The Collatz Chamber Lift: Parity-Vector Descent via Affine Residue Geometry

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Abstract

This note gives an expository chamber-coordinate presentation of classical parity-vector arithmetic for the accelerated Collatz map

$$T(n) = \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ (3n+1)/2, & n \equiv 1 \pmod 2. \end{cases}$$

For each depth k, the first k parities of an orbit form a parity word $w \in \{0,1\}^k$, and classical work of Terras, Everett, and Lagarias identifies these words with residue classes modulo 2^k. For a fixed word w, with $r(w)$ odd steps, the corresponding k-step iterate has the affine form

$$T_w(n) = \frac{3^{r(w)} n + b(w)}{2^k},$$

where $b(w)$ is an integer offset determined by the word. We re-present this standard machinery as a *Collatz chamber lift*: a finite coordinate system whose axes are depth k, odd-count layer r, parity word w, residue class modulo 2^k, affine offset $b(w)$, and signed gap $2^k - 3^{r(w)}$. In this language, descent at depth k is the elementary inequality

$$T_w(n) < n \quad \Longleftrightarrow \quad b(w) < \left(2^k - 3^{r(w)}\right) n,$$

with contraction only in the positive-gap regime $2^k > 3^{r(w)}$. The purpose of the note is organizational and visual: it does not propose a proof of the Collatz conjecture and does not claim novelty for the underlying parity-vector or affine-residue facts. Rather, it records a coordinate interface for viewing classical Collatz descent machinery as a binomially layered residue geometry.

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1 Introduction

The Collatz conjecture concerns the iteration of the map

$$C(n) = \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ 3n+1, & n \equiv 1 \pmod 2, \end{cases}$$

and asserts that every positive integer eventually reaches 1. A common accelerated version divides by 2 after each odd step and studies

$$T(n) = \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ (3n+1)/2, & n \equiv 1 \pmod 2. \end{cases} \tag{1}$$

This is the map used throughout this note.

The classical parity-vector viewpoint, developed in work of Terras, Everett, and Lagarias, encodes the first k steps of an orbit by the parity word

$$w = (w_0, w_1, \dots, w_{k-1}) \in \{0,1\}^k, \qquad w_j \equiv T^j(n) \pmod 2.$$

At fixed depth k, these parity words are in bijective correspondence with residue classes modulo 2^k. For a fixed word w, the k-step iterate is affine in the starting integer n:

$$T_w(n) = \frac{3^{r(w)}n + b(w)}{2^k}, \qquad r(w) = \sum_{j=0}^{k-1} w_j.$$

These statements are standard in the Collatz literature. The present paper does not claim otherwise.

The aim here is to re-coordinate these facts in a chamber language. The word *chamber* is used as an informal organizational term: a parity word w determines a finite residue chamber modulo 2^k, and words with the same odd-count r form a binomial layer of size $\binom{k}{r}$. The chamber lift records not only the word, but also its affine multiplier, offset, residue class, and descent threshold.

The payoff is not a new proof. The payoff is a cleaner visual bookkeeping system. The usual one-dimensional Collatz orbit is lifted into a finite coordinate atlas:

$$\boxed{(k, w, r(w), b(w), R_w, 2^k - 3^{r(w)})}$$

and then projected back to the one-dimensional descent condition

$$T^k(n) < n.$$

This is the same philosophy as a chamber/grid presentation in other arithmetic settings: restore an addressable geometry, then reconcile it with the classical one-dimensional formula.

Remark 1.1 (Scope). This note is expository and organizational. It does not prove the Collatz conjecture, does not improve known density results, and does not treat average drift or stochastic parity models as proofs. It repackages classical parity-vector and affine-residue arithmetic as a chamber-coordinate system.

2 The accelerated map and parity words

Definition 2.1 (Shortcut Collatz map)**.** Let $T : \mathbb{N} \to \mathbb{N}$ be the accelerated Collatz map defined by (1). For $n \in \mathbb{N}$, write

$$n_0 = n, \qquad n_{j+1} = T(n_j).$$

The length-k parity word of n is

$$w(n;k) = (w_0, w_1, \ldots, w_{k-1}) \in \{0,1\}^k, \qquad w_j \equiv n_j \pmod 2.$$

Definition 2.2 (Odd-count layer)**.** For a word $w = (w_0, \ldots, w_{k-1}) \in \{0,1\}^k$, define its odd-count

$$r(w) = \sum_{j=0}^{k-1} w_j.$$

The set of words with $r(w) = r$ is called the r-th parity layer at depth k.

At depth k, there are 2^k parity words. Grouping by odd-count gives the elementary binomial stratification

$$\#\{w \in \{0,1\}^k : r(w) = r\} = \binom{k}{r}.$$

This triangular structure is one of the reasons the chamber presentation is visually natural: a depth-k row splits into layers indexed by $r = 0, 1, \ldots, k$.

3 Parity chambers and residue classes modulo powers of two

The key classical fact is that the first k parities are determined exactly by the residue class of n modulo 2^k. Conversely, every length-k word occurs.

Theorem 3.1 (Parity-vector residue bijection, classical)**.** *For each $k \geq 1$, the map*

$$n \pmod{2^k} \longmapsto w(n;k) \in \{0,1\}^k$$

is a bijection from the residue classes modulo 2^k to the set of length-k parity words.

Remark 3.2 (Attribution). The residue-class dependence and parity-vector encoding are classical in the work of Terras and Everett, and are part of the standard Lagarias formulation of the $3x+1$ problem. In this note we call the residue class corresponding to w the *parity chamber* of w.

Definition 3.3 (Parity chamber)**.** Let $w \in \{0,1\}^k$. The parity chamber associated to w is the unique residue class

$$R_w \pmod{2^k}$$

whose positive representatives have first k shortcut parities equal to w.

Thus the set of positive integers is partitioned, at depth k, into 2^k residue chambers:

$$\mathbb{N} = \bigsqcup_{w \in \{0,1\}^k} \{n \in \mathbb{N} : n \equiv R_w \pmod{2^k}\}.$$

4 Affine residue geometry

For a fixed parity word, the shortcut map composes into an affine transformation.

Definition 4.1 (Affine offset)**.** Let $w = (w_0, \ldots, w_{k-1}) \in \{0,1\}^k$. Define

$$b(w) = \sum_{j=0}^{k-1} w_j \, 2^j \, 3^{w_{j+1}+\cdots+w_{k-1}}, \tag{2}$$

where the empty exponent sum is interpreted as 0. Equivalently, if the 1-positions of w are

$$0 \le d_1 < d_2 < \cdots < d_r \le k-1,$$

then

$$b(w) = \sum_{i=1}^{r} 2^{d_i} 3^{r-i}. \tag{3}$$

Lemma 4.2 (Recursive offset rule)**.** *The offset $b(w)$ is equivalently determined by*

$$b(\emptyset) = 0, \qquad b(u0) = b(u), \qquad b(u1) = 3b(u) + 2^{|u|},$$

where u is a finite prefix word and $|u|$ is its length.

Proof. Appending 0 adds no new odd step, so the affine offset is unchanged. Appending 1 applies the odd branch to the current affine form and introduces the new additive contribution at denominator $2^{|u|+1}$. Unrolling the recurrence gives (2), and grouping the 1-positions gives (3). □

Proposition 4.3 (Affine formula for a parity chamber)**.** *Let $w \in \{0,1\}^k$, let $r = r(w)$, and let $R_w \pmod{2^k}$ be the corresponding parity chamber. For every $n \equiv R_w \pmod{2^k}$,*

$$T^k(n) = T_w(n) = \frac{3^r n + b(w)}{2^k}. \tag{4}$$

Moreover the residue class is characterized by

$$3^r R_w + b(w) \equiv 0 \pmod{2^k}, \tag{5}$$

or equivalently

$$R_w \equiv -3^{-r} b(w) \pmod{2^k},$$

since 3^r is invertible modulo 2^k.

Proof. We prove the affine formula by induction on k. For $k = 0$, the empty word has $r = 0$, $b = 0$, and $T^0(n) = n$.

Assume the formula holds for a prefix u of length k. If the next parity is 0, then

$$T_{u0}(n) = \frac{1}{2} T_u(n) = \frac{3^{r(u)} n + b(u)}{2^{k+1}},$$

so $r(u0) = r(u)$ and $b(u0) = b(u)$. If the next parity is 1, then

$$T_{u1}(n) = \frac{3T_u(n) + 1}{2} = \frac{3(3^{r(u)} n + b(u)) + 2^k}{2^{k+1}} = \frac{3^{r(u)+1} n + 3b(u) + 2^k}{2^{k+1}},$$

so $r(u1) = r(u)+1$ and $b(u1) = 3b(u)+2^k$. This proves the recurrence and hence the affine formula.

For n in the chamber of w, $T^k(n)$ is an integer, so the numerator in (4) is divisible by 2^k. This gives (5). Since 3^r is odd, it is invertible modulo 2^k, giving the stated residue class. □

Remark 4.4 (Important normalization warning). For the fixed-length shortcut-map offset $b(w)$ used here, it is not generally true that $0 \le b(w) < 3^{r(w)}$. For example, $w = 00100$ has $r(w) = 1$ but $b(w) = 2^2 = 4 > 3$. Bounds of the form $0 \le F < 3^r$ occur in other odd-only or Syracuse normalizations and should not be transferred to the fixed-k shortcut parity-word formula without changing notation.

5 The chamber lift

The affine formula suggests a finite coordinate lift at depth k.

Definition 5.1 (Collatz chamber lift)**.** For a word $w \in \{0,1\}^k$, define its chamber-lift coordinates by

$$\mathcal{L}(w) = \big(k, w, r(w), b(w), R_w, G(w)\big),$$

where

$$G(w) = 2^k - 3^{r(w)}$$

is the signed gap associated to the word.

The coordinate $G(w)$ measures the difference between the denominator scale 2^k and the multiplicative odd-step scale $3^{r(w)}$. It is useful to separate three regimes:

$$G(w) > 0 \quad \text{positive-gap regime,}$$

$$G(w) = 0 \quad \text{neutral regime,}$$

$$G(w) < 0 \quad \text{expansive regime.}$$

For $k > 0$, the equality $2^k = 3^{r(w)}$ does not occur. Thus the finite chambers split into positive-gap and expansive chambers.

Definition 5.2 (Positive contraction gap)**.** When $G(w) > 0$, we call $G(w)$ the positive contraction gap of w. When $G(w) < 0$, we do not use the word contraction.

The boundary between the two regimes is controlled by

$$3^r < 2^k \quad \Longleftrightarrow \quad \frac{r}{k} < \frac{\log 2}{\log 3}.$$

Thus the triangular layer diagram indexed by (k, r) has a natural diagonal boundary:

$$r < k\frac{\log 2}{\log 3}.$$

This line is a visual aid, not a proof of global convergence.

6 Descent thresholds

The chamber lift projects back to the one-dimensional descent question through a simple inequality.

Proposition 6.1 (Affine descent threshold)**.** *Let $w \in \{0,1\}^k$ and let $n \equiv R_w \pmod{2^k}$. Then*

$$T^k(n) < n \quad \Longleftrightarrow \quad b(w) < \big(2^k - 3^{r(w)}\big)n.$$

In particular, if $G(w) = 2^k - 3^{r(w)} > 0$, then

$$T^k(n) < n \quad \Longleftrightarrow \quad n > \frac{b(w)}{G(w)}. \tag{6}$$

If $G(w) < 0$, the depth-k chamber cannot certify descent through this inequality.

Proof. Using Proposition 4.3,

$$T^k(n) < n \quad \Longleftrightarrow \quad \frac{3^{r(w)}n + b(w)}{2^k} < n.$$

Multiplying by 2^k and rearranging gives

$$b(w) < \big(2^k - 3^{r(w)}\big)n.$$

If $G(w) > 0$, division by $G(w)$ yields (6). If $G(w) < 0$, the right-hand side of the descent inequality is negative for $n > 0$, while $b(w) \geq 0$, so the inequality cannot hold. □

Remark 6.2 (Offset burden)*.* The term *offset burden* is used informally for $b(w)$ when comparing it with the positive contraction gap $G(w)$. In the positive-gap regime, descent occurs precisely when the contraction gap beats the offset burden at the starting scale n:

$$G(w)n > b(w).$$

This is terminology only; the algebra is classical.

7 Prefix descent and survivors

Depth-k descent should be distinguished from descent *within* k steps. A number can descend below its starting value at a prefix depth $j < k$ even if the length-k terminal word is being studied.

Definition 7.1 (Prefix word)**.** For $w = (w_0, \ldots, w_{k-1})$ and $1 \leq j \leq k$, write

$$w^{(j)} = (w_0, \ldots, w_{j-1})$$

for the prefix of length j. Let

$$r_j = r(w^{(j)}), \qquad b_j = b(w^{(j)}), \qquad G_j = 2^j - 3^{r_j}.$$

Proposition 7.2 (Prefix-survivor characterization)**.** *Let n have length-k parity word w. Then n does not descend below its starting value during the first k shortcut steps if and only if, for every $1 \leq j \leq k$,*

$$T^j(n) \geq n.$$

Equivalently, for every prefix $w^{(j)}$,

$$b_j \geq G_j n$$

whenever $G_j > 0$; prefixes with $G_j < 0$ cannot certify descent.

Proof. For each prefix $w^{(j)}$, Proposition 6.1 gives

$$T^j(n) < n \iff b_j < G_j n.$$

Thus failure to descend at every prefix is exactly the failure of this strict inequality at every positive-gap prefix. If $G_j < 0$, the inequality cannot hold because $b_j \geq 0$ and $n > 0$. □

Remark 7.3 (Finite residue atlas). For any fixed k, Proposition 7.2 gives a finite residue-class atlas: each residue class modulo 2^k has a word w, all prefixes of w have explicit affine data, and descent within k steps is checked by finitely many inequalities. This is the finite residue-sieve viewpoint used in computational verification strategies, restated here in chamber coordinates.

8 Heuristic drift and its limits

The chamber layer r determines the multiplicative part of the affine map:

$$\frac{3^r}{2^k}.$$

Ignoring the additive offset, the logarithmic drift attached to a word is

$$\lambda(w) = r(w)\log 3 - k\log 2.$$

A simple random parity model would expect roughly $r \approx k/2$, producing average per-step drift

$$\frac{1}{2}\log 3 - \log 2 = \frac{1}{2}\log\left(\frac{3}{4}\right) < 0.$$

This explains why the conjecture is statistically plausible in common heuristic models.

Remark 8.1 (Heuristic warning). The negative average drift is not a proof. The uniform distribution of length-k parity words over residue classes modulo 2^k is a finite theorem. It does not imply that parity decisions along a single arbitrary infinite trajectory behave as independent fair coin flips. The gap between density or logarithmic-density statements and the universal Collatz conjecture is the essential difficulty.

9 Relation to known results

This chamber-lift language is intentionally aligned with classical Collatz work.

- The parity-vector encoding and dependence on residues modulo powers of two are classical in Terras and Everett.
- Lagarias's survey and exposition provide the standard parity-vector and affine-iterate framework for the $3x + 1$ problem.
- Wirsching treats the $3n + 1$ map as a dynamical system and records many standard formulations.
- Lagarias and Weiss, and Sinai, study stochastic and statistical models; their work supports heuristic drift discussions, not proofs.

- Krasikov and Lagarias establish quantitative lower bounds for the set of integers known to reach 1.
- Tao proves a logarithmic-density almost-all result for the Collatz map, not the full conjecture.

The present note should be read as a coordinate filing system for these ingredients. It is not a replacement for the standard literature, and it is not a shortcut around the known barriers.

10 Known, packaged, and nonclaimed

Component	**Status**
Shortcut map $T(n)$	Classical
Parity word $w \in \{0,1\}^k$	Classical parity vector
Parity-vector residue bijection modulo 2^k	Classical Terras–Everett machinery
Affine formula $T_w(n) = (3^{r(w)}n + b(w))/2^k$	Classical
Offset formula for $b(w)$	Classical, rewritten in this note's indexing
Binomial layer count $\binom{k}{r}$	Elementary combinatorics
Signed gap $G(w) = 2^k - 3^{r(w)}$	Classical quantity, named here for organization
Positive contraction gap	Terminology for the regime $G(w) > 0$
Offset burden	Informal name for $b(w)$ in descent comparisons
Collatz chamber lift	Visual-coordinate packaging
Proof of the Collatz conjecture	Nonclaim
Improvement of density or computational bounds	Nonclaim
Global parity independence	Nonclaim, and unsafe as a theorem

11 Conclusion

The Collatz map is often presented as a one-dimensional iteration problem. The parity-vector viewpoint already shows that, at any fixed depth, the problem has a finite residue geometry modulo 2^k. This note packages that classical fact as a chamber lift.

At depth k, every parity word w determines a residue chamber, an odd-count layer, an affine offset, and a signed gap. The affine formula

$$T_w(n) = \frac{3^{r(w)}n + b(w)}{2^k}$$

then projects the chamber geometry back to the one-dimensional descent condition

$$b(w) < \left(2^k - 3^{r(w)}\right)n.$$

The resulting picture is not a proof of Collatz. It is a coordinate atlas for classical parity-vector descent machinery: a way to see the known arithmetic as a layered affine residue geometry.

A Symbol dictionary

Symbol	Meaning
T	accelerated Collatz map
n_j	j-th shortcut iterate, $n_{j+1} = T(n_j)$
k	depth, or number of shortcut steps
w	parity word of length k
w_j	parity bit of $T^j(n)$
$r(w)$	number of odd steps in w
$b(w)$	affine offset determined by the parity word
R_w	residue class modulo 2^k corresponding to w
$G(w)$	signed gap $2^k - 3^{r(w)}$
positive contraction gap	the case $G(w) > 0$
$w^{(j)}$	prefix of w of length j
$T_w(n)$	affine k-step update associated to parity word w
parity chamber	residue class modulo 2^k associated to a parity word
chamber lift	coordinate package (k, w, r, b, R_w, G)

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