ON THE MINORS OF A SKEW-SYMMETRICAL DETERMINANT

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1. The most general type of determinant that can arise as a minor of a skew-symmetrical determinant may be expressed in the form

$$\begin{vmatrix} [1x_1], & [1x_2], & \dots, & [1x_n] \\ [2x_1], & [2x_2], & \dots, & [2x_n] \\ \dots & \dots & \dots & \dots \\ [nx_1], & [nx_2], & \dots, & [nx_n] \end{vmatrix},$$

where we have, generally, $[rx_s]+[x_s r] = 0$.

In the most general type the two sets of numbers $1, 2, 3, \dots, n$ and $x_1, x_2, x_3, \dots, x_n$ are distinct, but all special cases may be deduced by making pairs out of the two sets identical.

Now we have

$$\begin{vmatrix} [1x_1], & [1x_2] \\ [2x_1], & [2x_2] \end{vmatrix} = [1x_1][2x_2]-[1x_2][2x_1] = -[12x_1x_2]+[12][x_1x_2].$$

Further, the determinant

$$\begin{vmatrix} [1x_1], & [1x_2], & [1x_3] \\ [2x_1], & [2x_2], & [2x_3] \\ [3x_1], & [3x_2], & [3x_3] \end{vmatrix}$$

is equal to

$$[1x_1]\{-[23x_2x_3]+[23][x_2x_3]\} -[1x_2]\{-[23x_1x_3]+[23][x_1x_3]\}$$
$$+[1x_3]\{-[23x_1x_2]+[23][x_1x_2]\}$$
$$= -[123x_1x_2x_3]+[12][3x_1x_2x_3]-[13][2x_1x_2x_3]+[23][1x_1x_2x_3].$$

Similarly, the determinant of four rows and columns is equal to

$$\begin{aligned}
&[1x_1]\{-[234x_2x_3x_4]+[23][4x_2x_3x_4]-[24][3x_2x_3x_4]+[34][2x_2x_3x_4]\}\\
&-[1x_2]\{-[234x_1x_3x_4]+[23][4x_1x_3x_4]-[24][3x_1x_3x_4]+[34][2x_1x_3x_4]\}\\
&+[1x_3]\{-[234x_1x_2x_4]+[23][4x_1x_2x_4]-[24][3x_1x_2x_4]+[34][2x_1x_2x_4]\}\\
&-[1x_4]\{-[234x_1x_2x_3]+[23][4x_1x_2x_3]-[24][3x_1x_2x_3]+[34][2x_1x_2x_3]\}\\
&=[1234x_1x_2x_3x_4]-\{[12][34x_1x_2x_3x_4]-[13][24x_1x_2x_3x_4]\\
&\qquad+[14][23x_1x_2x_3x_4]+[23][14x_1x_2x_3x_4]-[24][13x_1x_2x_3x_4]\\
&\qquad+[34][12x_1x_2x_3x_4]\}+[1234][x_1x_2x_3x_4].
\end{aligned}$$

Proceeding in this manner, we obtain, for the determinant of five rows and columns, the expression

$$\begin{aligned}
&[12345x_1x_2x_3x_4x_5]-\{[12][345x_1x_2x_3x_4x_5]-[13][245x_1x_2x_3x_4x_5]+\dots\\
&\qquad+[45][123x_1x_2x_3x_4x_5]\}\\
&\qquad+\{[1234][5x_1x_2x_3x_4x_5]-[1235][4x_1x_2x_3x_4x_5]+\dots\\
&\qquad+[2345][1x_1x_2x_3x_4x_5]\}.
\end{aligned}$$

Also, for that of six rows and columns, we should have

$$\begin{aligned}
&-[123456x_1x_2x_3x_4x_5x_6]\\
&+\{[12][3456x_1x_2x_3x_4x_5x_6]-\dots+[56][1234x_1x_2x_3x_4x_5x_6]\}\\
&-\{[1234][56x_1x_2x_3x_4x_5x_6]-\dots+[3456][12x_1x_2x_3x_4x_5x_6]\}\\
&+[123456][x_1x_2x_3x_4x_5x_6].
\end{aligned}$$

We are now in a position to state the general rule deducible from these special cases. For the determinant of n rows and columns we have an expression of the form

$$\begin{aligned}
&\pm\{[123\dots nx_1x_2\dots x_n]\\
&\quad-\Sigma\pm[rs][12\dots(r-1)(r+1)\dots(s-1)(s+1)\dots nx_1x_2\dots x_n]\\
&\quad+\Sigma\pm[pqrs][12\dots(p-1)(p+1)\dots(q-1)(q+1)\dots(r-1)(r+1)\dots\\
&\qquad\dots(s-1)(s+1)\dots nx_1x_2\dots x_n]-\dots\}.^* \quad (1)
\end{aligned}$$

* To obtain the sign attached to any particular product under one of the Σ's, we have the following rule:—Write down the numbers in the first Pfaffian of the product, and after them write down such of the numbers 1, 2, ..., n as occur in the second Pfaffian, preserving in each case the order in which the said numbers stand in the Pfaffians. Then, according as the number of displacements requisite to restore the numbers so obtained to their natural order be odd or even, so is the sign of the product negative or positive.

This expression needs some further explanation. Firstly, with regard to the ambiguous sign outside the double bracket, we have the following rule :—If n be even, the negative or positive sign is to be chosen according as $\frac{1}{2}n$ is odd or even; if n be odd, the negative or positive sign is to be chosen according as $\frac{1}{2}(n-1)$ is odd or even.*

For the typical sum-term within the double bracket, we have

$$(-1)^k \Sigma \pm [p_1 p_2 \dots p_{2k}]\, C[p_1 p_2 \dots p_{2k}],$$

where $p_1, p_2, \dots, p_{2k}$ are a set of $2k$ numbers, standing in their natural order, selected from the set $1, 2, \dots, n$, and the symbol $C[p_1 p_2 \dots p_{2k}]$ denotes the Pfaffian whose symbolical expression can be formed from the set of numbers $1, 2, \dots, n, x_1, x_2, \dots, x_n$ by leaving out the set $p_1, p_2, \dots, p_{2k}$.

If n be an even number, the last term within the bracket is

$$(-1)^{\frac{1}{2}n} [123 \dots n][x_1 x_2 x_3 \dots x_n].$$

If n be an odd number, then we have, for the last sum-term, the expression

$$(-1)^{\frac{1}{2}(n-1)} \Sigma \pm [123 \dots (p-1)(p+1) \dots n][p x_1 x_2 x_3 \dots x_n].$$

I propose to give a justification of the generality of this result by the method of induction. Thus, supposing the result to be true in the case of the determinant of n rows and columns, we proceed to establish its correctness for the case of the determinant of $n+1$ rows and columns. All the minors of this latter determinant will be expressible in the form (1); and, as the sign outside the double bracket will be the same for all, we may leave this out of account for the present.

Now
$$\begin{aligned}
&[1x_1][234 \dots (n+1) x_2 x_3 x_4 \dots x_{n+1}] \\
&-[1x_2][234 \dots (n+1) x_1 x_3 x_4 \dots x_{n+1}] \\
&+\dots \\
&+(-1)^n [1x_{n+1}][234 \dots (n+1) x_1 x_2 x_3 \dots x_n] \\
= \; &(-1)^n \{ \quad [123 \dots (n+1) x_1 x_2 x_3 \dots x_{n+1}] \\
&\qquad -[12][345 \dots (n+1) x_1 x_2 x_3 \dots x_{n+1}] \\
&\qquad +[13][245 \dots (n+1) x_1 x_2 x_3 \dots x_{n+1}] \\
&\qquad -\dots \\
&-(-1)^{n+1} [1(n+1)][234 \dots n x_1 x_2 x_3 \dots x_{n+1}] \}. \qquad (2)
\end{aligned}$$

* The referee points out that this sign may be accounted for by the inclusion of the factor $(-1)^{\frac{1}{2}n(n-1)}$.

This supplies us with the first term of the new expression, and with those portions of the first sum-term for which $r = 1$. Further, considering the typical term out of those composing the first sum-term, we have

$$[1x_1][234\dots(r-1)(r+1)\dots(s-1)(s+1)\dots(n+1)\,x_2x_3x_4\dots x_{n+1}]$$
$$-[1x_2][234\dots(r-1)(r+1)\dots(s-1)(s+1)\dots(n+1)\,x_1x_3x_4\dots x_{n+1}]$$
$$+\dots$$
$$+(-1)^n[1x_{n+1}][234\dots(r-1)(r+1)\dots(s-1)(s+1)\dots(n+1)\,x_1x_2x_3\dots x_n]$$
$$= (-1)^n\{\quad[123\dots(r-1)(r+1)\dots(s-1)(s+1)\dots(n+1)\,x_1x_2x_3\dots x_{n+1}]$$
$$-[12][345\dots(r-1)(r+1)\dots(s-1)(s+1)\dots(n+1)\,x_1x_2x_3\dots x_{n+1}]$$
$$+[13][245\dots(r-1)(r+1)\dots(s-1)(s+1)\dots(n+1)\,x_1x_2x_3\dots x_{n+1}]$$
$$-\dots$$
$$-(-1)^{n-1}[1(n+1)]$$
$$\times[234\dots(r-1)(r+1)\dots(s-1)(s+1)\dots n\,x_1x_2x_3\dots x_{n+1}]\}. \quad (3)$$

In this expression there will be no terms containing $[1r]$ or $[1s]$. The alternation of the positive and negative signs attached to the terms will, however, be steady throughout the expression.

It will thus be seen that, taking the latter terms on the right-hand side of equation (2), and all those included under the typical first term of the right-hand side of equation (3), we shall have the first sum-term of our new expression complete, viz.,

$$(-1)^n\{-\Sigma \pm [rs][123\dots(r-1)(r+1)\dots(s-1)(s+1)\dots$$
$$\dots(n+1)\,x_1x_2x_3\dots x_{n+1}]\}.$$

The remaining terms on the right-hand side of equation (3) will go towards the formation of the second sum-term. Thus, collecting all the terms containing

$$[234\dots(q-1)(q+1)\dots(r-1)(r+1)\dots(s-1)(s+1)\dots(n+1)\,x_1x_2x_3\dots x_{n+1}],$$

we have, for the coefficient of this expression,

$$(-1)^{n+1}\{(-1)^{q-1}[1q](-1)^{r+s-5}[rs]+(-1)^{r-2}[1r](-1)^{q+s-5}[qs]$$
$$+(-1)^{s-3}[1s](-1)^{q+r-5}[qr]\}$$
$$= (-1)^{n+1}(-1)^{q+r+s-6}\{[1q][rs]-[1r][qs]+[1s][qr]\}$$
$$= (-1)^n(-1)^{q+r+s-5}[1qrs] = (-1)^n(-1)^{q+r+s-9}[1qrs].$$

Thus these terms give us that portion of our next sum-term for which $p = 1$, viz.,

$$(-1)^n \{+\Sigma \pm [1qrs][123 \dots (q-1)(q+1) \dots (r-1)(r+1) \dots (s-1)(s+1) \dots$$
$$\dots (n+1)\, x_1 x_2 x_3 \dots x_{n+1}]\}.$$

Now, dealing with the typical sum-term, and making a slight and obvious modification of our notation, we have, on consideration of the coefficient of $[p_1 p_2 \dots p_{2k}]$,

$$[1x_1]\, C[1p_1 p_2 \dots p_{2k} x_1] - [1x_2]\, C[1p_1 p_2 \dots p_{2k} x_2] + \dots$$
$$+(-1)^n [1x_{n+1}]\, C[1p_1 p_2 \dots p_{2k} x_{n+1}]$$
$$= (-1)^n \{C[p_1 p_2 \dots p_{2k}] - [12]\, C[12p_1 p_2 \dots p_{2k}]$$
$$+[13]\, C[13p_1 p_2 \dots p_{2k}] - \dots$$
$$+(-1)^{n-2k} [1(n+1)]\, C[1p_1 p_2 \dots p_{2k} (n+1)]\}.$$

The contribution of the typical sum-term of expression (1) to the sum-term of the same order in our new expression is thus clear. It only remains to consider its contribution to the sum-term of the next order. Considering the portion containing $C[p_1 p_2 \dots p_{2k+1}]$, we have, for the coefficient of this expression,

$$(-1)^n (-1)^k \{\ (-1)^{p_1-1} [1p_1] (-1)^{h-p_1} [p_2 p_3 \dots p_{2k+1}]$$
$$+(-1)^{p_2-2} [1p_2] (-1)^{h-p_2} [p_1 p_3 p_4 \dots p_{2k+1}]$$
$$+(-1)^{p_3-3} [1p_3] (-1)^{h-p_3} [p_1 p_2 p_4 p_5 \dots p_{2k+1}]$$
$$+ \dots$$
$$+(-1)^{p_{2k+1}-(2k+1)} [1p_{2k+1}] (-1)^{h-p_{2k+1}} [p_1 p_2 \dots p_{2k}]\}$$
$$= (-1)^n (-1)^k (-1)^{h-1} \{[1p_1][p_2 p_3 \dots p_{2k+1}] - [1p_2][p_1 p_3 p_4 \dots p_{2k+1}]$$
$$+ \dots$$
$$+[1p_{2k+1}][p_1 p_2 \dots p_{2k}]\}$$
$$= (-1)^n (-1)^{k+1} (-1)^{h-2} [1p_1 p_2 \dots p_{2k+1}],$$

where $$h = p_1 + p_2 + \dots + p_{2k+1} - k(2k+3).$$

Further, $$h-2 = p_1 + p_2 + \dots + p_{2k+1} - (2k^2+3k+2),$$

and therefore

$$h - 2(k+1) = p_1 + p_2 + \dots + p_{2k+1} - (2k+1)(k+2).$$

Also $$(-1)^{h-2(k+1)} = (-1)^{h-2}(-1)^{2k} = (-1)^{h-2}.$$

Thus our expression becomes

$$(-1)^n(-1)^{k+1}(-1)^{p_1+p_2+\dots+p_{2k+1}-(2k+1)(k+2)}[1p_1 p_2 \dots p_{2k+1}].$$

Thus we obtain the contribution of these terms to the next sum-term, and observe that it is of the required form.

The only point remaining to be considered is the ambiguous sign outside the double bracket. We have shown that, for the determinant of $n+1$ rows and columns, we have an expression of the proper type affected with the sign $(-1)^n$. Further, if n be an even number, the ambiguous sign outside the double bracket in the expression (1) may be replaced by $(-1)^{\frac{1}{2}n}$. Thus our new expression is affected with the sign

$$(-1)^n(-1)^{\frac{1}{2}n} = (-1)^{\frac{1}{2}n} = (-1)^{\frac{1}{2}(m-1)},$$

where $m = n+1$, since n is even. As m is an odd number, this is of the proper form.

Taking n for an odd number, the above mentioned ambiguous sign may be replaced by $(-1)^{\frac{1}{2}(n-1)}$, and in this case our new expression will be affected with the sign

$$(-1)^n(-1)^{\frac{1}{2}(n-1)} = (-1)^{\frac{1}{2}(n+1)} = (-1)^{\frac{1}{2}m},$$

since n is odd. Further, m now denoting an even number, this also will be of the requisite form.

2. Having established the law for the general case, we proceed to obtain a few verifications, by application to special cases. It is a general rule that, if in the series of numbers involved in the symbolical expression of a Pfaffian any number be repeated, the corresponding expression vanishes. Thus, in some of the special cases our general expression becomes much reduced. Taking, for example, the case of the skew-symmetrical determinant of n rows and columns, we obtain this case by putting $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, ..., $x_n = n$; and we see that, if n be odd, it vanishes entirely. Further, if n be even, it reduces to $(-1)^n \{[123\dots n]\}^2$, that is, to $\{[123\dots n]\}^2$.

We next apply our result to the first minors of a skew-symmetrical determinant of n rows and columns, considering the minor of $[rs]$. In this case we shall have a determinant of $n-1$ rows and columns, obtained from the typical form by assuming $x_1 = 1$, $x_2 = 1$, ..., $x_{s-1} = s-1$,

$x_s = s+1$, $x_{s+1} = s+2$, ..., $x_{n-1} = n$, and by replacing the numbers r, $r+1$, ..., $n-1$ by the respective numbers $r+1$, $r+2$, ..., n.

If n be an even number, the expression for our minor becomes

$$(-1)^{\frac{1}{2}(n-2)}(-1)^{\frac{1}{2}(n-2)}(-1)^{n-s}[123 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n] \times [s123 \dots (s-1)(s+1) \dots n]$$

or $$(-1)^{\frac{1}{2}(n-2)}(-1)^{\frac{1}{2}(n-2)}(-1)^{n-s}[123 \dots (s-1)(s+1) \dots (r-1)(r+1) \dots n] \times [s123 \dots (s-1)(s+1) \dots n].$$

Now $$[s123 \dots (s-1)(s+1) \dots n] = (-1)^{s-1}[123 \dots n]$$

and $$(-1)^{\frac{1}{2}(n-2)}(-1)^{\frac{1}{2}(n-2)}(-1)^{n-s}(-1)^{s-1} = (-1)^{2n-3}.$$

Thus our minor is equivalent to either

$$-[123 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n][123 \dots n]$$

or $$-[123 \dots (s-1)(s+1) \dots (r-1)(r+1) \dots n][123 \dots n].$$

If n be an odd number, we obtain for our minor

$$(-1)^{\frac{1}{2}(n-1)}(-1)^{\frac{1}{2}(n-1)}[123 \dots (r-1)(r+1) \dots n][123 \dots (s-1)(s+1) \dots n]$$
$$= [123 \dots (r-1)(r+1) \dots n][123 \dots (s-1)(s+1) \dots n].$$

As a further illustration we will consider the minor of

$$\begin{vmatrix} [pr], & [ps] \\ [qr], & [qs] \end{vmatrix}$$

In this case we have a determinant of $n-2$ rows and columns, obtained by putting $x_1 = 1$, $x_2 = 2$, ..., $x_{r-1} = r-1$, $x_r = r+1$, $x_{r+1} = r+2$, ..., $x_{s-2} = s-1$, $x_{s-1} = s+1$, $x_s = s+2$, ..., $x_{n-2} = n$, and by replacing the numbers $p, p+1, \dots, q-2$ by the respective numbers $p+1, p+2, \dots, q-1$, and the numbers $q-1$, q, ..., $n-2$ by the respective numbers $q+1$, $q+2$, ..., n.

For simplicity we will take the case in which p, q, r, s are in order of magnitude. In this case, if n be even, we have

$$(-1)^{\frac{1}{2}(n-2)}\{(-1)^{\frac{1}{2}(n-4)}(-1)^{n-(r+s)+3}$$
$$\times [123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n]$$
$$\times [rs123 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n]$$
$$+(-1)^{\frac{1}{2}(n-2)}[123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots n]$$
$$\times [123 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n]\}$$
$$= -[123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n]$$
$$\times [123 \dots n] + [123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots n]$$
$$\times [123 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n].$$

On the other hand, if n be odd, we have

$$(-1)^{n-3}\{(-1)^{n-2}[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots n]\times[123\ldots(s-1)(s+1)\ldots n]$$
$$+(-1)^{n-1}[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(s-1)(s+1)\ldots n]\times[123\ldots(r-1)(r+1)\ldots n]$$
$$=-[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(r-1)(r+1)\ldots n]\times[123\ldots(s-1)(s+1)\ldots n]$$
$$+[123\ldots(p-1)(p+1)\ldots(q-1)(q+1)\ldots(s-1)(s+1)\ldots n]\times[123\ldots(r-1)(r+1)\ldots n].$$

In a similar manner we could determine those cases in which either or both of the numbers p and q are greater than either or both of the numbers r and s.

3. In the thirty-fourth volume of the *Proceedings* of the Society* I gave the following theorem :—

$$[12y_1z_1\ldots y_mz_m][345\ldots(2n)y_1z_1\ldots y_mz_m]$$
$$-[13y_1z_1\ldots y_mz_m][245\ldots(2n)y_1z_1\ldots y_mz_m]$$
$$+\ldots$$
$$+(-1)^{2n}[1(2n)y_1z_1\ldots y_mz_m][234\ldots(2n-1)y_1z_1\ldots y_mz_m]$$
$$=[y_1z_1\ldots y_mz_m][123\ldots(2n)y_1z_1\ldots y_mz_m].$$

This theorem leads directly to a generalization of a theorem given by Baker in a former volume of the *Proceedings*.† Thus, if we expand the Pfaffian $[123\ldots(2n)]$, and replace each element of the type $[rs]$ by an element of the form $[rsy_1z_1\ldots y_mz_m]$, then the value of the resulting expression will be

$$[y_1z_1\ldots y_mz_m]^{n-1}[123\ldots(2n)y_1z_1\ldots y_mz_m].$$

We can also apply our first quoted theorem to our typical determinant,

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* P. 149. I find that I had been anticipated in some of the theorems of this paper by Vivanti. See *Rendiconti del Circolo Matematico di Palermo*, t. XII., pp. 1–20.

† Vol. XXIX., p. 141.

and obtain an expression for the determinant derived from it by replacing each element of the type $[rx_s]$ by an element of the form $[rx_s y_1 z_1 \dots y_m z_m]$. The resulting expression will be of the form

$$\pm\Big[[123 \dots n x_1 x_2 \dots x_n y_1 z_1 \dots y_m z_m] \{[y_1 z_1 \dots y_m z_m]\}^{n-1}$$
$$+ \{[y_1 z_1 \dots y_m z_m]\}^{n-2} \{-\Sigma \pm [rs y_1 z_1 \dots y_m z_m]$$
$$\times [12 \dots (r-1)(r+1) \dots (s-1)(s+1) \dots n x_1 x_2 \dots x_n y_1 z_1 \dots y_m z_m] + \dots\}\Big].$$