On the Numerical Value of a Certain Series.

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1. It is known that, if n be uneven, the series

$$\frac{1}{1^n}-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\&c. \qquad (1)$$

admits of finite expression as a series of terms involving the first n Bernoullian numbers, and having π^n as a factor; but when n is even there is no such formula. If therefore the numerical value of the series (1) be required when n is even, there is apparently no method of obtaining it except from the series itself, or from some other infinite series into which it may admit of being transformed.

The most troublesome case to calculate directly is that of $n=2$, as the series

$$1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\&c. \qquad (2)$$

converges very slowly; and the calculation is not very easy even when recourse is had to Euler's formula,

$$\Sigma u_x = \text{const.}+\int u_x\,dx-\tfrac{1}{2}u_x+\frac{B_1}{1\,.\,2}\frac{du_x}{dx}-\frac{B_2}{1\,.\,2\,.\,3\,.\,4}\frac{d^3u_x}{dx^3}+\&c.$$

(see § 5); but I have recently obtained the value of the series (2) to 20 places of decimals in the following manner.

2. We know that, if z lie between 1 and -1,

$$\text{arc}\tan z = z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\frac{z^9}{9}-\&c.,$$

whence, putting $z=\tan t$;

$$t=\tan t-\tfrac{1}{3}\tan^3 t+\tfrac{1}{5}\tan^5 t-\tfrac{1}{7}\tan^7 t+\&c.;$$

and, dividing by $\sin t\cos t$,

$$\frac{t}{\sin t\cos t}=\sec^2 t-\tfrac{1}{3}\tan^2 t\sec^2 t+\tfrac{1}{5}\tan^4 t\sec^2 t-\&c.;$$

whence $$\int_0^x \frac{t}{\sin t\cos t}\,dt=\tan x-\tfrac{1}{9}\tan^3 x+\tfrac{1}{25}\tan^5 x-\&c.,$$

and the integral $$=\int_0^x\frac{2t}{\sin 2t}\,dt=\tfrac{1}{2}\int_0^{2x}\frac{t}{\sin t}\,dt;$$

so that $$\tfrac{1}{2}\int_0^{2x}\frac{t}{\sin t}\,dt=\tan x-\tfrac{1}{9}\tan^3 x+\tfrac{1}{25}\tan^5 x-\&c. \qquad (3).$$

Now $$\operatorname{cosec} t=\frac{1}{t}-\frac{1}{t-\pi}-\frac{1}{t+\pi}+\frac{1}{t-2\pi}+\frac{1}{t+2\pi}-\&c.,$$

whence
$$t \operatorname{cosec} t = 1 - \frac{2t^2}{t^2-\pi^2} + \frac{2t^2}{t^2-(2\pi)^2} - \frac{2t^2}{t^2-(3\pi)^2} + \&c.$$
$$= 1 + \frac{2}{\pi^2}\left(t^2 + \frac{t^4}{\pi^2} + \frac{t^6}{\pi^4} + \frac{t^8}{\pi^6} + \&c.\right)$$
$$- \frac{2}{2^2\pi^2}\left(t^2 + \frac{t^4}{2^2\pi^2} + \frac{t^6}{2^4\pi^4} + \frac{t^8}{2^6\pi^6} + \&c.\right)$$
$$+ \frac{2}{3^2\pi^2}\left(t^2 + \frac{t^4}{3^2\pi^2} + \frac{t^6}{3^4\pi^4} + \frac{t^8}{3^6\pi^6} + \&c.\right)$$
$$- \&c.$$
$$= 1 + 2s_2\frac{t^2}{\pi^2} + 2s_4\frac{t^4}{\pi^4} + 2s_6\frac{t^6}{\pi^6} + 2s_8\frac{t^8}{\pi^8} + \&c.,$$
where
$$s_{2n} = 1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \frac{1}{5^{2n}} - \&c.;$$
and, by integration,
$$\tfrac{1}{2}\int_0^{2x} \frac{t}{\sin t}\,dt = x + \frac{1}{3}\frac{s_2}{\pi^2}(2x)^3 + \frac{1}{5}\frac{s_4}{\pi^4}(2x)^5 + \frac{1}{7}\frac{s_6}{\pi^6}(2x)^7 + \&c. \;\ldots(4);$$
whence, from (3) and (4), on writing $\frac{1}{2}\pi x$ for x,
$$\tan\tfrac{1}{2}\pi x - \tfrac{1}{9}\tan^3\tfrac{1}{2}\pi x + \tfrac{1}{25}\tan^5\tfrac{1}{2}\pi x - \&c.$$
$$= \tfrac{1}{2}\pi\left(x + \tfrac{2}{3}s_2x^3 + \tfrac{2}{5}s_4x^5 + \tfrac{2}{7}s_6x^7 + \&c.\right) \;\ldots\ldots\ldots(5),$$
this equation being true if x does not lie beyond the limits $\frac{1}{2}$ and $-\frac{1}{2}$.

In (5) put $x = \frac{1}{2}$, and we find that
$$1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \&c.$$
$$= \tfrac{1}{2}\pi\left(\frac{1}{2} + \frac{1}{3}\frac{s_2}{2^2} + \frac{1}{5}\frac{s_4}{2^4} + \frac{1}{7}\frac{s_6}{2^6} + \&c.\right).$$

3. If we write
$$s'_{2n} = \frac{1}{2^{2n}} - \frac{1}{3^{2n}} + \frac{1}{4^{2n}} - \frac{1}{5^{2n}} + \&c.,$$
so that
$$s'_{2n} = 1 - s_{2n},$$
then
$$\frac{1}{2} + \frac{1}{3}\frac{s_2}{2^2} + \frac{1}{5}\frac{s_4}{2^4} + \frac{1}{7}\frac{s_6}{2^6} + \&c.$$
$$= \frac{1}{2} + \frac{1}{3}\frac{1}{2^2} + \frac{1}{5}\frac{1}{2^4} + \frac{1}{7}\frac{1}{2^6} + \&c.$$
$$- \frac{1}{3}\frac{s'_2}{2^2} - \frac{1}{5}\frac{s'_4}{2^4} - \frac{1}{7}\frac{s'_6}{2^6} - \&c.$$
$$= -\frac{1}{2} + \log\frac{1+\frac{1}{2}}{1-\frac{1}{2}} - \frac{1}{3}\frac{s'_2}{2^2} - \frac{1}{5}\frac{s'_4}{2^4} - \&c.$$
$$= -\frac{1}{2} + \log 3 - \frac{1}{3}\frac{s'_2}{2^2} - \frac{1}{5}\frac{s'_4}{2^4} - \frac{1}{7}\frac{s'_6}{2^6} - \&c. \;\ldots.(6).$$

Now the values of $s_2, s_4, \ldots s_{12}$, to 22 places, were contained in a paper* communicated to the Society at the last meeting (January 11, 1877); and on p. 56 of t. iv. of the *Proceedings*, the values of $s_{14}, \ldots s_{34}$ are given to 16 places. From these the values of $s'_2, s'_4, \ldots s'_{32}$ are at once derived; and by means of them the value of the series in (6) can be calculated to 20 places. For the term involving s'_{14} (which is that from which the smallest number of decimals is obtainable) is

$$0{\cdot}00006\ 08296\ 54020\ 3 \div (2\ 45760),$$

so that 21 places of the value of this term should be correct.

On performing the divisions the values of the terms are found to be as below:

0 . 01479	44138	81323	89848	03	(s_2)
66	20896	31284	42603	03	(s_4)
3	22520	28353	93949	98	(s_6)
	16349	81834	78823	82	(s_8)
	852	70987	36935	67	(s_{10})
	45	50684	64809	59	(s_{12})
	2	47516	49585	08	(s_{14})
		13675	27240	88	(s_{16})
		765	37491	45	(s_{18})
		43	29628	80	(s_{20})
		2	47112	33	(s_{22})
			14210	01	(s_{24})
			822	37	(s_{26})
			47	85	(s_{28})
			2	80	(s_{30})
				16	(s_{32})
				1	(s_{34})
0 . 01549	04806	06472	13111	86	

where a line marked (s_{2n}) contains the value of the term

$$\frac{1}{2n+1}\,\frac{s'_{2n}}{2^{2n}}.$$

The values are given to 22 places, but, as stated above, that of $\frac{1}{15}\,\frac{s'_{14}}{2^{14}}$ is not necessarily correct to more than 21 places.

We know that

$$\log 3 = 1.09861\ 22886\ 68109\ 69139\ 52\ \ldots,$$

* Page 145 of the present volume.

so that we obtain for the expression in (6) the value

$$0.58312\ 18080\ 61637\ 56027\ 66\ldots \qquad \ldots(7);$$

and, therefore, rejecting the last two figures, we find that, to 20 places,

$$\frac{2}{\pi}\left(1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\&c.\right)$$
$$= 0.58312\ 18080\ 61637\ 56028. \qquad \ldots(8).$$

On multiplying (7) by $\frac{1}{2}\pi$, and retaining 20 figures, the value of the series (2) is obtained, viz., we have

$$1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\&c.$$
$$= 0.91596\ 55941\ 77219\ 01505\ \ldots$$

4. Also, since

$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\&c. = \tfrac{1}{8}\pi^2$$
$$= 1.23370\ 05501\ 36169\ 82735\ \ldots,$$

we find that, to 20 places,

$$\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\&c. = 1.07483\ 30721\ 56694\ 42120,$$

$$\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{15^2}+\&c. = 0.15886\ 74779\ 79475\ 40615.$$

5. In the "Messenger of Mathematics," t. vi., pp. 75, 76, I have given ten-place values of these two series, calculated from the formulæ

$$\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}\ldots+\frac{1}{(4x-3)^2} = C_1-\frac{1}{4}\cdot\frac{1}{4x+1}-\frac{1}{2}\cdot\frac{1}{(4x+1)^2}$$
$$-\frac{2}{3}\cdot\frac{1}{(4x+1)^3}+\frac{32}{15}\cdot\frac{1}{(4x+1)^5}-\&c.\ldots(9),$$

$$\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}\ldots+\frac{1}{(4x-1)^2} = C_2-\frac{1}{4}\cdot\frac{1}{4x+3}-\frac{1}{2}\cdot\frac{1}{(4x+3)^2}$$
$$-\frac{2}{3}\cdot\frac{1}{(4x+3)^3}+\&c.\ldots\ldots\ldots(10).$$

Each was the result of a quadruple calculation, viz., putting in (9) $4x+1 = 101, 201, 301, 317$ respectively, the four values found for C_1 were

1.07483 3072, 1.07483 3073, 1.07483 3072, 1.07483 3071;

and putting, in (10), $4x+3 = 103, 203, 303, 319$ respectively, the four values found for C_2 were

0.15886 7479, 0.15886 7479, 0.15886 7478, 0.15886 7478.

The values adopted for the series were:

$$\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\text{\&c. } \textit{ad inf.} = 1.07483\ 3072,$$

$$\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+\text{\&c. } \textit{ad inf.} = 0.15886\ 7478;$$

and these are seen, on comparing them with the twenty-place values given above in § 4, to be correct to the last place, as also is the value of the series (2) which was given as 0.91596 5594.

6. In the paper in the "Messenger" cited in the last section, it is shown that

$$\frac{3^3 \cdot 7^7 \cdot 11^{11} \ldots (4n-1)^{4n-1}}{1^1 \cdot 5^5 \cdot 9^9 \ldots (4n-3)^{4n-3}} = (4n+1)^{2n}\, e^{-\frac{1}{2}+\frac{2}{\pi}\left(1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\text{\&c.}\right)} \ldots (11),$$

and it was in order to obtain the value of the constant in this formula that I was led to calculate the value of the series

$$1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\text{\&c.}$$

We may write (11) in the slightly more convenient form

$$\frac{3^3 \cdot 7^7 \cdot 11^{11} \ldots (4n-1)^{4n-1}}{1^1 \cdot 5^5 \cdot 9^9 \ldots (4n-3)^{4n-3}} = (4n)^{2n}\, e^{\frac{2}{\pi}\left(1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\text{\&c.}\right)},$$

which, using the value in (7),

$$= (4n)^{2n}\, e^{0.58312\ 18080\ 61637\ 56027 \ldots}$$

7. In connexion with the formulæ in § 2 it may be noted (see "Messenger," *loc. cit.*) that

$$\int_0^x \frac{t}{\sin t}\, dt = \sin x + \frac{2}{3}\,\frac{\sin^3 x}{3} + \frac{2 \cdot 4}{3 \cdot 5}\,\frac{\sin^5 x}{5} + \text{\&c.},$$

so that each side of the identical equation (5) is also

$$= \frac{1}{2}\left(\sin \pi x + \frac{2}{3}\,\frac{\sin^3 \pi x}{3} + \frac{2 \cdot 4}{3 \cdot 5}\,\frac{\sin^5 \pi x}{5} + \text{\&c.}\right)$$