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## A HISTORY OF THE THEORY OF ELASTICITY

*A History of the Theory of Elasticity and of the Strength of Materials, from Galilei to the Present Time.* By the late Isaac Todhunter, D.Sc., F.R.S. Edited and completed for the Syndics of the University Press by Karl Pearson, M.A., Professor of Applied Mathematics, University College, London. Vol. I. Galilei to Saint-Venant, 1639-1850. (Cambridge: at the University Press, 1886.)

THIS work was projected by the late Dr. Todhunter on the same lines as his well-known Histories of the "Theory of Probabilities," of the "Figure of the Earth," and of the "Calculus of Variations," and will doubtless equal them in usefulness to the mathematical student.

The first object of a writer in the preparation of such a work would be to draw up as complete a bibliography as possible of all books and papers relating to the subject, arranged in chronological order. Afterwards, in reading these memoirs, he would make copious notes, extracts, and criticisms; and then, on reaching the end of this self-imposed task, he would find his materials for a book like the present ready to place in the printer's hands. Incidentally, enough material and ideas would accumulate to form an independent treatise on the subject. Such a task was undertaken by Dr. Todhunter on the "History and Theory of Elasticity," from the standpoint of the mathematician, but he did not live, unfortunately, to complete it.

Prof. Karl Pearson explains in the preface the circumstances in which he undertook to edit and complete the work, and, from his own account, the labour thus devolved on him would have been sufficient to enable him to complete the "History" *ab initio.*

The present volume, like the previous "Histories," carries the subject and commentaries only to the year 1850, although Dr. Todhunter had analysed the chief mathematical memoirs from 1850 to 1870. The preparation of the second volume, to carry the history from 1850 up to date, is a task from which Prof. Pearson appears to recoil, with some justification; but it is to be hoped that he will enlist in his service some of the junior elasticians mentioned in his preface, and, by the application of the modern principle of the subdivision of labour, carry this invaluable work to its proper conclusion.

At the outset Prof. Pearson gives the palm to Galileo Galilei (1638) as the founder of the subject of elasticity and the strength of materials, while Dr. Todhunter asserts in § 18 that "the first work of genuine mathematical value on our subject is due to James Bernoulli . . . 1695." Galileo treated only the question of the breaking moment of a beam, or rather what we should call the *bending* moment, exactly as is done now in calculating the *stresses* in a structure, before proceeding to determine the consequent *strains* and deformations.

At this point the law enunciated by Hooke (1678) must intervene, which goes by his name, "Ut tensio, sic vis," originally published by him, in the fashion of those times, as an anagram, *c e i i i n o s s s t t u u.* Stated in the modern form, this law asserts that

$$\frac{\text{tension}}{\text{extension}} = \frac{\text{pressure}}{\text{compression}} = \frac{\text{stress}}{\text{strain}} = \text{modulus of elasticity},$$

and is the law universally employed to connect mathematically the corresponding stresses and strains in an elastic substance, as pointed out by Saint-Venant [8].

When the stresses and strains are large enough for variations on Hooke's law to become observable, a fresh set of phenomena depending on the ductility and viscosity of the substance came into play, and the previous mathematical investigations no longer hold. Much of the confusion pointed out by Dr. Todhunter in the treatment of the subject by experimentalists is due to the fact that in experiments it has been usual to test the strength of structures to the breaking-point, and hence the use of the term *breaking* instead of *bending* moment. The modern experiments of Wöhler show that this point, at which ductility manifests itself, is much sooner reached than was formerly supposed; consequently, modern engineering practice is much less bold than formerly in large iron structures like bridges. For this reason, the diagrams of the frontispiece, though physically extremely interesting, cannot be considered to bear on the mathematical theory.

Returning again to the treatment of the subject by the mathematicians, we find a picturesque diagram given by Galileo (p. 2) of a beam built into an old wall and supporting a weight, the cross-grained character of the wood of the beam being carefully shown; so that it is not surprising that Galileo does not attempt any molecular theory to account for the flexure of the beam. This theory, supplied by Hooke's law, was applied by Mariotte, Leibnitz, De Lahire, and Varignon; but they neglect the compression of the fibres, and so place the neutral plane in the lower face of Galileo's beam. The true position of the neutral plane was assigned by James Bernoulli in 1695, who, in his investigation of the simplest case of the bent beam, was led to the consideration of the curve called the "elastica." This "elastica" curve speedily attracted the attention of the great Euler (1744), and must be considered to have directed his attention to the elliptic integrals. Probably the extraordinary divination which led Euler to the formula connecting the sum of two elliptic integrals, thus giving the fundamental theorem of the addition equation of elliptic functions, was due to mechanical considerations concerning the "elastica" curve; a good illustration of the general principle that the pure mathematician will find the best materials for his work in the problems presented to him by natural and physical questions. The result obtained by Euler for the thrust at which a straight column begins to bend, when the corresponding "elastica" differs from a straight line very slightly in a curve of sines, is of the utmost importance to the architect and engineer; and, as Prof. Kennedy can testify, is employed with the greatest confidence in the design of the highest columns and pillars.

It is interesting to find the complete treatment of the problem of lateral vibrations of elastic bars is also due to Euler, though the analytical difficulties of the *period equations* seem to have puzzled him. If we employ the modern notation of the *hyperbolic* functions, we shall find his period equations all reduced to the form—

$$\cos\omega \cosh\omega = \pm 1,$$

or,

$$\tanh\omega = \pm \sin\omega\,;$$

and this again is equivalent to

$$\tanh\tfrac{1}{2}\omega = \pm\tan\tfrac{1}{2}\omega, \text{ or } \mp\cot\tfrac{1}{2}\omega,$$



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whence a graphical determination of the values of ω is easily inferred (pp. 50, 51, footnote).

Another interesting paper due to Euler is "De altitudine columnarum sub proprio pondere corruentium" (1778), investigating the height at which a mast or tree will begin to bend under its own weight. To this paper he might well have prefixed the old German proverb, quoted by Goethe in "Wahrheit und Dichtung":—"Es ist dafür gesorgt, dass die Baüme nicht in dem Himmel wachsen." We know now that the functions of Bessel are required for the complete analytical solution of this question, though the *Theorema maxime memorabile* enunciated by Euler, "Maxima altitudo, qua columnæ cylindricæ ex eadem materia confectæ, proprium pondus etiamnunc sustinere valent, tenet rationem subtriplicatam amplitudinis," is interesting as one of the first applications of the principle of mechanical similitude, showing why the proportions of the giant of the forest are stunted compared with those of the young tree, and also why it is hopeless to attempt the problem of human flight while $g$ is 32.

Lagrange considered the same subject in "Sur la figure des colonnes" (1770), examining and disproving the dictum of Vitruvius that the *renflement* of a column was necessary for strength: the dictum can hardly be called an architectural fallacy, as the *renflement* corrects the tendency, due to *irradiation*, of a perfectly cylindrical column to appear attenuated in the middle; for a similar reason it is necessary to slightly blunt the neighbourhood of the point of a Gothic spire to avoid the appearance of concavity.

Coulomb, a well-known name to electricians, is mentioned by Saint-Venant as giving about this time (1780), in "Remarques sur la rupture des corps," the true position of the neutral line of a beam, although it is asserted by Dr. Todhunter that the ancient erroneous idea prevailed into the present century.

In Chapter II. the work of Young, Gregory, Eytelwein, Plana, Dupin, Belli, Binet, Biot, Rennie, Barlow, Tredgold, Fourier, Nobili, Bordoni, Hodgkinson, and others is analysed. Of these the English writers, who generally were experimentalists as well as theorists, are severely handled by Dr. Todhunter for their heresies on the neutral axis. Considering that the neutral axis is a mathematical fiction, depending on an ignoration of the shearing stress, and the consequent warping of the normal sections of a beam, this treatment of Dr. Todhunter is too severe, compared with the leniency with which he views the metaphysical speculations of the pure theorists. These experimentalists were trusted in their advice on important constructions, and took care their formulæ erred on the right side of strength.

To Navier (1821) we are first indebted for the general mathematical equations of the equilibrium and vibrations of an elastic solid, to be satisfied in the interior and at the surface, and henceforth the researches of mathematicians take a bolder flight from the treatment of the simple beam of former investigators.

Mlle. Sophie Germain's "Recherches sur la théorie des surfaces élastiques" (1821) appears to afford Dr. Todhunter gratification in showing that sex can make itself apparent even in mathematics. However, it is dangerous to argue from this instance, as hardly any mathematician has yet written on elastic surfaces without falling into error in the boundary conditions, and the subject is even now not yet certainly settled.

The vibration of elastic surfaces is important in its bearing on acoustics and music, and received about this time experimental and theoretical treatment from Chladni, Strehlke, Pagani, and Savart.

Chapters IV. and V. give an account of the treatment of the subject by the celebrated mathematicians Poisson and Cauchy, who practically exhausted the soluble problems, if we except the torsion questions considered by Saint-Venant. Poisson's results are generally expressed by means of definite integrals, most of which we see now can be classified as Bessel's functions. Both Poisson and Cauchy appear to have considered the subject of elasticity principally in its bearing on the new theory of physical optics, then receiving such important experimental and theoretical treatment at the hands of Fresnel.

Henceforth the theory receives development at the hands of so many writers that it is possible only to specify the honoured names of Gerstner, Green, McCullagh, Poncelet, and Maxwell as having contributed important advance to the subject.

Lamé's "Theory of Elasticity," carefully analysed in Chapter VII., still remains a standard text-book, in conjunction with the treatises of F. Neumann and Clebsch.

The volume concludes with an account of Saint-Venant's researches before 1850, the subsequent work to be recorded in the second volume. Saint-Venant is the name most honoured by practical elasticians and engineers, inasmuch as he has developed his theories from the definite practical problems presented by the large and daring constructions in iron and steel which mark the middle of this century.

In the appendix Mr. Pearson has carefully analysed the conflicting notations of different writers, and proposed a very convenient terminology and notation, which would save great trouble if universally adopted. He has also given an account of experiments carried out by Prof. Kennedy in his mechanical laboratory, which have an important bearing on the limitations of the truth of Hooke's law, or, in the language of elasticity, the constancy of the ratio of stress to corresponding strain.

The present volume is an indispensable hand-book of reference for the mathematician and the engineer, and in the editing and printing must be considered a very fitting tribute to the wonderful industry and application of its projector, the late Dr. Todhunter.

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## THE ENCYCLOPÆDIA BRITANNICA

*The Encyclopædia Britannica.* Vol. XX. Pru—Ros. Vol. XXI. Rot—Sia. (Edinburgh: A. and C. Black, 1886.)

THE leading scientific articles in these two volumes are mainly biological. In Vol. XX. Prof. A. Newton contributes the articles on the various important groups of birds; and in those on the Quail, Screamer, Secretary Bird, Seriema or Cariama, it is truly surprising to find so many facts condensed into so small a compass. Mr. C. T. Newton's article on Pterodactyles gives us the newest information on this strange group of fossil reptiles. In the article on Reproduction only the broadest aspects of the phenomena attending it are glanced at, Mr. P. Geddes treating of the Animal, and Mr. S. H.

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