On the Solution of certain Partial Differential Equations of the Second Order having more than Two Independent Variables.

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The equations in question are included in the form

$$\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} V_{ij}\frac{d^2z}{dx_i dx_j}+V_0=0 \quad\text{..........................(1)},$$

where V_{ij}, V_0 are functions of $x_1 \dots x_n$, x, $p_1\left(\equiv\frac{dz}{dx_1}\right)$, ... $p_n\left(\equiv\frac{dz}{dx_n}\right)$.

"Suppose that there exists a solution

$$F\{x_1, x_2 \dots x_n,\ z,\ \phi(u_1, u_2 \dots u_{n-1}) \dots\}=0$$

involving an arbitrary function ϕ of the $\overline{n-1}$ quantities $u_1, u_2, \dots u_{n-1}$, functions of $z, x_1, x_2 \dots x_n$, whose values are independent of the form of ϕ. It is obvious that the arguments $u_1, u_2 \dots u_{n-1}$ (although they may without loss of generality be regarded as determinate functions of $z, x_1, x_2, \dots x_n$) are not of necessity determinate functions of $z, x_1 \dots x_n$; for, taking $v_1, v_2, \dots v_{n-1}$ any particular values of $u_1, u_2, \dots u_{n-1}$, then the general values are $u_1, u_2 \dots u_{n-1}$, each of them equal to an arbitrary function of $v_1, v_2 \dots v_{n-1}$; and it hence follows that $u_1, u_2 \dots u_{n-1}$ are each of them a solution of one and the same partial differential equation of the first order, viz., this is the equation

$$\begin{vmatrix} \left(\frac{du}{dx_1}\right), & \left(\frac{du}{dx_2}\right), & \dots & \left(\frac{du}{dx_n}\right) \\ \left(\frac{dv_1}{dx_1}\right), & \left(\frac{dv_1}{dx_2}\right), & \dots & \left(\frac{dv_1}{dx_n}\right) \\ \dots & \dots & & \dots \\ \left(\frac{dv_{n-1}}{dx_1}\right), & \left(\frac{dv_{n-1}}{dx_2}\right), & \dots & \left(\frac{dv_{n-1}}{dx_n}\right) \end{vmatrix}=0,$$

where $v_1, \dots v_{n-1}$ are regarded as given functions of $z, x_1, \dots x_n$, and $\left(\frac{dv_1}{dx_1}\right)$ means $\frac{dv_1}{dx_1}+p_1\frac{dv_1}{dz}$, &c. ....

"We propose in the first instance to investigate the conditions which must be satisfied in order that (1) may have a solution of the form above referred to; and when the conditions are satisfied, to determine the solution. It will appear as a result that in certain cases there will be two sets of arguments—say $u_1, u_2, \dots u_{n-1}$ the arguments of an arbitrary function ϕ, and $\bar{u}_1, \bar{u}_2, \dots \bar{u}_{n-1}$ the arguments of a second arbitrary function $\bar{\phi}$, where in like manner $\bar{u}_1, \bar{u}_2, \dots \bar{u}_{n-1}$ are not of necessity determinate functions of $z, x_1, \ldots x_n$, but are the solutions of one and the same partial differential equation obtained by consideration of a set of particular solutions $\overline{v}_1, \overline{v}_2, \ldots \overline{v}_{n-1}$; or that there will be a solution

$$F\{x_1, x_2, \ldots x_n, z, \phi(u_1, u_2, \ldots u_{n-1}), \overline{\phi}(\overline{u}_1, \overline{u}_2, \ldots \overline{u}_{n-1})\} = 0,$$

or, it may be, involving also the integrals of these arbitrary functions in regard to each or any of the arguments.

"The arguments $u_1, \ldots u_{n-1}$ and $\overline{u}_1, \overline{u}_2, \ldots \overline{u}_{n-1}$ may without loss of generality be regarded as determinate functions of $z, x_1, x_2, \ldots x_n$; and it is convenient so to regard them. Say therefore that there are $2n-2$ arguments $u_1 \ldots u_{n-1}$ and $\overline{u}_1 \ldots \overline{u}_{n-1}$, each of them a determinate function of $z, x_1, x_2, \ldots x_n$." *

Three cases arise for discussion.

1. If $\overline{n-1}$ of these arguments be independent, the given equation can be transformed to an ordinary equation of the second order. The arbitrary constants in the solution of this are to be replaced by arbitrary functions, of each of which the arguments will be the $\overline{n-1}$ independent quantities. In this case we have for all values of i, j,

$$V_{ii}V_{jj} - V_{ij}^2 = 0.$$

2. If n of the arguments be independent, the equation is always reducible to the form

$$\sum_{i=1}^{i=n-1} A_i\left(\frac{d^2\zeta}{d\xi_i d\eta_1} + P\frac{d\zeta}{d\xi_i}\right) + Q\frac{d\zeta}{d\eta_1} + Z' = 0,$$

where ξ_i, η_1 are the n independent arguments. In this equation A_i, P, Q are functions of $\xi_i \ldots \eta_1$ only, and the coefficients A_i do not involve η_1.

If Z' be linear in ζ ($\equiv Z\zeta + W$ say), and the conditions

$$\Sigma A_i \frac{dP}{d\xi_i} = Z - PQ = \frac{dQ}{d\eta_1}$$

be satisfied, the solution contains the arbitrary functions only.

If Z' be linear in ζ, but these conditions are not fulfilled, the solution is of the form

$$\zeta = \zeta_0 + \sum_{j=0}^{j=k} X_j \left(\Sigma_i A_i \frac{d}{d\xi_i}\right)^{k-j} \Phi(\xi_1 \ldots \xi_{n-1})$$
$$+ \sum_{j=0}^{j=l} H_j \left(\frac{d}{d\eta_1}\right)^{l-j} \Psi(\eta_1, \eta_2 \ldots \eta_{n-1});$$

and the quantities X_j, H_j, $\eta_2 \ldots \eta_{n-1}$ are determined by linear equations of the first order.

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* I have to thank Prof. Cayley for this statement of the question proposed for investigation in the present paper.

If Z' be not linear in ζ, the solution, if it exist, must involve integrals of the arbitrary functions (which may be replaced by derivatives if the solution is finite). This case is not discussed in the present paper.

3. The case in which $n+1$ of the arguments are independent is also omitted from the present investigation. It is noticeable that this case has no analogue in the case of equations with two independent variables.

No other cases can arise, since the $\overline{n-1}$ arguments of either function must be different. Hence there cannot be less than $\overline{n-1}$ independent arguments. Neither can there be more than $\overline{n+1}$ independent arguments, since these arguments are by hypothesis functions of only $n+1$ independent quantities $x_1 \ldots x_n$, z.

When r of the arguments of one function are identical with r of the other, (1) can be transformed to an equation having only $\overline{n-r}$ independent variables.

The paper concludes with a note on the application of a similar method to equations of an order higher than the second.

Determination of the Arguments.

1. Let one of the arguments of one of the arbitrary functions (ϕ say) in the solution be u. We may assume that none of the derivatives of ϕ with respect to u occur in the solution. For, supposing $\frac{d^r\phi}{du^r}$ to be the highest derivative, we may replace it by an arbitrary function ψ, so that any lower derivative $\frac{d^s\phi}{du^s}$ would be represented by the indefinite integral $\frac{d^{s-r}}{du^{s-r}}\psi(u)$. The solution may therefore be written in the form

$$F\{x_1, x_2 \ldots x_n, z, \phi(u, \ldots), \ldots\} = 0 \quad \text{.................(2)},$$

where F involves none of the derivatives of ϕ with respect to u.

Differentiate (2) with respect to x_i, x_j in succession, and retain only the terms which involve the derivative $\frac{d^2\phi}{du^2}$, either directly, or implicitly in $\frac{d^2z}{dx_i dx_j}$. Thus

$$\frac{d^2z}{dx_i dx_j}\left\{\frac{dF}{dz}+\frac{dF}{d\phi}\cdot\frac{d\phi}{du}\cdot\frac{du}{dz}+\&c.\ldots\right\}+\ldots+\frac{dF}{d\phi}\cdot\frac{d^2\phi}{du^2}\left(\frac{du}{dx_i}\right)\left(\frac{du}{dx_j}\right)+\ldots=0,$$

where $\left(\frac{du}{dx_i}\right)\equiv\frac{du}{dx_i}+p_i\frac{du}{dz}$.

Multiply by V_{ij}, and take the sum of all the equations formed by giving to i, j all values from 1 to n inclusive. Since (2) is a solution of (1) this sum may be written in the form

$$-V_0\left\{\frac{dF}{dz}+\frac{dF}{d\phi}\cdot\frac{d\phi}{du}\cdot\frac{du}{dz}+\dots\right\}+\dots+\frac{dF}{d\phi}\cdot\frac{d^2\phi}{du^2}\cdot\Sigma_i\Sigma_j V_{ij}\left(\frac{du}{dx_i}\right)\left(\frac{du}{dx_j}\right)$$
$$+\ \&c.\ \dots\dots = 0.$$

From this equation all the second derivatives of z have disappeared, and since, by (2), $p_1 \dots p_n$ do not involve $\frac{d^2\phi}{du^2}$, the only term in which this arbitrary function occurs is the second of those written above. The coefficient of $\frac{d^2\phi}{du^2}$ in this term must therefore vanish; that is, since $\frac{dF}{d\phi}$ is not equal to 0,

$$\left.\begin{aligned} &\Sigma_i\Sigma_j V_{ij}\left(\frac{du}{dx_i}\right)\left(\frac{du}{dx_j}\right)=0,\\ \text{or}\quad &V_{11}\left(\frac{du}{dx_1}\right)^2+V_{22}\left(\frac{du}{dx_2}\right)^2+\dots+2V_{12}\left(\frac{du}{dx_1}\right)\left(\frac{du}{dx_2}\right)+\dots=0\end{aligned}\right\}\quad\dots(3),$$

an equation for the determination of u.

2. Now if $u_1, u_2 \dots$ are the arguments of ϕ, viz., if they are particular values of u, the general value of u is

$$u = \phi\,(u_1, u_2 \dots),$$

where ϕ is arbitrary. Thus u satisfies an equation of the form

$$A_1\left(\frac{du}{dx_1}\right)+A_2\left(\frac{du}{dx_2}\right)+\dots=0,$$

namely, the equation

$$\left.\begin{vmatrix} \left(\frac{du}{dx_1}\right), & \left(\frac{du}{dx_2}\right), & \dots & \left(\frac{du}{dx_n}\right) \\ \frac{du_1}{dx_1}+p_1\frac{du_1}{dz}, & \frac{du_1}{dx}+p_2\frac{du_1}{dz}, & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{du_{n-1}}{dx_1}+p_1\frac{du_{n-1}}{dz}, & \dots & \dots & \dots \end{vmatrix}=0.\right\}\quad\dots(4).$$

Hence the expression on the left of (3) breaks up into two factors linear with respect to $\left(\frac{du}{dx_1}\right)$, &c., each of which gives a set of arguments.

It thus appears that the $\frac{n(n+1)}{1.2}-1$ ratios $\frac{V_{ii}}{V_{11}}, \frac{V_{ij}}{V_{11}}$ are expressible in terms of $2n-2$ ratios $\frac{A_2}{A_1}, \frac{A_i}{A_1} \dots, \frac{B_2}{B_1}, \frac{B_i}{B_1} \dots$, and are therefore connected by $\frac{n(n+1)}{1.2}-1-(2n-2)$ or $\frac{(n-1)(n-2)}{1.2}$ equations of condition. These are evidently the $\frac{(n-1)(n-2)}{1.2}$ different equations that can be formed by giving to i, j all values from 2 to n inclusive in the equation

$$\begin{vmatrix} V_{11}, & V_{1i}, & V_{1j} \\ V_{i1}, & V_{ii}, & V_{ij} \\ V_{j1}, & V_{ji}, & V_{jj} \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

For this is the condition that

$$V_{11} = \lambda A_1 B_1, \quad V_{ii} = \lambda A_i B_i, \quad V_{jj} = \lambda A_j B_j,$$

$$2V_{ij} = \lambda(A_i B_j + A_j B_i), \quad 2V_{j1} = \lambda(A_j B_1 + A_1 B_j), \quad 2V_{i1} = \lambda(A_1 B_i + A_i B_1);$$

and if this be true for all values of i, j, the expression on the left of (3) is identically equal to

$$\lambda \left\{ A_1 \left(\frac{du}{dx_1}\right) + \ldots \right\} \left\{ B_1 \left(\frac{du}{dx_1}\right) + \ldots \right\}.$$

3. Let us now examine the form of A, &c. in the factors of (3). From (4) it is clear that the coefficient of $\left(\frac{du}{dx_i}\right)$ does not contain p_i. Moreover, it is linear with respect to p_1, p_2, &c. For, if we write down the coefficient of such a product as $p_1 p_2$ in the coefficient of $\left(\frac{du}{dx_i}\right)$, we find it to be a determinant having the column $\frac{du_1}{dz}$, $\frac{du_2}{dz}$, &c. repeated; viz, that it is identically zero. Hence we may write

$$A_i = a_i + a_{i1} p_1 + \ldots + a_{ij} p_j + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and

$$a_{ii} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

The $n^2 - 1$ ratios between $a_i \ldots$, $a_{ii} \ldots$, $a_{ij} \ldots$, $a_{ji} \ldots$ are, by (4), expressible in terms of $n-1$ quantities $u_1 \ldots u_{n-1}$, and are therefore connected by $n(n-1)$ equations of condition. These are, of course, implied in (4), and might be obtained thence. We prefer to adopt a different course.

Since u_1, &c. are functions of $x_1 \ldots x_n$, z only, they must satisfy the equation for u independently of the values of $p_1 \ldots p_n$; that is, the coefficients of the powers and products of $p_1 \ldots p_n$ must separately vanish.

The product $p_i p_j$ only occurs in the terms $A_i \left(\frac{du}{dx_i}\right)$, $A_j \left(\frac{du}{dx_j}\right)$, and expressing that its coefficient in (4) vanishes we get the equation

$$a_{ij} + a_{ji} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

The conditions that the coefficient of p_i and the part independent of p_i, &c. vanish, are

$$\left.\begin{array}{l} a_1 \dfrac{du}{dx_1} + \ldots\ldots + a_n \dfrac{du}{dx_n} = 0 \\ \ldots\ldots \qquad \ldots\ldots \\ (\mathrm{I} \equiv)\ a_i \dfrac{du}{dz} + \ldots\ldots + a_{in} \dfrac{du}{dx_n} = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

The equations (9) must be satisfied by the $\overline{n-1}$ different integrals $u_1 \ldots u_{n-1}$. But the system (9) only contains $\overline{n+1}$ independent variables $(x_1 \ldots x_n, z)$. Hence it must be equivalent to two equations only. Suppose these to be $\mathrm{I}=0$, $\mathrm{J}=0$; then any other equation $\mathrm{K}=0$ must be of the form $\mathrm{K} \equiv \lambda \mathrm{I} + \mu \mathrm{J}$.

Expressing that the coefficients of $\dfrac{du}{dx_i}, \dfrac{du}{dx_j}, \dfrac{du}{dx_l}$ on each side of this identity are equal, we get the equations

$$a_{ki} = \mu a_{ji}, \quad a_{kj} = \lambda a_{ij}, \quad a_{kl} = \lambda a_{il} + \mu a_{jl};$$

whence, eliminating λ, μ, and reducing by (8),

$$a_{ij} a_{kl} + a_{ik} a_{lj} + a_{il} a_{jk} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

From this we get $(n-1)^2$ different conditions by giving to k, l all values from 0 to n inclusive, except i, j. These include (8) as a special case, viz., when $k=l$.

The conditions that $\mathrm{I}=0$, $\mathrm{J}=0$ should have $\overline{n-1}$ common integrals have been given by Boole (Supplementary Vol., chap. XXV.). If we write

$$\Delta_i \equiv \frac{d}{dx_j} + \frac{a_i}{a_{ij}} \cdot \frac{d}{dz} + \ldots + \frac{a_{ik}}{a_{ij}} \cdot \frac{d}{dx_k} + \ldots,$$

$$\Delta_j \equiv \frac{d}{dx_i} + \frac{a_j}{a_{ji}} \cdot \frac{d}{dz} + \ldots + \frac{a_{jk}}{a_{ji}} \cdot \frac{d}{dx_k} + \ldots,$$

the conditions are

$$\left.\begin{array}{l} \Delta_i \dfrac{a_j}{a_{ji}} = \Delta_j \dfrac{a_i}{a_{ij}} \\ \ldots\ldots \qquad \ldots\ldots \\ \Delta_i \dfrac{a_{jk}}{a_{ji}} = \Delta_j \dfrac{a_{ik}}{a_{ij}} \\ \ldots\ldots \qquad \ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

The $(n-1)$ conditions (11) and the $(n-1)^2$ conditions (10) together make up the $n(n-1)$ conditions which must be satisfied by the coefficients a. The sufficiency of the conditions is proved by the fact that they insure, first, that two independent equations of the system (9) shall have $n-1$ common integrals, which are the arguments required; second, that these integrals shall satisfy all the other equations (9).

4. *Example:*

$$(p_2-p_3)\left(\frac{d^2z}{dx_1^2}-\frac{d^2z}{dx_2dx_3}\right)+(p_3-p_1)\left(\frac{d^2z}{dx_2^2}-\frac{d^2z}{dx_3dx_1}\right)$$
$$+(p_1-p_2)\left(\frac{d^2z}{dx_3^2}-\frac{d^2z}{dx_1dx_2}\right)+V=0.$$

This equation satisfies the single condition (5), and the factors of (3) are found to be
$$\left(\frac{du}{dx_1}\right)+\left(\frac{du}{dx_2}\right)+\left(\frac{du}{dx_3}\right)=0,$$
$$(p_2-p_3)\left(\frac{du}{dx_1}\right)+(p_3-p_1)\left(\frac{du}{dx_2}\right)+(p_1-p_2)\left(\frac{du}{dx_3}\right)=0.$$

The first of these gives us
$$\frac{du}{dx_1}+\frac{du}{dx_2}+\frac{du}{dx_3}=0,$$
$$\frac{du}{dz}=0,$$
simultaneously, whence $u=\phi\,(x_2-x_3,\ x_3-x_1)$.

The second gives $a_1=a_2=a_3=0$,
$$a_{23}=1=-a_{32},\quad a_{31}=1=-a_{13},\quad a_{12}=1=-a_{21};$$
and the equations (9) reduce to
$$\frac{du}{dx_1}=\frac{du}{dx_2}=\frac{du}{dx_3};$$
whence $u=\psi\,(x_1+x_2+x_3,\ z)$.

In fact, the solution of the given equation, when $V=0$, is
$$\phi\,(x_2-x_3,\ x_3-x_1)+\psi\,(x_1+x_2+x_3,\ z)=0.$$

Transformation of the Variables.

5. Suppose that one of the arguments is identical with one of the independent variables, say x_1, so that x_1 satisfies the equations (9). This requires that the coefficient of $\frac{du}{dx_1}$ should vanish in each of the equations, or $a_1=a_{12}=a_{13}=\ldots=a_{1n}=0$.

Also, by (8), $a_{21}=a_{31}=\ldots=a_{n1}=0$;

that is, $A_1=0$, and none of the coefficients A involve p_1.

So if r of the arguments be $x_1, \ldots x_r$ the coefficients $A_1 \ldots A_r$ vanish, and the remaining coefficients do not involve $p_1 \ldots p_r$.

By using these results we can simplify the equation (1) without undertaking the laborious task of actually transforming the variables.

6. If (3) is a perfect square—viz., if for all values of i, j,

$$V_{ii}V_{jj} - V_{ij}^2 = 0,$$

the two sets of arguments become identical. Let them be represented by $\xi_1 .. \xi_{n-1}$. Take two functions of $x_1 \dots x_n, z$ which are independent of these and each other. Represent them by ξ_n, ζ. Now transform the equation, taking $\xi_1 \dots, \zeta$ as variables instead of $x_1 \dots z$. By Art. 5 each of the factors into which (3) splits up becomes simply

$$\left(\frac{du}{dx_n}\right) = 0.$$

Thus all the coefficients V of the transformed equation vanish except V_{nn}, and the transformed equation is therefore of the form

$$\frac{d^2\zeta}{d\xi_n^2} + V_0 = 0 \qquad (12).$$

7. If (3) be not a perfect square, so that the two sets of arguments are not identical, let these arguments be represented by $\xi_1 \dots \xi_{n-1}$, $\eta_1 \dots \eta_{n-1}$. For the sake of generality we may assume that r of these arguments are identical with r of the other set; *e. g.*,

$$\xi_1 = \eta_1, \quad \dots\dots \quad \xi_r = \eta_r.$$

Take for new independent variables $\xi_1 \dots \xi_{n-1}, \eta$, the last being any one of the arguments $\eta_{r+1} \dots \eta_{n-1}$.

For dependent variable take ζ such that $\xi_1 \dots \xi_{n-1}, \eta, \zeta$ considered as functions of $x_1 \dots x_n, z$ may be mutually independent.

In this case the equations into which (3) splits up are of the form

$$\left(\frac{du}{d\eta}\right) = 0,$$

$$A_{r+1}\left(\frac{du}{d\xi_{r+1}}\right) + \dots + A_{n-1}\left(\frac{du}{d\xi_{n-1}}\right) = 0,$$

where $A_{r+1} \dots A_{n-1}$ do not involve $\frac{d\zeta}{d\xi_1}, \dots \frac{d\zeta}{d\xi_r}$.

The transformed equation is therefore of the form

$$A_{r+1}\frac{d^2\zeta}{d\xi_{r+1}d\eta} + \dots + A_{n-1}\frac{d^2\zeta}{d\xi_{n-1}d\eta} + V_0 = 0 \qquad (13).$$

It is to be noted that there is a considerable range of choice in the new independent variables. These may be either $\xi_1 \dots \xi_{n-1}$ with any one of the arguments $\eta_{r+1} \dots \eta_{n-1}$, or they may be $\eta_1 \dots \eta_{n-1}$ with any one of the arguments $\xi_{r+1} \dots \xi_{n-1}$.

The two equations (12), (13) are together equivalent to (1) limited by the conditions of Arts. 2, 3. We shall therefore examine the form of V_0 as it occurs in these equations instead of in the general form.

On the Form of V_0.

8. Consider first the equation

$$\frac{d^2\zeta}{d\xi_n^2}+V_0=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

If this be capable of exact solution in the form (2), V_0 cannot involve $\frac{d\zeta}{d\xi_1}\ldots\frac{d\zeta}{d\xi_{n-1}}$. For $\frac{d\zeta}{d\xi}$, for instance, involves the arbitrary function $\frac{d\phi}{d\xi_1}$, which does not occur in any other part of (12). Thus, (2) cannot be a solution of (12) if V_0 contain $\frac{d\zeta}{d\xi_1}$, unless some condition is imposed upon the form of ϕ; *i. e.*, unless ϕ ceases to be arbitrary. Hence V_0 is a function of $\frac{d\zeta}{d\xi_n}$ and the variables $\xi_1 \ldots \xi_n . \zeta$ only, or (12) is an ordinary differential equation of the second order. The solution of this equation involves two arbitrary constants; that is, two arbitrary quantities whose values do not change when ξ_n changes; that is, two arbitrary functions of $\xi_1 \ldots \xi_{n-1}$. The solution of the original equation is, of course, obtained from this by writing for $\xi_1 \ldots \xi_n . \zeta$ their values in terms of $x_1 \ldots x_n$, z.

9. In the second form of equation, viz.,

$$A_{r+1}\frac{d^2\zeta}{d\xi_{r+1}d\eta}+\ldots+A_{n-1}\frac{d^2\zeta}{d\xi_{n-1}d\eta}+V_0=0 \quad \ldots\ldots\ldots\ldots(13),$$

$A_{r+1}, \ldots A_{n-1}$ do not involve $\frac{d\zeta}{d\xi_1}\ldots\frac{d\zeta}{d\xi_r}$. Hence, by precisely the same reasoning as in the last article, V_0 cannot involve these quantities, and the equation (13) is therefore reduced to an equation with $\overline{n-r}$ independent variables. We shall discuss the equation under the supposition $r=0$, since, by the above, (13) is, in fact, of the form of an equation in which this condition is fulfilled. We write the equation, then, in the form

$$A_1\frac{d^2\zeta}{d\xi_1 d\eta_1}+\ldots+A_{n-1}\frac{d^2\zeta}{d\xi_{n-1}d\eta_1}+V_0=0 \quad \ldots\ldots\ldots\ldots\ldots(14).$$

The arguments are $\quad \xi_1 \ldots \xi_{n-1},$

$\quad \eta_1 \ldots \eta_{n-1},$

where $\eta_1 \ldots \eta_{n-1}$ are $\overline{n-1}$ independent integrals of

$$\sum_{i=1}^{i=n-1} A_i\frac{d\eta}{d\xi_i}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15).$$

Two cases arise. If of tho $2n-2$ arguments n only are mutually independent when considered as functions of $x_1 \ldots x_n$, z, then none of the quantities $\eta_2 \dots \eta_{n-1}$ involve the new dependent variable ζ, for this, considered as function of $x_1 \dots x_n$, z, is independent of $\xi_1, \dots \xi_{n-1} . \eta$. If, however, $\overline{n+1}$ of the arguments are mutually independent, then $\eta_2 \dots \eta_{n-1}$ or some of them must involve ζ. We omit the discussion of this case in the present paper, and confine our attention to the first alone; viz., we assume that in the transformed equation the arguments do not involve the dependent variable, and that accordingly the coefficients A_i do not involve ζ or any of its first derivatives $\frac{d\zeta}{d\xi_i}, \dots \frac{d\zeta}{d\eta_1}$.

10. Under these circumstances ζ must occur explicitly in F in (2); thus we may solve for ζ and write the solution in the more convenient form

$$\zeta = \mathrm{F}\{\xi \dots \xi_i \dots \eta_1, \ \phi(\xi \dots), \ \psi(\dots \eta \dots), \dots\} \quad \text{..........(16).}$$

Form from this the derivatives $\frac{d\zeta}{d\xi_i}, \frac{d\zeta}{d\eta}, \frac{d^2\zeta}{d\xi_i d\eta}$, and retain only the terms which involve the derivatives of ϕ, ψ. Thus,

$$\left.\begin{aligned} \frac{d\zeta}{d\xi_i} &= \frac{d\mathrm{F}}{d\xi_i} + \frac{d\mathrm{F}}{d\phi} \cdot \frac{d\phi}{d\xi_i} + \dots + \frac{d\mathrm{F}}{d\psi}\left(\frac{d\psi}{d\eta_2} \cdot \frac{d\eta_2}{d\xi_i} + \dots\right) + \dots \\ \frac{d\zeta}{d\eta_1} &= \frac{d\mathrm{F}}{d\eta_1} + \frac{d\mathrm{F}}{d\psi}\left(\frac{d\psi}{d\eta_1} + \frac{d\psi}{d\eta_2} \cdot \frac{d\eta_2}{d\eta_1} + \dots\right) + \dots \end{aligned}\right\} \text{......(17);}$$

$$\begin{aligned} \frac{d^2\zeta}{d\xi_i d\eta_1} = \ & \frac{d^2\mathrm{F}}{d\xi_i d\eta_1} + \frac{d^2\mathrm{F}}{d\xi_i d\psi}\left\{\frac{d\psi}{d\eta_1} + \frac{d\psi}{d\eta_2} \cdot \frac{d\eta_2}{d\eta_1} + \dots\right\} + \dots \\ & + \frac{d^2\mathrm{F}}{d\eta_1 d\phi} \cdot \frac{d\phi}{d\xi_i} + \frac{d^2\mathrm{F}}{d\psi d\phi} \cdot \frac{d\phi}{d\xi_i}\left\{\frac{d\psi}{d\eta_1} + \frac{d\psi}{d\eta_2} \cdot \frac{d\eta_2}{d\eta_1} + \dots\right\} + \dots \\ & + \frac{d^2\mathrm{F}}{d\eta_1 d\psi}\left\{\frac{d\psi}{d\eta_2} \cdot \frac{d\eta_2}{d\xi_i} + \dots\right\} . \\ & + \frac{d^2\mathrm{F}}{d\psi^2}\left\{\frac{d\psi}{d\eta_2} \cdot \frac{d\eta_2}{d\xi_i} + \dots\right\}\left\{\frac{d\psi}{d\eta_1} + \frac{d\psi}{d\eta_2} \cdot \frac{d\eta_2}{d\eta_1} + \dots\right\} + \dots \\ & + \frac{d\mathrm{F}}{d\psi}\left\{\frac{d^2\psi}{d\eta_1 d\eta_2} \cdot \frac{d\eta_2}{d\xi_i} + \dots + \frac{d\psi}{d\eta_2} \cdot \frac{d^2\eta_2}{d\eta_1 d\xi_i} + \dots\right\} + \dots \end{aligned}$$

A remark as to the completeness of this value of $\frac{d^2\zeta}{d\xi_i d\eta_1}$ is necessary. Since two differentiations of ζ are performed, it is clear that derivatives of ϕ and ψ of the first order may arise from integrals of these functions in ζ; and as a matter of fact such derivatives will occur in $\frac{d^2\zeta}{d\xi_i d\eta_1}$, but will disappear from the equation (14). For this equation, as far as terms of the second order are concerned, may be written in the form

$$\Sigma_i \mathrm{A}_i \frac{d}{d\xi_i} \cdot \frac{d}{d\eta_1} \cdot \zeta = 0.$$

Now of these two operators $\frac{d}{d\eta_1}$ does not affect ϕ, for the arguments of this function are independent of η_1, and $\Sigma A_i \frac{d}{d\xi_i}$ does not affect ψ, since each of the arguments of this function satisfies the equation (15). Hence neither of the arbitrary functions will be differentiated twice, and their derivatives of the first order can only arise from the part of ζ involving ϕ, ψ.

An illustration occurs which may throw some light on this point. In the value of $\frac{d^2\zeta}{d\xi_i d\eta_1}$ we have the function $\frac{d^2\psi}{d\eta_1 d\eta_2}$. By what has been stated above, this function must disappear from (14), and we find its coefficient in (14) to be

$$\frac{dF}{d\psi} \cdot \Sigma A_i \frac{d\eta_2}{d\xi_i},$$

of which the second factor vanishes, because η_2 is a solution of (15).

In virtue of the same condition, other parts of the value of $\frac{d^2\zeta}{d\xi_i d\eta}$ will disappear on substitution in (14). If we retain only those terms which involve the derivatives and which will be represented in (14) we obtain the simplified equation

$$\frac{d^2\zeta}{d\xi_i d\eta_1} = \frac{d^2F}{d\phi\, d\psi} \cdot \frac{d\phi}{d\xi_i}\left(\frac{d\psi}{d\eta_1}\right) + \frac{d^2F}{d\eta_1 d\phi} \cdot \frac{d\phi}{d\xi_i}$$
$$+ \frac{d^2F}{d\xi_i d\psi}\left(\frac{d\psi}{d\eta_1}\right) + \frac{dF}{d\psi} \cdot \sum_{j=2}^{j=n-1} \frac{d\psi}{d\eta_j} \cdot \frac{d^2\eta_j}{d\eta_1 d\xi_i} + \dots \qquad \dots\dots (18),$$

where $$\left(\frac{d\psi}{d\eta_1}\right) \equiv \frac{d\psi}{d\eta_1} + \frac{d\psi}{d\eta_2} \cdot \frac{d\eta_2}{d\eta_1} + \dots.$$

In order that ζ should satisfy (14) it is necessary that, when we substitute (17) and (18) in that equation, the coefficient of each power and product of the arbitrary functions should separately vanish. The conditions hence obtained will serve in great measure to determine the form of V_0 in (14), and of F in (16).

11. In the first place, V_0 must not involve powers or products of the arbitrary functions other than those which occur in (18). Hence the most general form of V_0 is

$$\Sigma_i B_i \frac{d\zeta}{d\xi_i} \cdot \frac{d\zeta}{d\eta_1} + \Sigma_i P_i \frac{d\zeta}{d\xi_i} + Q \frac{d\zeta}{d\eta_1} + Z',$$

where B_i, P_i, Q, Z′ do not involve $\frac{d\zeta}{d\xi_i}$ or $\frac{d\zeta}{d\eta_1}$. Now, equating to zero the coefficient of $\frac{d\phi}{d\xi_i}\left(\frac{d\psi}{d\eta_1}\right)$ in (14), we get

$$A_i \frac{d^2F}{d\phi\, d\psi} + B_i \frac{dF}{d\psi} \cdot \frac{dF}{d\phi} = 0 \dots\dots\dots\dots\dots\dots (19)$$

Hence the ratio $A_i : B_i$ is independent of i; viz., (14) may be written

$$\Sigma_i A_i \left(\frac{d^2\zeta}{d\xi_i d\eta_1} + k\frac{d\zeta}{d\xi_i}\cdot\frac{d\zeta}{d\eta_1}\right) + \ldots = 0.$$

We shall now show that we can make k vanish by a change of the dependent variable only. Take as new dependent variable

$$\zeta' \equiv f(\zeta, \xi, \ldots \xi_{n-1}, \eta_1);$$

whence

$$\frac{d\zeta'}{d\xi_i} = \frac{df}{d\zeta}\cdot\frac{d\zeta}{d\xi_i} + \frac{df}{d\xi_i},$$

$$\frac{d\zeta'}{d\eta_1} = \frac{df}{d\zeta}\cdot\frac{d\zeta}{d\eta_1} + \frac{df}{d\eta_1},$$

$$\frac{d^2\zeta'}{d\xi_i d\eta_1} = \frac{df}{d\zeta}\cdot\frac{d^2\zeta}{d\xi_i d\eta_1} + \frac{d^2f}{d\zeta^2}\cdot\frac{d\zeta}{d\xi_i}\cdot\frac{d\zeta}{d\eta_1}$$

$$+ \frac{d^2f}{d\xi_i d\zeta}\cdot\frac{d\zeta}{d\eta_1} + \frac{d^2f}{d\eta_1 d\zeta}\cdot\frac{d\zeta}{d\xi_1} + \frac{d^2f}{d\xi_i d\eta_1}.$$

Hence the expression

$$A_i\frac{d^2\zeta'}{d\xi_i d\eta_1} + P_i'\frac{d\zeta'}{d\xi_i} + Q_i'\frac{d\zeta'}{d\eta_1} + Z_i''$$

is equivalent to the expression

$$A_i\frac{d^2\zeta}{d\xi_i d\eta_1} + A_i k\frac{d\zeta}{d\xi_1}\cdot\frac{d\zeta}{d\eta_1} + P_i\frac{d\zeta}{d\xi_i} + Q_i\frac{d\zeta}{d\eta_1} + Z_i',$$

provided we have

$$\frac{d^2f}{d\zeta^2} = k\frac{df}{d\zeta},$$

$$\frac{d^2f}{d\eta_1 d\zeta} + P_i'\frac{df}{d\zeta} = P_i\frac{df}{d\zeta},$$

$$\frac{d^2f}{d\xi_i d\zeta} + Q_i'\frac{df}{d\zeta} = Q_i\frac{df}{d\zeta},$$

$$\frac{d^2f}{d\xi_i d\eta_1} + P_i'\frac{df}{d\xi_i} + Q_i'\frac{df}{d\eta_1} + Z_i'' = Z_i'\frac{df}{d\zeta}.$$

The first of these determines ζ'; viz.,

$$\zeta' \equiv f = \int e^{\int k dz}\, dz,$$

while the others determine the values of P_i', Q_i', Z_i'' when P_i, Q_i, Z_i' are given.

Now this value of ζ' is independent of i, and therefore, if (14) be changed to its equivalent with ζ' as dependent variable, the resulting equation will be linear with respect to $\frac{d\zeta'}{d\xi_i}$, $\frac{d\zeta'}{d\eta_1}$. We shall therefore assume for the future that $B_i = 0$, since (14) may always be reduced to an equation in which this is true, if it be soluble in the assumed form.

On this assumption it follows from (19) that $\frac{d^2F}{d\varphi\, d\psi} = 0$; viz., in (16) the two arbitrary functions are separated.

12. From the coefficient of $\frac{d\phi}{d\xi_i}$ we learn that

$$A_i \frac{d^2F}{d\eta_1\, d\varphi} + P_i \frac{dF}{d\varphi} = 0,$$

so that the ratio $A_i : P_i$ is the same for all values of i; and we may write (14) in the form

$$\Sigma A_i \left(\frac{d^2\zeta}{d\xi_i\, d\eta_1} + P \frac{d\zeta}{d\xi_i}\right) + Q \frac{d\zeta}{d\eta_1} + Z' = 0 \quad \text{...........} \quad (20).$$

The conditions for all values of i then become

$$\frac{d^2F}{d\eta_1\, d\varphi} + P \frac{dF}{d\varphi} = 0 \quad \text{.......................} \quad (21).$$

Differentiate this with respect to ψ, and notice that $\frac{d^2F}{d\phi\, d\psi}$ vanishes; we shall obtain $\frac{dP}{d\psi} \cdot \frac{dF}{d\varphi} = 0,$

of which the first factor alone can vanish. Thus P does not contain ψ; *i.e.*, P does not contain ζ.*

The integration of (21) with respect to η_1 and ϕ in succession gives

$$\zeta \equiv F = e^{-\int P\, d\eta_1} \cdot \phi\,(\xi_1, \ldots \xi_{n-1}) + \text{parts not involving } \phi.$$

13. The coefficient of $\left(\frac{d\psi}{d\eta_1}\right)$ gives the condition

$$\Sigma A_i \frac{d^2F}{d\xi_i\, d\psi} + Q \frac{dF}{d\psi} = 0 \quad \text{.....................} \quad (22).$$

Since $A_i \ldots$ do not involve ζ we can show by differentiating this equation with respect to ϕ, that $\frac{dQ}{d\phi}$ vanishes, or that Q does not contain ζ.

Also, from (22), we have $\frac{dF}{d\psi} = H,$

$$F = H\psi\,(\eta_1, \eta_2, \ldots \eta_{n-1}) + \text{a part not involving } \psi,$$

* This is wrong. Suppose the solution involves a function ψ_1 defined by the equation $\frac{d}{d\eta_1}\psi_1 = \psi$. Then in $\frac{d\zeta}{d\eta_1}$ we have a term $\frac{dF}{d\psi_1}\psi$; and in $\frac{d^2\zeta}{d\xi_i d\eta_1}$ a term $\frac{d^2F}{d\psi_1\, d\phi}\psi \frac{d\phi}{d\xi_i}$. Thus the first equation of Art. 12 should be

$$A_i \left\{ \frac{d^2F}{d\eta_1 d\phi} + \frac{d^2F}{d\psi_1 d\phi}\psi + \ldots \right\} + P_i \frac{dF}{d\phi} = 0.$$

This does not affect the first result of the article; but the second result is true if, and only if, $\frac{d^2F}{d\psi_1 d\phi}$ vanishes. A corresponding correction must be made in Art. 13. (*July*, 1876.)

where H is a solution of the equation

$$\Sigma A_i \frac{dH}{d\xi_i} + QH = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (23).$$

14. The coefficient of $\frac{d\psi}{d\eta_j}$ gives

$$\Sigma_i A_i \left\{ \frac{d^2\eta_j}{d\xi_i\, d\eta_i} + P \frac{d\eta_j}{d\xi_i} \right\} = 0.$$

The second set of terms vanishes in virtue of (15), of which η_j is a solution. The remaining condition

$$\Sigma_i A_i \frac{d^2\eta_j}{d\xi_i\, d\eta_1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

may also be simplified by the use of (15); viz.,

$$\Sigma_i A_i \frac{d\eta_j}{d\xi_i} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

From this we get $\Sigma_i A_i \frac{d^2\eta_j}{d\eta_1\, d\xi_i} + \Sigma \frac{dA_i}{d\eta_1} \cdot \frac{d\eta_j}{d\xi_i} = 0,$

so that (24) reduces to $\Sigma_i \frac{dA_i}{d\eta_1} \cdot \frac{d\eta_j}{d\xi_i} = 0.$

Now this equation has $n-2$ integrals $(\eta_2, \ldots \eta_{n-1})$ in common with (15), and it has the same $\overline{n-1}$ independent variables. Hence it must be identical with (15), or $\mu A_i = \frac{dA_i}{d\eta_1}$;

whence the quantities A_i either are, or can, by moving a factor common to all, be made independent of η_1. The same is therefore true of $\eta_2, \ldots \eta_{n-1}$, the integrals of (15).

15. We have now to examine the form of Z′ in (20). It will be convenient to consider first the case in which the solution contains the arbitrary functions ϕ, ψ unaccompanied by any of their indefinite integrals. In this case the complete solution is from Arts. 12, 13,

$$\zeta = \zeta_0 + e^{-\int P d\eta_1} . \phi(\xi_1, \ldots \xi_{n-1}) + H\psi(\eta_1, \ldots \eta_{n-1}) \ldots\ldots\ldots (25).$$

Here ζ, and therefore its derivatives, are linear with respect to ϕ, ψ, and A_i, P, Q do not involve ϕ, ψ. Hence Z′ must also be linear with respect to ϕ, ψ; that is, it is a linear function of ζ.

Accordingly (20) is of the form

$$\Sigma_i A_i \left(\frac{d^2\zeta}{d\xi_i\, d\eta_1} + P \frac{d\zeta}{d\xi_i} \right) + Q \frac{d\zeta}{d\eta_1} + Z\zeta + W = 0 \ldots\ldots\ldots (26),$$

where A_i, P, Q, Z, W do not involve ζ.

Now let us express that the coefficient of ϕ in (26) is zero. We obtain the condition $$\Sigma_i A_i \frac{dP}{d\xi_i} = Z - PQ \quad \text{......................} \quad (27).$$

Again, from the coefficient of ψ we get, after an easy reduction by means of (23), the equation $$\frac{dQ}{d\eta_1} = Z - PQ \quad \text{........................} \quad (28).$$

Lastly, taking the part independent of arbitrary functions, we find that ζ_0 is any particular integral of (26).

The conditions (27), (28) may be obtained in two other ways, which put in evidence very clearly the sufficiency and necessity of these relations.

In the first place they are the conditions that the coefficients of the arbitrary functions in (25) satisfy (26) when $W=0$. Thus they are necessary, since we may put either ϕ or ψ equal to 1, and the other equal to zero.

Secondly, they express the conditions that we can find a new dependent variable ζ' such that the transformed equation is

$$\Sigma A_i \frac{d}{d\xi_i} \cdot \frac{d\zeta'}{d\eta_1} = 0,$$

the solution of which is

$$\zeta' = \phi(\xi_1, \dots \xi_{n-1}) + \psi(\eta_1, \dots \eta_{n-1}).$$

Hence the sufficiency.

[The last mode of viewing the question gives also the meaning of these conditions separately in a convenient form. We can always transform (26) so that the P or Q or Z of the transformed equation may vanish (viz., by taking $\zeta e^{\int P d\eta_1}$, $\frac{\zeta}{H}$ or $\frac{\zeta}{\zeta_0}$ as new dependent variable). Then (27) expresses that in the transformed equation P, Z simultaneously vanish; (28) that Q, Z simultaneously vanish; and

$$\Sigma_i A_i \frac{dP}{d\xi_i} = \frac{dQ}{d\eta_1} \quad \text{..........................} \quad (29)$$

expresses the same for the P, Q of the new equation.]

16. We now proceed to discuss (20) when its solution involves the integrals of ϕ, ψ. In this case z' need not be linear in ζ; for instance, it may be an exponential function of ζ. In the present paper, however, we shall limit the discussion to that of (26) when the conditions (27) (28) are not satisfied.

If the condition (27) be not satisfied, the coefficient of ϕ in (26) does not vanish; the complete value of ζ must therefore involve some quantity such that the differentiations implied in (26) will give rise to a new term of the form $k \cdot \phi$. But the operator $\frac{d}{d\eta_1}$ does not affect ϕ. Thus, the new part of ζ must involve some function of ϕ_1 where ϕ_1 is defined by the equation

$$\Sigma_i A_i \frac{d}{d\xi_i} . \phi_1 = \phi.$$

It is easily shown that ϕ_1 occurs in the first power in ζ. For suppose we had

$$\zeta = X_0\phi + X_1\phi_1 + X^1 f(\phi_1) + \dots$$

where $X_0 \equiv e^{-\int P d\eta_1}$ and $f(\phi_1)$ is not linear in ϕ_1. The coefficient of ϕ in (26) corresponding to this amended value of ζ consists of two parts, one of which involves the arbitrary factor $f'(\phi_1)$. Each of these must vanish separately, and we get

$$\frac{dX'}{d\eta_1} + PX' = 0,$$

whence

$$X' = e^{-\int P d\eta_1} = X_0,$$

and

$$\frac{dX_1}{d\eta_1} + Px_1 + \Sigma_i A_i \left(\frac{d^2 X_0}{d\xi_i d\eta_1} + P\frac{dX_0}{d\xi_i}\right) + Q\frac{dX_0}{d\eta_1} + ZX_0 = 0.$$

The first of these equations shows that any non-linear part of the new terms in ζ may be merged in φ. The second gives an equation for the determination of X_1 the coefficient of ϕ_1 in ζ.

If now the coefficient of ϕ_1 in (26) vanishes, the equation is complete: if not, we must add a term involving ϕ_2, where

$$\Sigma A_i \frac{d}{d\xi_i}\phi_2 \equiv \phi_1 \quad \text{or} \quad \left(\Sigma A_i \frac{d}{d\xi_i}\right)^2 \phi_2 \equiv \varphi.$$

This may be shown to be linear as above; and the coefficient is calculated in the same way.

Precisely the same reasoning leads to similar results for the series commencing with $H\psi$. We may write the complete solution, therefore, in the form

$$\zeta = \zeta_0 + \sum_{j=0}^{j=k} X_j \phi_j + \sum_{j=0}^{j=l} H_j \psi_j \quad \dots\dots\dots\dots\dots\dots (30),$$

where

$$\left(\Sigma A_i \frac{d}{d\xi_i}\right)^j \phi_j \equiv \phi_0,$$

$$\left(\frac{d}{d\eta_1}\right)^j \psi_j \equiv \psi_0,$$

$$\frac{dX_0}{d\eta} + PX_0 = 0, \quad \Sigma A_i \frac{dH_0}{d\xi_i} + QH_0 = 0,$$

$$\frac{dX_{j+1}}{d\zeta_1} + PX_{j+1} + \overline{X}_j = 0,$$

$$\Sigma_i A_i \frac{dH_{j+1}}{d\xi_i} + QH_{j+1} + \overline{H}_j = 0,$$

where we have written $\overline{X}_j$, $\overline{H}_j$ for the result of substituting X_j, H_j in (26) when $W=0$.

Lastly, we must have X_k, H_e satisfying the equation (26) when $W=0$.

As it is unsatisfactory to employ inverse operators in the solution, we will point out that these may at once be removed if the solution be finite. For if we replace ϕ_k and ψ_e by new arbitrary functions Φ, Ψ, (30) may be written in the form

$$\zeta = \zeta_0 + \sum_{j=0}^{j=k} X_j \left(\Sigma_i A_i \frac{d}{d\xi_i}\right)^{k-j} \Phi\,(\xi_1, \dots \xi_{n-1})$$
$$+ \sum_{j=0}^{j=l} H_j \left(\frac{d}{d\eta_1}\right)^{l-j} (\Psi\eta_1, \eta_2 \dots \eta_{n-1}) \quad \dots\dots\dots\dots (31),$$

a solution in which only differentiations of the arbitrary functions occur.

17. *Example.* $\dfrac{d^2z}{dx_1\,dx_2} + \dfrac{d^2z}{dx_1\,dx_3} - \dfrac{4z}{(x_1+x_2+x_3)^2} = 0.$

Using the notation of Art. 16, we have

$$P = 0, \quad Q = 0,$$
$$X_0 = 1, \quad H_0 = 1,$$
$$\overline{X}_0 = \overline{H}_0 = \frac{-4}{(x_1+x_2+x_3)^2},$$

so that neither series of integrals is complete in one term.

For X_1 we have

$$\frac{dX_1}{dx_1} - \frac{4}{(x_1+x_2+x_3)^2} = 0,$$
$$X_1 = -\frac{4}{x_1+x_2+x_3},$$
$$\text{and } \overline{X}_1 = 0,$$

so that the solution is complete as far as one series of integrals is concerned.

For H_1 the equation

$$\frac{dH_1}{dx_2} + \frac{dH_1}{dx_2} - \frac{4}{(x_1+x_2+x_2)^2} = 0$$

gives $$H_1 = -\frac{2}{x_1+x_2+x_3},$$

whence $$\overline{H}_1 = 0,$$

or H_1 satisfies the original equation.

The solution is therefore

$$z = \phi\,(x_2, x_3) - \frac{4}{x_1+x_2+x_3}\left(\frac{d}{dx_2}+\frac{d}{dx_3}\right)^{-1}\phi\,(x_2, x_3)$$
$$+ \psi\,(x_1, x_2-x_3) - \frac{2}{x_1+x_2+x_3}\left(\frac{d}{dx_1}\right)^{-1}\psi\,(x_1, x_2-x_3)\,;$$

or changing the arbitrary functions, and arranging,

$$z = \frac{2}{x_1+x_2+x_3}\{\phi(x_2, x_3) + \psi(x_1, x_2-x_3)\} - \frac{1}{2}\cdot\frac{d\phi}{dx_1} - \frac{1}{2}\frac{d\phi}{dx_2} - \frac{d\psi}{dx_1}.$$

18. In conclusion, I may be permitted to notice that a similar method is applicable without difficulty, though not without a considerable amount of labour in the general case, to equations of an order higher than the second. An equation of the m^{th} order, and having n independent variables, when it is completely soluble in terms of arbitrary functions of definite functions of the variables, will involve m arbitrary functions each having $n-1$ arguments. If the arguments are the same in all the functions, the given equation is transformable into an ordinary equation of the m^{th} order.

If n of the arguments are independent, and none of the integrals appear in the solution, it appears likely that the equation can be expressed in the form

$$\Delta_1 . \Delta_2 \ldots \Delta_m \zeta = 0,$$

where
$$\Delta \equiv \mathrm{A}\frac{d}{dx_1} + \mathrm{B}\frac{d}{dx_2} + \ldots + \mathrm{N}\frac{d}{dx_n}$$

and where Δ_1, Δ_2, &c., are commutative. That this is sufficient is clear from the fact that the solution can be found, viz., it is

$$\zeta = \sum_{k=1}^{k=m} \phi_k,$$

where ϕ_k is an arbitrary function whose arguments are the integrals of

$$\Delta_k u = 0.$$

The necessity of this condition, however, is not obvious.

If i of the arguments should be common to all the functions ϕ_k, the given equation is reducible to an equation of the m^{th} order having $m-i$ independent variables.

January 13th, 1876.

LORD RAYLEIGH, F.R.S., Vice-President, in the Chair.

Major J. R. Campbell, Mr. R. Forsyth Scott, and Professor H. W. Lloyd Tanner, were admitted into the Society.

The following communications were made to the Society:—Mr. J. W. L. Glaisher, "On an Elliptic Function Identity;"—Professor Tanner, on "The Solution of Partial Differential Equations of the second order with any number of variables, when there is a complete first integral;"—Professor Clifford on "Free Motion of a rigid system in an n-fold homaloid; expression of the velocities by Abelian functions;"—Lord Rayleigh on "The approximate solution of certain Potential Problems."

The following presents were received:—

"Une réforme géométrique: introduction à la géométrie descriptive des cristalloïdes, par le comte Léopold Hugo." Paris, 1874.

"Géométrie hugodomoidale anhellénique, mais philosophique et architectonique: la question de l'équidomoïde et des cristalloïdes géométriques, par le comte Léopold Hugo," Paris, 1875: from the Author.

"Ueber die v. Staudt'schen Würfe, von Rud. Sturm in Darmstadt" (Math. Annalen, Oct. 1875).

"Bulletin de la Société Mathématique de France." Tome iii., Juin, No. 2. Paris, 1875.

"Proceedings of Royal Society." Vol. xxiv., Nos. 164, 165.

"Bulletin des Sciences Mathématiques et Astronomiques." Tome ix., Sept., Oct., 1875.

"Annali di Matematica." Serie ii^a. Tomo vii° Fasc. 2°. (Nov. 1875, Milano.)

"Ueber eine Relation zwischen den Singularitäten einer algebraischen Curve, von Felix Klein in München" (aus den Sitzungsberichten der Physik.-Med.-Societät zu Erlangen, 13 December, 1875).

"Ueber den Zusammenhang der Flächen, von F. Klein in München." From the author.

"The Theory of Screws; a Study in the Dynamics of a Rigid Body." By Dr. R. Stawell Ball. 1876. From the author.

"Monatsbericht," Sept., Oct., Nov., 1875.

"Crelle." 81 Band, zweites Heft, 1875.

On an Elliptic-Function Identity. By J. W. L. Glaisher, M.A., F.R.S.

[*Read January 13th*, 1876.]

1. The identity in question is that

$$e^{4\left\{\frac{q}{1-q}\cos x+\frac{1}{3}\frac{q^3}{1-q^6}\cos 3x+\frac{1}{5}\frac{q^5}{1-q^{10}}\cos 5x+\&c.\right\}}$$

$$=\frac{1+\dfrac{4q}{1+q^2}\cos x+\dfrac{4q^2}{1+q^4}\cos 2x+\dfrac{4q^3}{1+q^6}\cos 3x+\&c.}{1-\dfrac{4q^2}{1+q^4}+\dfrac{4q^4}{1+q^8}-\dfrac{4q^6}{1+q^{12}}+\&c.}\quad\ldots\ldots\ldots(1);$$

and it is noteworthy on account of the peculiarity, that although the exponent on the left-hand side changes sign if q be replaced by q^{-1}, yet