The Syllabus of Geometrical Conics

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THE SYLLABUS OF GEOMETRICAL CONICS.

THE Sixth General Meeting of the Association for the Improvement of Geometrical Teaching appointed a Committee for Geometrical Conics, and one for "Higher Plane Geometry, including such subjects as Transversals, Projection, Anharmonic Ratio, etc."

Dr. Hirst in his Presidential Address has said of the latter subjects, "Until these notions become more familiar ones, I, for my part, believe that Geometrical Conics will always remain in its present unsatisfactory condition. . . . It will be, of course, a question for this Association to decide whether, pending the introduction of the more thorough treatment based on the notions to which I have alluded, some improvement may not be introduced into the subject of Geometrical Conics, as at present understood, with a view of enabling examiners at all events to examine with greater facility and purpose. At present the definitions are so multiform and the sequence of propositions so varied in different text-books, that it is found to be an exceedingly difficult task to examine satisfactorily in the subject at all."

Before considering how to give effect to the principle of modernising the subject in a practical way, let me deprecate any narrowness of view with respect to elementary geometry. It is not merely a first step to a higher geometry. It is also, beside being an instrument of mental culture, an introduction to applied mathematics and the solution of physical problems; and the clear insight into the laws and processes of nature enjoyed by Kepler and Newton, and in our own days by such men as Adams and Maxwell, is inseparable from their previous mastery of the geometry of Euclid, Archimedes, and Apollonius.

That I should be proposing just now to discuss the Association's Syllabus of Geometrical Conics is owing partly to the fact that I have, as I think, arrived at something like finality in my own view of the way in which the subject should be approached. I may add that the Committee, before reporting, did me the honour to ask me to join it as a co-opted member, so that I have been, as I may say, invited to express an opinion on the subject. It seems to me that I can do this rather better now than I could have done then.

To go back to the beginning of my work at geometrical conics. The first thing that I wrote, namely in 1862, was an article in the *Oxford, Cambridge, and Dublin Messenger of Mathematics* (vol. i. p. 250), which commenced with the remark that the results of orthogonal projection can be proved in one plane, namely, by what the A. I. G. T. Syllabus aptly terms *Reduction of Ordinates*. This is, I am sure, a safer construction for the beginner, but orthogonal projection must of course come in somewhere. Let me read a paragraph from the Thirty-Sixth Annual Report of the Cambridge Local Examinations and Lectures Syndicate, dated 10th March 1894 (see the *Cambridge University Reporter* for 1893-94, p. 596).

"*Conic Sections.* Seniors:—A few very good papers were sent up in this subject, but most of the candidates were only able to write out one or two of the pieces of book-work. Many candidates attempted to prove one of the geometrical theorems by projections; most of these proofs were, however, valueless, for all that was done was to prove at unnecessary length a well-known property of the circle, no appeal being made to any of the fundamental principles of projection."

The same article in the *Oxford, Cambridge, and Dublin Messenger* contains the proposition, which I had not seen stated anywhere, that, at any point of a central conic,—

The projection of the normal, terminated by the minor axis, upon either focal distance, is equal to half the major axis.*

Let the circle through the foci S, H, and a point P on the curve cut the minor axis in g and t.

The lines Pg, Pt are the bisectors of the angle SPH.

Let Pg be the internal bisector, and let the conic be an ellipse. Then Pg is the normal at P.

Draw gE, gK normal † to SP, HP respectively.

Then evidently $g\mathrm{E} = g\mathrm{K}$.

* Note that the square of the true conjugate axis of a hyperbola is negative, and therefore less than the square of the transverse axis.

† The expression "normal" is a convenient substitute for "perpendicular," and there is precedent for its use in this sense.

Also $$gS = gH.$$

Hence, in the right-angled triangles gES, gKH,
$$SE = HK,$$
or $$SP - PE = PK - HP.$$

Hence, PK being equal to PE,
$$PE \text{ (or } PK) = \tfrac{1}{2}(SP + HP).$$

Next, let the conic be a hyperbola.

In this case Pt is the normal at P.

Draw tE', tK' normal to SP, HP; then it may be shown similarly that
$$PE' \text{ (or } PK') = \tfrac{1}{2}(SP \sim HP).$$

In either case the projection of the normal upon SP or HP is equal to CA, or half the major axis.

Corollaries.

1. Given that EK is the diameter parallel to the tangent at P, it follows by right-angled triangles, if Pg, EK cross at F, that
$$PF \cdot Pg = PE^2 = CA^2.$$

2. Given, further, that the projection PL of the normal terminated by the major axis upon SP is equal to CB^2/CA, then similarly
$$PF \cdot PG = PE \cdot PL = CB^2.$$

3. By similar triangles, if CD be the radius conjugate to CP,
$$PG \cdot Pg = SP \cdot HP = CD^2.$$

4. If PN be the ordinate of P, and e denote the eccentricity, then, because CG, SE and CS, PE subtend equal angles at g,
$$CG : SE = CS : PE,$$
or $$CG = e \cdot SE = e^2 \cdot CN.$$

In 1863 was published my *Geometrical Conics, including Anharmonic Ratio and Projection*, in which a chapter upon "Conics" in general came before the chapters on the three separate curves. The chapter on "Conics" began with the tangent, and gave the well-known construction for tangents by Professor Adams.

Looking back at an article on "The Principles of Geometrical Conics," in the *O. C. D. Messenger*, vol. v. (1871), I find this note on what I called Adams' theorem: "I have varied the proof. That given by the discoverer being indirect."

Professor Adams, as I am reminded by this note, took Euclid's definition of a tangent, and obtained some of his results in a very indirect way.*

From any point T on a certain line through P on a conic he drew (Fig. 2) perpendiculars TM, TN to SP and the directrix, and proved that
$$SM = e \cdot TN.$$

Hence, ST being greater than TN, and T therefore lying without the conic, except when T is at P, the line TP is the tangent at P.

To draw the tangents from a given point T, draw the circle with centre S and radius equal to $e \cdot TN$; draw tangents TM, TM′ to the circle, and SR, SR′ parallel to them to meet the directrix. This may be shortened a little by calling the circle the Adams circle.

Then TR, TR′ are the required tangents; and TP, TP′ subtend equal (or supplementary) angles at S, because in the circle the angles TSM, TSM′ are equal.

This is a variation upon the construction of Boscovich, now well known to members of the Association through Mr. Langley's articles upon the "Eccentric Circle," as I have proposed to call Boscovich's circle.

The general chapter on conics in my *Geometrical Conics* of 1863 contained a proof of the property of diameters, of which the following was the original form.

Let PQ be a chord given in direction, O its middle point, R its intersection with the directrix (Fig. 3).

Draw SY normal to the chord, and let it meet the directrix in D.

Then $$SP : PR = SQ : QR = \text{a constant ratio } (c).$$

Hence $$c^2 (PR^2 \sim QR^2) = SP^2 \sim SQ^2 = PY^2 \sim QY^2.$$

This reduces to $$OY/OR = c^2;$$

whence it follows that the locus of O is a straight line through D.

This led up to a new proof that
$$CV \cdot CT = CP^2,$$
including PV = PT in the parabola.

I will now submit some observations upon the Syllabus, beginning with its traditional treatment of the diameters of a hyperbola which do not meet the curve.

The Conjugate Hyperbola.

Mr. Charles Smith, Master of Sidney Sussex College, writes in the preface to his recently published *Geometrical Conics* (1894): "I have discarded the usual method of treating a hyperbola as if it were *two* conics, namely, the curve itself and the conjugate hyperbola. This will, I hope, meet with the approval of teachers."

But in this there is no novelty. In the preface to edition 1, part 1 of the *Elementary Geometry of Conics*, which was dated April 1872, I wrote: "No allusion has yet been made to the Conjugate Hyperbola, which may be viewed as a contrivance for giving a false definiteness to the student's conceptions, and perpetuating his illusion that the Hyperbola is a discontinuous curve."

Eventually the conjugate hyperbola was mentioned in an Appendix. In the present edition it is defined only in a footnote to Problem 259, after mention as below in the last paragraph of the text (p. 108): "The diameters of a hyperbola which do not meet it in real points are sometimes said to be terminated by the conjugate hyperbola (Prob. 259). But this brings in an unnecessary curve and is apt to mislead the student."

A little before the year 1881 I found—and wrote in the prolegomena to the *Ancient and Modern Geometry of Conics*—that Boscovich condemned the conjugate hyperbola.

* Later writers have taken them (with the proof that $SG = e \cdot SP$) in the form in which they found them in my *Geometrical Conics* (1863). But see the preface to Mr. Richardson's *Geometrical Conics*.

This unnecessary curve is commonly brought in to prove that, with the usual lettering,

$$CP^2 \sim CD^2 = CA^2 \sim CB^2;$$

that is to say, $CP^2 \sim PT^2 = CA^2 - CB^2$,

if the tangent at P meets an asymptote in T.

Leaving out the curve altogether, we have to prove that if CT, C*t* be lines given in position, and if the rectangle CT, C*t* be constant, then

$$CP^2 \sim (\tfrac{1}{2}Tt)^2 = \text{a constant.}$$

Bisect CT, T*t* in O, P, and draw P*y* normal to CT.

Then
$$\begin{aligned} CP^2 \sim PT^2 &= Cy^2 \sim Ty^2 \\ &= 4OC \,.\, Oy, \\ &= \text{a constant,} \end{aligned}$$

because (in the triangle OP*y*, whose sides are given in direction) O*y* varies as OP, and therefore inversely as OC.*

Having proved in this way that $CP^2 \sim CD^2$ is constant, we can dispense with the preceding proposition in the Syllabus, and with the eight excellent four-branch figures given by Messrs. Cockshott and Walters in their chapter on the Hyperbola.

We may then show by the eccentric circle, as in the *Elementary Geometry of Conics* (p. 67 note), how the two branches of the hyperbola join at infinity, thus modernising the geometry of that curve, while we also gain greatly in simplicity.

The conjugate hyperbola is an antiquated stumbling-block in the way of sound teaching.

The Parabola.

The proposition $QV^2 = 4SP \,.\, PV$

is presumably meant to be proved by tangent-properties, and Messrs. Cockshott and Walters, who follow the Syllabus, do so prove it. But it can be proved very simply indeed without them, by steps which can be retraced with equal ease, as in a corollary in the *Elementary Geometry of Conics* (p. 35).

It is worth while to show the student of mechanics how to work back from "QV^2 varies as PV" to the focus and directrix of the parabola.

The Ellipse.

It does not appear from the Syllabus how

$$QV^2 : PV \,.\, P'V = CD^2 : CP^2$$

is to be proved,—whether by tangent-properties or without them,—if any of the preceding propositions may be used. There should be the same rule for the ellipse and the parabola. But the Syllabus does not appear to rest as a whole upon any principles.

The Hyperbola.

Mr. Cockshott, speaking of the Syllabus at the Ninth General Meeting of the Association, said that "The asymptotes were introduced into the hyperbola rather earlier than usual. They really got very little idea of the shape of the hyperbola without the asymptote" (Report for 1883, p. 21). About this there can be no doubt. Probably the hyperbola was discovered from the asymptotes as the locus "$xy =$ a constant," as is argued in a Paper printed in the Association's Report for 1884.

The *New Treatment of the Hyperbola*, printed in the Report for 1890, and afterwards in the *Elementary Geometry of Conics*, brings in the asymptotes still earlier, and uses them to prove that

$$PN^2 + CB^2 : CN^2 = CB^2 : CA^2.$$

First it is shown that, if an asymptote meets the directrix in Y and the ordinate in O,

$$OY = SP;$$

and then, by applications of Euclid I. 47, that

$$ON^2 - PN^2 = SY^2 \text{ (or } CB^2\text{)},$$

whence we get $PN^2 + CB^2$ in terms of CN^2 by parallels.

This construction is perhaps the simplest possible. Thus far we want only one quadrant of the curve and one vertex; and the steps of the proof being reversible, it is easy to work back by it from "$xy =$ a constant" to the focus and directrix.

In the Paper of 1884 it was said of the conjugate hyperbola that "It comes down to us from times when the doctrine of continuity in relation to infinity had not as yet been formulated, and in so far as it is regarded as useful or necessary it is a hindrance to sound teaching in the matter of infinity, and to a right conception of the genesis of the hyperbola."

Inventio Orbium.

Newton showed how to determine conics, which he regarded as planetary orbits, from given points upon them and other conditions. In pure geometry it is equally necessary to be able to show that a conic is determined, for example, by five points; and a Syllabus is incomplete without at least this primary problem.

Attention is called to this grave defect in the Syllabus of Geometrical Conics by a recent work on Modern Plane Geometry, in a footnote to the proof that "the locus of a point at which four fixed points subtend a constant cross ratio is a conic," thus: "There is, of course, an incompleteness in this proof if it may not be assumed that a conic can be described through five given points, and this proposition *is not proved in treatises on Geometrical Conics.*"

This is true of a book which follows the Syllabus, but not of all books upon the subject. The young student, however, who cannot be expected to know better than his text-book, is given to understand that an absolutely essential proposition is to be taken for granted, and that it passes the wit of man to prove it in a presentable way. Is this "the more thorough treatment" which modern geometers desiderate?

Professor Adams found by experience as an examiner

* Another form of the proof is as follows. Complete the parallelogram TC*t*K, and let x, y denote its sides. Then $CK^2 - Tt^2 = x^2 + y^2 + 2xy \cos C - (x^2 + y^2 - 2xy \cos C)$, which is constant when xy and C are given.

that a little knowledge of modern geometry, with its "circular points at infinity," is a fruitful source of error.

Recent Works.

As in 1871 (*O. C. D. Messenger*, vol. v. p. 143), I should take as the first principle in an elementary work, that propositions should be proved *as nearly from the definition and with as few assumptions as possible.* This carries with it that chord-properties should in general be proved without the help of tangent-properties.

A recent writer completely adopts my way of working out the latter principle in the case of the parabola, and follows the *Geometry of Conics* in other respects.

Of all possible improvements in detail in the Syllabus the disuse of the conjugate hyperbola seems the most important. It has been already mentioned that Mr. C. Smith discards it.

Messrs. Milne and Davis make no use of the conjugate hyperbola.

When this has been got rid of, the student can be introduced by means of the eccentric circle to the leading ideas of modern geometry.

C. Taylor.

MATHEMATICAL WORTHIES.

II. John Dee.

John Dee, commonly though improperly called Dr. Dee, whose life extended over more than eighty years (1527-1608), was a mathematician and astronomer for more than fifty years, and a magician and raiser of spirits in his old age. It is by these latter practices that he is generally best known, and full accounts of them are given in the *Dictionary of National Biography* and in the life of Sir Edward Kelley in the recently published book, *Twelve Bad Men*, edited by Mr. Seccombe.

Here we have to do with him as a mathematician. He was one of the original Fellows of Trinity College, Cambridge, being then but twenty, and seems to have incurred the reproach of practising magical arts through having devised some clever stage tricks for a representation of the Pax of Aristophanes. He went abroad and lectured on Euclid at Louvain and Rheims, being so popular at the latter place that those for whom there was no room struggled for places to look in at the windows.

The most noteworthy contribution that he made to mathematical science was his preface and notes to the splendid translation into English by Sir H. Billingsley of the fifteen books ascribed to Euclid. De Morgan has stated his belief that this translation was really Dee's work. The preface, an account of the state of mathematical science at that time, is worth reading from its historical interest. Dee disliked the word geometry, and would have preferred megethology in its place. His catalogue of applications of geometry is perspective, astronomy, music, cosmography, astrology, static, anthropography, trochilike, helicosophy, pneumatithmy, menadry, hypogeiody, hydragogy, horometry, zography, architecture, thaumaturgike, archimastry. It may be as well to explain that menadry treats of the mechanical powers, hypogeiody refers to the projection on the irregular surface of the earth of the course of a mining passage, and that zography is painting. In his account of astronomy we find it stated that the sun is $161\frac{7}{8}$ times greater than the earth, and the earth is $42\frac{7}{8}$ times greater than the moon; that, estimated in radii of the earth, the sun's greatest distance is 1179, the moon's greatest distance is $68\frac{1}{3}$ and least $52\frac{1}{4}$, while the distance of the starry sky is $20081\frac{1}{2}$. De Morgan in the companion to the *Almanack* 1837 and 1855 notices that there is no statement here of his adherence to the Copernican hypothesis, but he thinks that Recorde, Dee, and Digges almost certainly held that view, though unavowedly.

Queen Elizabeth found Dee very useful to her. He was a man of ability, and under the character of one interested in science and astrology could really engage, without being suspected, in secret political missions. Lilly distinctly says he received a regular salary, but it is more likely that he was economically paid with mere promises.

When the Pope was considering the subject of the reformation of the Calendar, Dee was engaged by the Queen to report upon the matter, and in Halliwell's collection of letters on science there is a memorial of Lord Burghley's concerning this report. So far as England was concerned, the feeling against Popery was too strong to allow the reformation to be made.

Dee acted in anticipation of modern times in the petitions he addressed to both Mary and Elizabeth for the preservation and recovery of ancient writings and monuments. He died in 1608, and was buried at Mortlake.

G. Heppel.

NOTES.

18. On the hyperbolic functions.

Hyperbolic functions may be used in problems relating to a single branch of the hyperbola, and they are specially useful in investigating the properties of a chord, since by their aid the hyperbolic formulæ become similar to those of the ellipse.

It may therefore be of service to give a list of cases where the use of these functions is preferable to the substitution $x = a \sec \phi$, $y = b \tan \phi$.

A point P on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ may be taken as $(a \cosh u, b \sinh u)$, where $u = 2$ area ACP$/ab$.

The tangent at (u) is $\frac{x}{a}\cosh u - \frac{y}{b}\sinh u = 1$.

The normal at (u) is $ax \operatorname{sech} u + by \operatorname{cosech} u = a^2 + b^2$.

The chord QQ′ through $(u \pm v)$ is $\frac{x}{a}\cosh u - \frac{y}{b}\sinh u = \cosh v$.