. It is a coordinate device, used to interpret the discrepancy variable and to make divisibility constraints on motion explicit. A reader interested only in the discrepancy bound may proceed directly to Section 5.

Why renormalize. In the ordinary multiplication grid, every cell is a unit square, so a unit step has the same geometric scale everywhere. The chamber grid is different. Chamber n has internal spacing $1/n$, so its cells shrink as n grows. To compare chambers on equal footing we rescale each chamber's cells to unit width.

The blow-up coordinate. Recall the raw chamber coordinate

$$a(n,k) = n - 1 + \frac{k}{n}, \qquad 1 \leq k \leq n,$$

and the normalized chamber coordinate $u = k/n \in (0, 1]$. Partition $(0, 1]$ into the half-open cells

$$I_{n,k} = \left(\frac{k-1}{n}, \frac{k}{n} \right], \qquad 1 \leq k \leq n,$$

each of width $1/n$. For $u \in I_{n,k}$ define the local cell coordinate

$$r = nu - (k-1) \in (0, 1],$$

so that

$$u = \frac{k-1+r}{n}, \qquad x = n-1+\frac{k-1+r}{n}.$$

This is a reversible change of coordinates

$$(n,k,r) \longleftrightarrow x = n-1+\frac{k-1+r}{n}.$$

Definition 4.1 (Chamber blow-up map)**.** For $u \in (0,1]$ let $k = \lceil nu \rceil$, so that $u \in I_{n,k}$. The chamber blow-up map is

$$\mathcal{B}_n : (0,1] \to (0,1], \qquad \mathcal{B}_n(u) = nu - (k-1) = nu - \lceil nu \rceil + 1,$$

with inverse

$$\mathcal{B}_n^{-1}(k,r) = \frac{k-1+r}{n}.$$

The raw chamber position is recovered by $x = n-1+\mathcal{B}_n^{-1}(k,r)$.

The map $\mathcal{B}_n$ sends each cell $I_{n,k}$ bijectively onto $(0,1]$; geometrically it enlarges every fractional cell of width $1/n$ to unit width.

Arithmetic resolution. The reduced denominator of the grid point k/n is recovered from (n,k) alone:

$$q(n,k) = \frac{n}{(n,k)},$$

independent of the within-cell coordinate r. Thus the triple (n,k,r) separates the continuous position r (within a cell), the cell index k, and the chamber resolution n from the arithmetic datum $q(n,k)$. One may carry out geometric operations in the normalized unit cell and then return to the chamber to read off the divisor structure.

Normalized motion and legal moves. A normalized displacement $u \mapsto u + \Delta u$ corresponds to the index shift

$$k \mapsto k + n\,\Delta u.$$

This is a move between grid points exactly when $n\,\Delta u \in \mathbb{Z}$. For instance, the half-chamber shift $\Delta u = \frac{1}{2}$ is a legal grid move if and only if n is even. The blow-up coordinate therefore makes parity and divisibility constraints on motion visible: the same geometric move is or is not arithmetically admissible depending on n.

Example 4.2. Take $n = 6$, $k = 4$. Then $u = 4/6 = 2/3$ and the reduced denominator is $q = 6/(6,4) = 3$. A point halfway inside the cell $I_{6,4} = (3/6, 4/6]$ has $r = \frac{1}{2}$, giving

$$u = \frac{3+\frac{1}{2}}{6} = \frac{7}{12}, \qquad x = 5 + \frac{7}{12}.$$

The endpoint $r = 1$ returns the grid point $u = 4/6$, whose reduced denominator is 3.

Relation to the discrepancy variable. The discrepancy functional $\Delta_{n,D}(x)$ of Section 5 is a function of $x \in [0,1]$, and this x is precisely the normalized chamber coordinate u. The blow-up refines u into a cell index k and a within-cell coordinate r. As made precise in Lemma 5.3 below, $\Delta_{n,D}$ is piecewise linear with downward slope $-\Phi_D$ and upward unit jumps exactly at the grid points $u = k/n$ whose reduced denominator lies in D; in blow-up coordinates these are the cell boundaries $r = 1$. Consequently the supremum in Theorem 5.2 and its corollaries is always attained at, or in a one-sided limit at, such cell boundaries. For the prime-power shell of Proposition 8.1, for example, the extremum occurs at $x = (p-1)/p^a$, the right endpoint of the cell $I_{p^a,p-1}$.

In the spirit of a coordinate device, the blow-up map can be summarized as: *make the geometry uniform, then watch the arithmetic change.*

5 Main theorem

Lemma 5.1 (Legendre–Möbius counting identity)**.** *For* $q \geq 1$ *and real* $y \geq 0$*,*

$$C_q(y) = \sum_{d \mid q} \mu(d) \left\lfloor \frac{y}{d} \right\rfloor .$$

Proof. The coprimality indicator satisfies $\mathbf{1}_{(a,q)=1} = \sum_{d \mid (a,q)} \mu(d)$. Therefore

$$C_q(y) = \sum_{1 \leq a \leq y} \sum_{\substack{d \mid q \\ d \mid a}} \mu(d) = \sum_{d \mid q} \mu(d) \, \#\{ 1 \leq a \leq y : d \mid a \} = \sum_{d \mid q} \mu(d) \left\lfloor \frac{y}{d} \right\rfloor .$$

This is valid for real $y \geq 0$, since the number of positive multiples of d not exceeding y is $\lfloor y/d \rfloor$. ☐

Theorem 5.2 (Divisor-window discrepancy formula and bound)**.** *Let* $n \geq 1$*, let* $D \subseteq \mathrm{Div}(n)$*, and let* $0 \leq x \leq 1$*. Then*

$$A_{n,D}(x) = \sum_{q \in D} \sum_{d \mid q} \mu(d) \left\lfloor \frac{qx}{d} \right\rfloor .$$

Furthermore,

$$\Delta_{n,D}(x) = \sum_{q \in D} \sum_{d \mid q} \mu(d) \left(\left\lfloor \frac{qx}{d} \right\rfloor - \frac{qx}{d} \right),$$

with $\Delta_{n,D}(0) = \Delta_{n,D}(1) = 0$*. Finally,*

$$\sup_{0 \leq x \leq 1} |\Delta_{n,D}(x)| \leq \sum_{q \in D} 2^{\omega(q)}.$$

Proof. Apply Lemma 5.1 with $y = qx$:

$$C_q(qx) = \sum_{d \mid q} \mu(d) \left\lfloor \frac{qx}{d} \right\rfloor .$$

Summing over $q \in D$ gives the stated formula for $A_{n,D}(x)$.

Next, the standard identity

$$\varphi(q) = q \sum_{d \mid q} \frac{\mu(d)}{d} = \sum_{d \mid q} \mu(d) \, \frac{q}{d}$$

gives $x\varphi(q) = \sum_{d \mid q} \mu(d) \, \frac{qx}{d}$. Subtracting this from the formula for $C_q(qx)$, and then summing over $q \in D$, yields

$$\Delta_{n,D}(x) = \sum_{q \in D} \sum_{d \mid q} \mu(d) \left(\left\lfloor \frac{qx}{d} \right\rfloor - \frac{qx}{d} \right).$$

At $x = 0$, every term is zero. At $x = 1$, since $d \mid q$ the value q/d is an integer, hence $\lfloor q/d \rfloor - q/d = 0$. Thus $\Delta_{n,D}(0) = \Delta_{n,D}(1) = 0$.

For the uniform bound, use $| \lfloor t \rfloor - t| < 1$ for non-integer t, with the value 0 at integer t. Hence

$$|\Delta_{n,D}(x)| \leq \sum_{q \in D} \sum_{d \mid q} |\mu(d)| = \sum_{q \in D} 2^{\omega(q)},$$

because the number of $d \mid q$ with $\mu(d) \neq 0$ is the number of squarefree divisors of q, namely $\sum_{d \mid q} |\mu(d)| = 2^{\omega(q)}$. Taking the supremum over $0 \leq x \leq 1$ proves the result. ☐

Lemma 5.3 (Piecewise-linear structure)**.** *For fixed n and $D \subseteq \mathrm{Div}(n)$, the function $\Delta_{n,D}(x)$ is right-continuous and piecewise linear on $[0,1]$. Away from the finitely many points*

$$\frac{a}{q}, \qquad q \in D, \quad 1 \le a \le q, \quad (a,q) = 1,$$

it has slope $-\Phi_D$. At each such point it has an upward jump of size 1. Since each such reduced fraction a/q with $q \mid n$ equals k/n for a unique k, these jump points are precisely the chamber cell boundaries whose reduced denominator lies in D.

Proof. The function $A_{n,D}(x)$ is a nondecreasing step function. It increases by 1 exactly when x crosses a reduced fraction a/q with $q \in D$ and $(a,q) = 1$, using the closed upper endpoint convention of Convention 2.1. Between such jump points, $A_{n,D}(x)$ is constant, while $x\Phi_D$ is linear with slope Φ_D. Hence $\Delta_{n,D}(x) = A_{n,D}(x) - x\Phi_D$ has slope $-\Phi_D$ between jumps and upward jumps of size 1 at the listed points. Proposition 3.1 identifies each listed point with a unique chamber grid point k/n, and therefore with a unique cell boundary in the blow-up coordinate. $\square$

Remark 5.4. The bound is intentionally elementary. It ignores cancellation among the fractional-part terms, and it uses the crude per-term estimate $|\lfloor t \rfloor - t| < 1$. It is therefore a safe baseline, not an optimal general estimate.

6 The full chamber

The full divisor window $D = \mathrm{Div}(n)$ is special. In that case the reduced-denominator decomposition recombines into the regular grid.

Corollary 6.1 (Full chamber collapse)**.** *For every $n \ge 1$ and $0 \le x \le 1$,*

$$A_{n,\mathrm{Div}(n)}(x) = \lfloor nx \rfloor .$$

Consequently, $\Delta_{n,\mathrm{Div}(n)}(x) = \lfloor nx \rfloor - nx$, and

$$\sup_{0 \le x \le 1} |\Delta_{n,\mathrm{Div}(n)}(x)| \le 1.$$

Since $\Phi_{\mathrm{Div}(n)} = n$, the normalized discrepancy satisfies

$$\frac{1}{\Phi_{\mathrm{Div}(n)}} \sup_{0 \le x \le 1} |\Delta_{n,\mathrm{Div}(n)}(x)| \le \frac{1}{n}.$$

Proof. By Proposition 3.1, the map $k \mapsto \frac{k}{n}$ is a bijection from $\{1 \le k \le \lfloor nx \rfloor\}$ to the set of reduced fractions $a/q \le x$ with $q \mid n$: reducing k/n gives a denominator $q \mid n$, and the inverse map from a reduced fraction a/q with $q \mid n$ is $k = a\,\frac{n}{q}$. Therefore the total count is exactly $\lfloor nx \rfloor$. The discrepancy formula follows, and the normalization uses Gauss's identity $\sum_{q \mid n} \varphi(q) = n$. $\square$

Remark 6.2. This corollary is the main calibration point of the note. The whole chamber is not an irregular Farey object: it is the regular grid. The arithmetic signal appears only after resolving the chamber by denominator windows.

7 Exact shells and reduced residues

The exact shell $D = \{n\}$ recovers the familiar reduced-residue counting problem.

Corollary 7.1 (Exact shell)**.** *For $D = \{n\}$,*

$$\Delta_{n,\{n\}}(x) = \sum_{d|n} \mu(d) \left(\left\lfloor \frac{nx}{d} \right\rfloor - \frac{nx}{d} \right).$$

Equivalently, with $\{t\} = t - \lfloor t \rfloor$,

$$\Delta_{n,\{n\}}(x) = -\sum_{d|n} \mu(d) \left\{ \frac{nx}{d} \right\}.$$

Hence

$$\sup_{0 \le x \le 1} |\Delta_{n,\{n\}}(x)| \le 2^{\omega(n)}.$$

Proof. This is Theorem 5.2 with $D = \{n\}$. The fractional-part form follows from $\lfloor t \rfloor - t = -\{t\}$. □

Remark 7.2. This is the classical elementary error term for counting reduced residues:

$$\#\{ 1 \le a \le y : (a, n) = 1 \} = y \, \frac{\varphi(n)}{n} + O\Big(2^{\omega(n)}\Big).$$

Thus the exact shell is not a new theorem. It is the standard reduced-residue discrepancy written in chamber language.

Remark 7.3 (Size of the bound). The quantity $2^{\omega(n)}$ counts the squarefree divisors of n. By the Hardy–Ramanujan theorem [3], $\omega(n)$ has normal order $\log \log n$, so $2^{\omega(n)}$ is typically of size $(\log n)^{\log 2}$; its maximal order is governed by Wigert's estimate [4] for the divisor function, giving the worst case $2^{(1+o(1)) \log n / \log \log n}$. The bound of Corollary 7.1 is therefore small for typical n and grows only subpolynomially even in the worst case.

8 Prime-power shells and ladders

The elementary bound of the main theorem can be sharpened in certain structured windows. The simplest case is a prime-power denominator.

Proposition 8.1 (Prime-power shell)**.** *Let $q = p^a$ with p prime and $a \ge 1$. Then*

$$\Delta_{p^a,\{p^a\}}(x) = \{p^{a-1}x\} - \{p^a x\},$$

and

$$\sup_{0 \le x \le 1} |\Delta_{p^a,\{p^a\}}(x)| = 1 - \frac{1}{p},$$

the supremum being attained, for instance at $x = (p-1)/p^a$.

Proof. The only squarefree divisors of p^a are 1 and p, so the exact-shell formula of Corollary 7.1 gives

$$\Delta_{p^a,\{p^a\}}(x) = \big(\lfloor p^a x \rfloor - p^a x\big) - \big(\lfloor p^{a-1} x \rfloor - p^{a-1} x\big) = \{p^{a-1}x\} - \{p^a x\}.$$

Put $t = p^a x \in [0, p^a]$, so that $\{p^{a-1}x\} - \{p^a x\} = \{t/p\} - \{t\}$. Write $t = m + u$ with $m \in \mathbb{Z}$ and $0 \le u < 1$, and let $r \in \{0, 1, \dots, p-1\}$ be the residue of m modulo p. Then $\{t\} = u$ and, since $0 \le r + u < p$,

$$\left\{\frac{t}{p}\right\} = \frac{r+u}{p}.$$

Hence

$$\left\{\frac{t}{p}\right\} - \{t\} = \frac{r}{p} - \left(1 - \frac{1}{p}\right) u.$$

For fixed r this is maximized at $u = 0$, with value $r/p \le (p-1)/p$, the bound $1 - 1/p$ being attained at $r = p-1$; and it is minimized as $u \to 1^-$, with infimum $(r - (p-1))/p \ge -(p-1)/p$, the symmetric value $-(1 - 1/p)$ being approached (but not attained) at $r = 0$. Therefore

$$\sup_{0 \le x \le 1} |\Delta_{p^a, \{p^a\}}(x)| = 1 - \frac{1}{p},$$

attained, e.g., at $x = (p-1)/p^a$ (where $t = p - 1$, so $r = p - 1$ and $u = 0$). ☐

Corollary 8.2 (Prime-power ladder)**.** *Let* $D = \{p, p^2, \dots, p^a\}$. *Then*

$$\Delta_{p^a, D}(x) = \{x\} - \{p^a x\},$$

and hence

$$\sup_{0 \le x \le 1} |\Delta_{p^a, D}(x)| = 1 - \frac{1}{p^a}.$$

Proof. For each $1 \le j \le a$, Proposition 8.1 gives $\Delta_{p^j, \{p^j\}}(x) = \{p^{j-1}x\} - \{p^j x\}$. Summing over j telescopes:

$$\Delta_{p^a, D}(x) = \sum_{j=1}^{a} \big(\{p^{j-1}x\} - \{p^j x\}\big) = \{x\} - \{p^a x\}.$$

For the supremum, set $N = p^a$ and repeat the argument of Proposition 8.1 with p replaced by N. For $x \in [0, 1)$ one has $\{x\} = x$, and writing $t = Nx = m + u$ with $m \in \{0, \dots, N-1\}$ and $0 \le u < 1$,

$$\{x\} - \{Nx\} = \frac{m}{N} - \left(1 - \frac{1}{N}\right) u,$$

whose absolute value has supremum $1 - 1/N = 1 - 1/p^a$, attained at $x = (p^a - 1)/p^a$. The endpoint $x = 1$ gives $\{1\} - \{p^a\} = 0$. ☐

Remark 8.3. If the divisor 1 is also included, so $D = \{1, p, p^2, \dots, p^a\} = \mathrm{Div}(p^a)$, then the ladder recombines into the full chamber: for $x \in [0, 1)$, $\Delta_{p^a, \{1\}}(x) = \lfloor x \rfloor - x = -x = -\{x\}$, so

$$\Delta_{p^a, \mathrm{Div}(p^a)}(x) = \big(-\{x\}\big) + \big(\{x\} - \{p^a x\}\big) = -\{p^a x\} = \lfloor p^a x \rfloor - p^a x,$$

in agreement with Corollary 6.1.

9 Other divisor windows

The main theorem also gives immediate bounds for common divisor-window choices.

Corollary 9.1 (High-denominator bands)**.** *Fix* $0 < \theta \le 1$*, and let* $D_\theta(n) = \{\, q \mid n : q \ge \theta n \,\}$*. Then*

$$|D_\theta(n)| \le \left\lfloor \frac{1}{\theta} \right\rfloor, \qquad \sup_{0 \le x \le 1} |\Delta_{n,D_\theta(n)}(x)| \le \left\lfloor \frac{1}{\theta} \right\rfloor 2^{\omega(n)}.$$

Proof. If $q \mid n$ and $q \ge \theta n$, then the complementary divisor $c = n/q$ satisfies $c \le 1/\theta$. The map $q \mapsto n/q$ injects $D_\theta(n)$ into the set of positive integers $c \le 1/\theta$, so $|D_\theta(n)| \le \lfloor 1/\theta \rfloor$. Since $q \mid n$ implies $\omega(q) \le \omega(n)$, Theorem 5.2 gives

$$\sup |\Delta_{n,D_\theta(n)}| \le \sum_{q \in D_\theta(n)} 2^{\omega(q)} \le |D_\theta(n)|\, 2^{\omega(n)} \le \lfloor 1/\theta \rfloor\, 2^{\omega(n)}. \qquad \square$$

Corollary 9.2 (Divisor intervals)**.** *Let* $D = [Q_1, Q_2] \cap \mathrm{Div}(n)$*. Then*

$$\sup_{0 \le x \le 1} |\Delta_{n,D}(x)| \le 2^{\omega(n)}\, \#\{\, q \mid n : Q_1 \le q \le Q_2 \,\}.$$

Proof. Immediate from $\sum_{q \in D} 2^{\omega(q)} \le \sum_{q \in D} 2^{\omega(n)} = 2^{\omega(n)} |D|$. $\square$

Remark 9.3. Sharper bounds for general divisor windows would require controlled cancellation among the fractional-part terms in the Möbius formula of Theorem 5.2. We leave the optimal general-window constant as an open finite problem.

10 Relation to global Farey discrepancy

Let $F_Q = \{\alpha_{1,Q} < \cdots < \alpha_{N_Q,Q}\}$ denote the Farey fractions of order Q in $(0,1]$, with $N_Q = \sum_{q \le Q} \varphi(q)$, up to the standard endpoint convention. Classical work of Franel [5] and Landau [6] relates the discrepancy of the globally ordered Farey sequence to the Riemann hypothesis. In one common normalization, the rank-error terms

$$d_{j,Q} = \alpha_{j,Q} - \frac{j}{N_Q}$$

satisfy RH-equivalent estimates such as

$$\sum_{j=1}^{N_Q} d_{j,Q}^2 = O\big(Q^{-1+\varepsilon}\big) \qquad \text{and} \qquad \sum_{j=1}^{N_Q} |d_{j,Q}| = O\Big(Q^{1/2+\varepsilon}\Big)$$

for every $\varepsilon > 0$, depending on the precise normalization and endpoint convention. Authoritative treatments include Edwards [7] and Broughan [8], and the unconditional local-discrepancy rate $D_Q \asymp 1/Q$ is due to Niederreiter [9] and Dress [10].

The present note does not study that global object. Instead, it fixes a single integer n and restricts denominators to the finite divisor lattice $\mathrm{Div}(n)$, or to a window $D \subseteq \mathrm{Div}(n)$. Thus the global denominator condition $q \le Q$ is replaced by the local arithmetic condition $q \mid n$. The distinction is essential:

$$\text{global Farey: all } q \le Q, \qquad \text{divisor chamber: only } q \mid n.$$

This finite localization is why the present estimates are elementary. It also means they should not be described as improving global Farey discrepancy bounds or bypassing the Mertens function. They address a different object.

Remark 10.1 (Safe positioning). The chamber framework is a denominator-filtered re-coordinate of reduced-fraction and Farey structure. Franel–Landau supplies the classical global discrepancy/RH bridge. The divisor-window discrepancy studied here is a finite arithmetic restriction and carries no implication for the Riemann hypothesis. Related restricted-denominator Farey families have been studied globally elsewhere — for example with odd denominators [14] or under divisibility constraints [15, 16], and through maximum mean discrepancies [13] and rank/local-discrepancy formulas [11, 12] — but those works concern the global, order-Q object, not the finite divisor-window object of this note.

11 Computational verification plan

The results above are algebraic and do not require computation. Nevertheless, a small computational ledger is useful for figures and implementation checks. A minimal verification pipeline is the following.

1. Generate $\mathrm{Div}(n)$, and tabulate $\mu(q)$, $\omega(q)$, and $\varphi(q)$ for all divisors of selected n.
2. Choose windows such as $D = \{n\}$, $D = \mathrm{Div}(n)$, $D_\theta(n) = \{q \mid n : q \geq \theta n\}$, and prime-power ladders $D = \{p, p^2, \ldots, p^a\}$.
3. Evaluate $A_{n,D}(x) = \sum_{q \in D} \#\{1 \leq a \leq qx : (a, q) = 1\}$ at all breakpoints $x = a/q$ with $q \in D$ and $0 \leq a \leq q$ (the supremum is attained at or beside these jumps, equivalently at the cell boundaries $r = 1$ of Section 4).
4. Plot $A_{n,D}(x)$ against the linear expectation $x\,\Phi_D$.
5. Record $\sup_x |\Delta_{n,D}(x)|$ and the normalized value $\Phi_D^{-1} \sup_x |\Delta_{n,D}(x)|$, and build heatmaps over n and over window families D.
6. Confirm the closed forms

$$A_{n,\mathrm{Div}(n)}(x) = \lfloor nx \rfloor, \qquad \sup_x |\Delta_{p^a,\{p^a\}}| = 1 - \frac{1}{p}, \qquad \sup_x |\Delta_{p^a,\{p,\ldots,p^a\}}| = 1 - \frac{1}{p^a}.$$

Computation can verify the implementation and generate rank plots, but it is not a substitute for the proofs above.

12 Known, new, empirical, and nonclaims

Component	**Status**
Legendre–Möbius inversion for $C_q(y)$	Classical
Gauss identity $\sum_{q\mid n}\varphi(q)=n$	Classical
Full chamber equals the regular grid	Elementary consequence
Exact shell $D=\{n\}$ as reduced-residue discrepancy	Classical, in chamber notation
Chamber blow-up coordinate (n,k,r)	Coordinate device (new packaging)
General divisor-window functional $\Delta_{n,D}$	New packaging of classical ingredients
Prime-power shell bound $1-1/p$	Minor quantitative refinement
Prime-power ladder telescoping $1-1/p^a$	Minor quantitative refinement
Computational heatmaps and rank plots	Empirical visualization of proven formulas
RH implication	Nonclaim
Improvement of global Farey discrepancy	Nonclaim
Optimal general-window constant	Open finite subproblem

13 Motivational note on reciprocal-floor shelling

The arithmetic construction above was motivated in part by a broader reciprocal-floor viewpoint. In a separate zeta-zero diagnostic, one may define, at a simple nontrivial zero $\rho=\frac{1}{2}+i\gamma$,

$$P_\rho=|\zeta'(\rho)|\,h_\rho,$$

where h_ρ is the nearest-neighbor half-gap, and then form the integer shell index

$$m_{\text{isolate}}(\rho)=\left\lfloor\frac{1}{P_\rho}\right\rfloor+1.$$

This has the same reciprocal-floor chamber shape as the chamber indices used above, and it is the source of the term "shell." For orientation only, both h_ρ and $|\zeta'(\rho)|$ have natural height-dependent scales, so the product P_ρ was introduced as a finite-range diagnostic in a separate project.

This statistic plays no role in the proof of the divisor-window discrepancy theorem. It is not a zero locator, it is not a Riemann-hypothesis criterion, and it is not used here as evidence for any spectral claim. The joint law of h_ρ and $|\zeta'(\rho)|$ is a separate future project. The arithmetic note stands on its own.

14 Conclusion

The chamber grid $a(n,k)=n-1+\frac{k}{n}$ has two faces. As a point set, chamber n is just the regular grid. As a reduced-denominator decomposition, it is the finite Farey-like family of reduced fractions whose denominators divide n. This note isolates the second face by studying divisor-window discrepancy, and it records a blow-up coordinate that renormalizes the chambers so they may be compared on equal footing.

The main formula

$$\Delta_{n,D}(x) = \sum_{q\in D}\sum_{d\mid q}\mu(d)\left(\left\lfloor\frac{qx}{d}\right\rfloor - \frac{qx}{d}\right)$$

is elementary, but it gives a clean bookkeeping coordinate for local reduced-fraction discrepancy inside a chamber. The full window recombines to the regular grid; the exact shell recovers reduced-residue discrepancy; prime-power windows reveal small telescoping refinements with exact suprema $1-1/p$ and $1-1/p^a$.

The resulting framework is modest by design. It changes the object from global Farey discrepancy to a finite divisor-window discrepancy problem. That makes the problem elementary and local, not RH-adjacent. The value of the construction is not that it unlocks global asymptotic mysteries, but that it makes the arithmetic decomposition of a chamber explicit and measurable.

A Symbol dictionary

For convenience, the recurring notation of this note is collected here.

Symbol	**Meaning**
n	chamber index, $n \geq 1$
$a(n,k)$	raw chamber coordinate $n-1+k/n$, for $1 \leq k \leq n$
k	step index inside a chamber, $1 \leq k \leq n$
u	normalized chamber coordinate $u = k/n \in (0,1]$ (equals the discrepancy variable x)
$I_{n,k}$	cell $\big((k-1)/n,\ k/n\big]$, of width $1/n$
r	within-cell coordinate, $r = nu-(k-1) \in (0,1]$
$\mathcal{B}_n$	chamber blow-up map $\mathcal{B}_n(u) = nu-(k-1)$, with $k = \lceil nu \rceil$
$\mathrm{Div}(n)$	set of positive divisors of n
D	divisor window, $D \subseteq \mathrm{Div}(n)$
q	a reduced denominator; here $q \mid n$
(a,q)	greatest common divisor of a and q
φ	Euler totient function
μ	Möbius function
$\omega(q)$	number of distinct prime divisors of q (with $\omega(1)=0$)
$C_q(y)$	$\#\{1 \leq a \leq y : (a,q)=1\}$, the coprime-counting function
$A_{n,D}(x)$	$\sum_{q\in D} C_q(qx)$, the divisor-window counting function
Φ_D	$\sum_{q\in D}\varphi(q)$
$\Delta_{n,D}(x)$	$A_{n,D}(x) - x\,\Phi_D$, the divisor-window discrepancy
$\{t\},\ \lfloor t\rfloor$	fractional part and floor of t
$P_\rho,\ m_{\text{isolate}}$	motivational zeta-shell statistics (Section 13), not used in proofs

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