

ON LATITUDE VARIATION IN A RIGID EARTH, AS ILLUSTRATED  
BY MAXWELL'S DYNAMICAL TOP.<sup>1</sup>

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THE remarkably large and accurate series of observations recently published by Professor George Davidson<sup>2</sup> has given additional interest to the question of latitude variation.

His paper raised the question, in my mind, as to the possibility of explaining this, now fairly well proven, phenomenon in terms of elementary dynamics.

The following affirmative answer is simply one which I looked up for my own satisfaction, and, though it contains no new contribution to knowledge, I have thought it worthy of presentation both on account of the renewed interest in this very old astronomical inquiry and also for the sake of calling your attention to this beautiful, but much neglected top, which Maxwell first spun at Edinburgh some forty years ago.

First of all we are not considering the geocentric or geodetic, but simply the astronomical latitude, viz., the angle between the plumb line and the celestial equator, or if you choose, the altitude of "the celestial pole."

For the pole, *as observed by upper and lower culminations*, is a direction which is fixed in space so far as the motion of the earth is concerned. It is what we shall later call the "invariable pole." The plumb line, on the contrary, is fixed in the earth, considered as a rigid body, and hence moves with it.

Having then one line fixed in space and another in the moving body, one can, by actually measuring the angle between them, detect any change in the axis about which the body is rotating; that is, one can detect whether or not the earth shifts about the axis of original spin with any motion other than that of daily rotation. For such a determination observations by Talcott's Method are appropriate, since they give the zenith distances (or what, for our purpose, is the same thing—*differences of zenith distances*) of stars of known declination. And declinations, it must be remembered, are referred entirely to an outside system, viz., the celestial equator and the fixed stars.

<sup>1</sup> Being a paper read before the Science Club of Northwestern University, 8th of February, 1895.

<sup>2</sup> Astronomical Jour., No. 323.

Having now acquired a perfectly definite understanding of what is *meant* by latitude variation, our next inquiry shall be what right have we, on dynamical grounds, to *expect* any such variation as that described?

Before taking up this question it will be necessary to state the

*Conditions of the Problem.*

When compared with the most general motion of a solid body, the question is a very simple one; for we shall assume, first, what Mr. Chandler and Professor Newcomb<sup>1</sup> have recently shown is not allowable, if we wish to predict the motion as completely as possible.

(1) *that there is no relative motion of the different parts of the earth.* That is, all questions of elasticity and viscosity are ruled out.

(2) Secondly, *that no forces act upon the body.* All accelerations are, therefore, ruled out; and velocities must be expressible in terms of the constants of the body, including among these the constants which determine the initial circumstances.

(3) The third assumption is that the *body is fixed at one point*, viz., *the center of mass.* Thus robbed of three degrees of freedom, it can only spin about an axis through the fixed point. And, in virtue of the assumption of no forces, this rotation must occur without precession or nutation.

*Illustration of the Hoop.*

In order that everyone may clearly grasp the phenomenon we are studying, I may recall the very apt illustration employed by Maxwell, and afterwards used by Thomson and Tait, viz., that the motion of the earth is practically that of a circular hoop rolling, *but not slipping*, on a stick of circular cross-section. I say "practically" because the earth is elliptical, and not quite circular, in sections parallel to the equator.

Now imagine the axis of the stick fixed in space. And let us, therefore, call it the "*invariable axis.*"

The line of contact of stick and hoop is the axis about which the hoop is really rotating at any given instant. It is, therefore, called the "*instantaneous axis.*" The path of this axis in space is evidently the circumference of the *stick*; but its path *in the body* is the section of the hoop.

As to the path of the invariable axis in the body, it is also a circle, viz., one whose center is the center of the hoop and whose radius is the distance between centers of hoop and stick.

Now to these two axes we must add still a third, viz., the axis of the hoop which passes through its center and is perpendicular to its plane. This is known as the "*axis of figure,*" and here represents the axis of figure of the earth. This is the axis whose motion *in space* we are tracing.

<sup>1</sup> Monthly Notices, p. 336-341 (1892).

It is frequently called the "true axis," and is what is meant in general when the "axis" or "pole" of the earth is spoken of without further modification.

(Experiment with hoop shown.) Observe that whenever the hoop rotates about the stick, the line joining centers of stick and hoop changes direction *in the hoop, i. e.*, the plane containing the three axes just described shifts position in the rotating body. This amounts to saying that the axis of figure has changed its azimuth with reference to the invariable axis—that the axis of the earth no longer points to the same place among the fixed stars; or, still again, if you choose, that the plumb line has shifted with reference to the celestial equator; or, in general, the latitude of every point changes as the body rotates. All this, of course, provided the motion of the earth *is* that of the hoop rolling on the stick.

#### *Rate of Rotation of Pole of Figure.*

So far we have considered only the *path* of the invariable axis in the body, and not at all its rate of rotation. The illustration, however, is competent to give us a clear picture of this part of the solution also. It is evident that, during each rotation of the hoop, the instantaneous axis advances along the hoop by one circumference of the stick. The rate, therefore, at which the invariable axis proceeds through the body is *nearly* one circumference of the stick "per day;" and one complete revolution of the invariable axis about the axis of figure will occupy *exactly* as many "days" as there are radii of the stick in one internal radius of the hoop.

This number has been computed for the earth, considered as a rigid body, and found to be approximately 306 days. Such observations as those of Professor Davidson show that here the circle described by the pole of figure is *less than 100 feet* in diameter. Corresponding to this, the diameter of the stick upon which the earth may be considered as rolling, is but a few inches.

#### *Mathematical Theory.*

What now remains for us is to show from simple dynamical principles that this hoop *does* represent the motion of a freely rotating rigid solid, fixed as its center of mass. The method employed by Maxwell<sup>1</sup> makes it necessary to introduce only two physical considerations, viz.:

##### (1) *The constancy of angular momentum of the rotating body.*

It is important to remember that this law is true only for rigid bodies, as illustrated by the fact that a cat always rights upon its feet, however it be dropped.

<sup>1</sup> Scientific Papers, Vol. I., pp. 248–262.

(2) *The constancy of kinetic energy of the rotating body.*

But these two experimental facts will be useful only after they have been given algebraic expression, so that one can see their consequences.

*Axes of Reference.*

It is well known that in any rigid body there can be described an ellipsoid whose center shall coincide with the center of mass, and such that the moment of inertia of the body about any radius vector of the ellipsoid will be numerically equal to the inverse squares of this radius vector. In any rigid body, there will be therefore, three mutually perpendicular directions about which the moments of inertia will be respectively maximum, minimum, and intermediate. These are the so-called "principal axes." Imagine such an ellipsoid described about the center of the earth, or, if you please, about the center of mass of this top.

In what follows, we shall use these three axes as axes of reference. Under no circumstances will any other be used; so that all angles and distances will be unambiguous. Call them  $x, y, z$ . Since these axes move in space, as the body moves, they are called "moving axes;" but for an inhabitant of the body they are fixed.

*Definitions.*

Next we shall need a few definitions. They are tedious, I know, but conversation without them is very difficult.

Let  $A, B, C$  = Moments of Inertia about  $x, y, z$ , respectively, taken in order of diminishing magnitude.

$\omega_1, \omega_2, \omega_3$  = Angular velocities about  $x, y, z$ , respectively.

$l, m, n$ , = Direction cosines of axis of original impulse, *i.e.*, of the direction of angular momentum.

$$\text{Then } A^2 \omega_1^2 + B^2 \omega_2^2 + C^2 \omega_3^2 = H^2 = \text{constant} \quad (1)$$

expresses the first of the two dynamical principles assumed above; and

$$\left. \begin{aligned} l &= \frac{A\omega_1}{H} \\ m &= \frac{B\omega_2}{H} \\ n &= \frac{C\omega_3}{H} \end{aligned} \right\} \text{define the } \textit{invariable axis}.$$

For the kinetic energy, we have

$$\frac{1}{2} (A\omega_1^2 + B\omega_2^2 + C\omega_3^2) = \frac{V}{2} = \text{constant} \quad (2)$$

which expresses the other of the two physical considerations involved.

Here  $A, B, C$  are constants which are given so soon as the body is determined upon; but  $l, m, n$ , though defining the invariable axis, are themselves variable, being referred to moving axes; so also is the case with  $\omega_1, \omega_2, \omega_3$ .

While we are not here especially interested in the instantaneous axis, it may make matters clearer to remark parenthetically that its position in the body is defined by the direction cosines  $p, q, r$ , where

$$\left. \begin{aligned} p &= \frac{\omega_1}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}} \\ q &= \frac{\omega_2}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}} \\ r &= \frac{\omega_3}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}} \end{aligned} \right\} \text{Instantaneous axis.}$$

In accordance with our convention above, we now choose for the axis of  $x$  the *axis of figure* in the case of the earth, and then proceed to find the

*Path of the Invariable Axis in the Body.*

Here, pursuing the ordinary method, the time coördinates must be eliminated in order to obtain a general relation between the space coördinates. Choosing for this purpose Eq. (2) and eliminating the angular velocities by means of the defining equations for  $l, m, n$ , we have

$$\frac{l^2}{A} + \frac{m^2}{B} + \frac{n^2}{C} = \frac{V}{H^2} \quad (3)$$

Let us, for a moment place

$$\frac{1}{A} = a^2$$

$$\frac{1}{B} = b^2$$

$$\frac{1}{C} = c^2$$

$$\frac{V}{H^2} = e^2$$

Then Eq. (3) becomes

$$a^2 l^2 + b^2 m^2 + c^2 n^2 = e^2$$

But if  $x, y, z$ , be any point on the invariable axis, at distance  $r$  from the origin, then

$$l = \frac{x}{r}$$

$$m = \frac{y}{r}$$

$$n = \frac{z}{r}$$

and equation (3) becomes again

$$a^2x^2 + b^2y^2 + c^2z^2 = e^2r^2$$

or

$$(a^2 - e^2)x + (b^2 - e^2)y^2 + (c^2 - e^2)z^2 = 0$$

which you see is the equation of a cone having its apex at the origin. The physical meaning of this is that the invariable axis, always passing through the fixed point, describes a cone in the body; or, in other words, the axis of figure describes a cone in space.

Knowing now the path of the invariable axis, it is a matter of the utmost ease to determine the path of the invariable pole in any plane we may choose, say the  $yz$  plane—a plane tangent at the north pole, if you like. The equation to such a plane is

$$x = k = \text{constant.}$$

Under this condition, Eq. (3) becomes

$$(b^2 - e^2)y^2 + (c^2 - e^2)z^2 = (e^2 - a^2)k^2 = \text{constant.}$$

And the same *form* of equation is obtained for any other plane. Just what curve this equation represents depends upon the relation between

the moments of inertia  $\frac{1}{b^2}$ ,  $\frac{1}{c^2}$  and the quantity  $\frac{1}{e^2}$

It must not be forgotten that  $e$  ranges in value only between  $a$  and  $c$ , since

$$a^2l^2 + b^2m^2 + c^2n^2 = e^2$$

where

$$l^2 + m^2 + n^2 = 1.$$

If  $b > e$ , the path of the invariable *pole* is evidently an ellipse. The physical meaning of this is that if, at the beginning of the motion, the invariable pole does not quite coincide with the pole of figure, it will never coincide. On the other hand, these two poles will never part company *very widely* except when  $b$  approaches very near the limit of equality with  $e$ , under which condition ellipticity of the path becomes very great.

If, in addition to the condition  $b > e$ , we have also  $c = b$ , the path becomes a circle. These are practically the conditions fulfilled by the earth, which is very nearly symmetrical with reference to its maximum principal axis, the so-called axis of figure. What one might expect, therefore, in the case of the earth, is to find a variation of latitude corresponding to this circular motion of the true pole about the pole of the heavens—provided

of course, the earth were not originally set spinning exactly about its axis of figure. And recent observations show almost, if not quite, conclusively that it was not.

There is another special case which happily is not realized in the case of the earth, but is easily shown with this top. Imagine the body set spinning about the  $y$  axis, the mean principal axis.

The path of the invariable pole in the plane  $y = k$ , a constant, will be

$$(a^2 - e^2)x^2 + (c^2 - e^2)z^2 = (e^2 - b^2)k^2.$$

But this is the equation of a hyperbola since always  $a < e < c$ . Since the hyperbola is a curve with infinite branches, it is evident that, if a rigid body be spun about its intermediate axis of inertia, the invariable pole *will start off toward infinity*, so to speak, along one of these branches. In other words, the motion becomes unstable. The physical fact, however, that the actual path of the pole must be a closed curve constitutes an interesting experimental demonstration of the theorem that the spherical projection of these hyperbolas are closed curves. The physical significance of these closed curves is that the invariable pole always follows one of them *away* from the mean axis and *about* a new axis—an axis about which it *can* rotate in stable equilibrium.

This dynamic instability is to be carefully distinguished from static instability such as results from displacing the center of support underneath the center of gravity.

Indeed, one can easily imagine a set of highly improbable circumstances under which the water of our planet might be diverted into polar ice caps to such an extent as to produce this very dynamic instability.

#### *Period of Rotation of Invariable Axis about Axis of Figure.*

We have now seen that the true pole once displaced from the invariable pole remains displaced and moves, in the case of the earth, in an ellipse about it. It remains to sketch hastily the method by which the angular velocity ("mean motion") of the invariable axis in the body may be determined. Consider any point,  $P$ , on the invariable axis at unit distance from the origin. Its coördinates will be  $l, m, n$ . Let  $V_x, V_y, V_z$  be the linear speed of this point along  $x, y, z$ , respectively. Its angular velocity, say  $\omega_x$ , about the axis of  $x$  then is

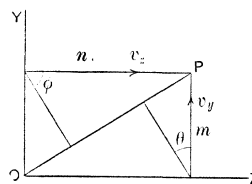


Fig. 1.

$$\begin{aligned}\omega_x &= v_y \frac{\cos \theta}{\sqrt{m^2 + n^2}} - v_z \frac{\cos \varphi}{\sqrt{m^2 + n^2}} \\ &= v_y \frac{n}{m^2 + n^2} - v_z \frac{m}{m^2 + n^2}\end{aligned}$$

where  $\theta$  and  $\varphi$  have meanings indicated in figure.

But

$$\begin{aligned}v_x &= n\omega_2 - m\omega_3 \\v_y &= l\omega_3 - n\omega_1 \\v_z &= m\omega_1 - l\omega_2\end{aligned}$$

Hence

$$\begin{aligned}\omega_x &= \frac{ln\omega_3 - n^2\omega_1 - m^2\omega_1 + lm\omega_2}{m^2 + n^2} \\&= -\frac{\omega_1}{1 - l^2} + \frac{l(l\omega_1 + m\omega_2 + n\omega_2)}{1 - l^2}\end{aligned}$$

or substituting<sup>1</sup> for  $\omega_1, \omega_2, \omega_3$ , in terms of  $l, m, n$ , we have

$$\begin{aligned}\omega_x &= -Hl \frac{a^2 - e^2}{1 - l^2} \\&= \frac{\omega_1}{1 - l^2} \left( \frac{e^2 - a^2}{a^2} \right)\end{aligned}$$

This solution is complete from a mathematical standpoint; but the physical meaning is clearer when  $\omega_x$  is expressed in terms of days. What we want to know is this: How many rotations does the earth perform while the invariable axis describes a complete cone, or, in other words, what is the period of latitude variation in term of days? This can be answered as soon as we know the area of the complete ellipse which the invariable pole describes and the area which it sweeps out during one revolution of the body. The ellipse, the base of our original cone, is

$$\frac{b^2 - e^2}{e^2 - a^2} y^2 + \frac{c^2 - e^2}{e^2 - a^2} z^2 = 1$$

Accordingly its area is

$$\pi \sqrt{\frac{e^2 - a^2}{b^2 - e^2}} \sqrt{\frac{e^2 - a^2}{c^2 - e^2}}$$

To determine the remaining factor, let  $\theta$  be the angle which the invariable axis makes with the axis of  $x$ ; and let  $r$  be the perpendicular let fall from the extremity of the invariable axis (of unit length) upon the axis of  $x$ . If  $\omega_r$  be the angular velocity of the radius vector  $r$  about  $x$ ,

$$\begin{aligned}\omega_r &= \sin \theta \cdot \omega_x \\&= \sqrt{1 - l^2} \cdot \omega_x\end{aligned}$$

The area described per second by  $r$  is

<sup>1</sup> It may be well here to recall the distinction between  $\omega_1$ , and  $\omega_x$ . The former is the  $x$  component of the total angular velocity, *i. e.*, the angular velocity about the instantaneous axis, while the latter is the  $x$  component about the invariable axis.



$$\begin{aligned}
&= \frac{1}{2} r \omega_r \\
&= \frac{1}{2} (1 - l^2) \omega_z \\
&= \frac{1}{2} \omega_1 \left( \frac{e^2 - a^2}{a^2} \right)
\end{aligned}$$

But the body rotates once in  $\frac{2\pi}{\omega_1}$  seconds, whence the area described during one rotation of the body is

$$\pi \frac{e^2 - a^2}{a^2}$$

The ratio of these two areas is the quantity we want, viz., the period of the latitude variation. Let us call it  $T$ , then

$$\begin{aligned}
T &= \frac{\pi \sqrt{\frac{e^2 - a^2}{b^2 - e^2}} \sqrt{\frac{e^2 - a^2}{c^2 - e^2}}}{\pi \frac{e^2 - a^2}{a^2}} \text{ days.} \\
&= \frac{a^2}{\sqrt{b^2 - e^2} \sqrt{c^2 - e^2}} \text{ days.} \\
&= \frac{a^2}{\sqrt{b^2 - a^2} \sqrt{c^2 - a^2}} \text{ very approximately.}
\end{aligned}$$

The degree of approximation here introduced is quite allowable, since it only involves the assumption that  $l$ , the cosine of maximum latitude variation, is unity.

If we assume, what in the case of the earth is also very nearly true, that  $b = c$ , then

$$T = \frac{a^2}{b^2 - a^2} = \frac{B}{A - B}$$

the value of which is approximately 306 days, the Eulerian period. (Latitude variation, dynamical instability, and other phenomena shown with the top.)

It is a matter of some surprise that this beautiful instrument has not been more widely used in this country; for it has proven itself an invaluable aid in presenting the whole subject of dynamics of rotation. This particular top, due to both the skill and the kindness of our fellowtownsman, Mr. A. T. Merriman, runs from 10 to 15 minutes after having been spun at a moderate speed.

Maxwell's directions for making the top are so explicit that one cannot go astray in following them.

The phenomena of nutation and precession are shown with this quite as well as with the ordinary gyroscope; and in addition it exhibits num-

erous other phenomena such, for instance, as this. Adjusting the top so that it spins stably when carrying a paper color disc ; the paper disc is replaced by one of sheet metal, just heavy enough to make the axis of spin the intermediate axis of inertia. You see that instability results ; and we have an illustration of the effects of the possible, if highly improbable, heaping up of ice about the two poles of the earth.

Note added February, 1898.

*Adjustment of Maxwell's Top.*

I. After oiling bearing, set the six horizontal screws (See Plate) so that they each lack about  $\frac{1}{8}$ " of being screwed in clear up to the head.

II. Set the bob (large nut) about 1" below the top *thread* of the steel axle. Clamp it with a check-nut.

III. Adjust the steel axle in the body of the top so that its lower end lies just a trifle (say, half a turn) above the center of gravity of the top. This adjustment is made, of course, by balancing the top on the supporting pillar. This adjustment made, clamp with check-nut.

IV. Take any two screws at opposite ends of a diameter, and unscrew them until they lack about  $\frac{1}{2}$ " or  $\frac{3}{4}$ " of being screwed in clear up to the head.

This will produce marked inequality among the three principal axes of the inertia ellipsoid.

V. Put the color disc on the top. It is now ready to spin *slowly*, unless possibly the color disc has made the top statically unstable, in which case the bell of the top must be lowered.

*In beginning*, it is wise always to use the spinning fork, never to spin the top very fast and always to be ready to catch the top with the hand in case dynamical instability occurs.

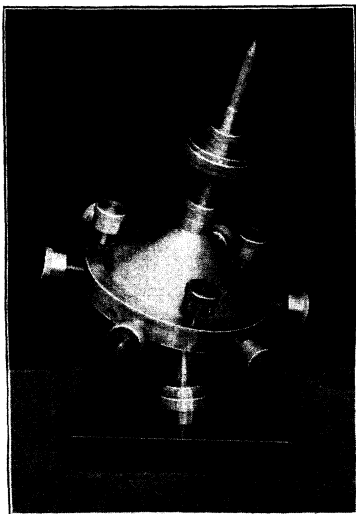
VI. *Adjustment of axle to coincide with axis of figure.*

When the top is first spun, as above described, it will, in general show a certain amount of "wobble." This is due to the mass not being symmetrically distributed about the axle.

To remedy this, observe the color in the disc into which the instantaneous axis runs. Then, if the steel axle be the axis about which the moment of inertia is a *maximum*, one must screw *down* that one of the three vertical screws which lies on the same side of the top as the color into which the instantaneous axis makes its excursion.

But if the steel axis, the axis of *least* moment of inertia, then one must screw *up* (*i. e.*, unscrew) this same screw. One easily discovers by trial which is the proper adjustment to make. Continue this adjustment until, on spinning, the top shows no "wobble." The top is now in adjustment.

In demonstrating the top to a large audience, the disc should be il-



luminated by a strong light from above, the eyes of the audience being shielded from its direct rays.

The following phenomena are easily shown :

I. *Precession*—clockwise by screwing bob in one way. Counterclockwise by screwing bob in other way.

II. *Statical stability conferred by rotation.*

The top is set so as to be in evident static instability : on rotation it becomes stable.

III. *Dynamical Instability conferred by rotation.*

The top is adjusted to statical stability by lowering the bell. On setting the “bob” up or down a little, one soon finds a position where the axle is the *intermediate* axis of inertia. This position is at once recognized by the fact that the top now refuses to rotate about the axis of original spin.

The top is now stable only when at rest ; and great care must be used not to allow the top to wreck itself.

A very simple way to make the axle the axis of maximum, intermediate, and minimum moment of inertia successively is as follows : Entirely remove the check out of the bob. Screw the bob down near the bell of the top. The top now spins stably ; and, if it has been spun in the right direction, one can, by simply holding the bob between thumb and forefinger, bring it to any desired position on the axle, *without stopping the top*. As the bob moves up, the moment of inertia about the axle becomes *relatively* less and less, until it has passed from maximum to minimum, through the intermediate value.

IV. *Variation of Latitude.*

Spin the top slowly. Tap the steel axis smartly with a lead pencil. The instantaneous axis immediately leaves the center of the color disc and travels around the disc in an ellipse.

By screwing the “bob” into its *other* position of stability, the instantaneous axis travels around the disc in the opposite direction.

V. *Effect of Polar Icecap.*

Remove the color disc. Adjust the top so that it spins stably, but is *on the verge of dynamic instability*. Set the top spinning, and while it is still spinning, gently slip on the color disc, having in the meantime imagined sufficient brass taken from the equatorial region of the top to make up the actual color disc used.

If the top has been properly adjusted dynamic instability will result, *i. e.*, the top will try to rotate about an axis lying in the equatorial plane.

VI. In addition to the above, the top is useful in illustrating other phenomena connected with rotational dynamics—especially in representing the parallel treatment of translational and rotational dynamics.

