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THE THEORY OF FUNCTIONS.

Theory of Functions of a Complex Variable. By Dr. A. R. Forsyth. (Cambridge University Press, 1893.)

WHAT is the theory of functions about? This question may be heard now and again from a mathematical student; and if, by way of a partial reply, it be said that the elements of the theory of functions forms the basis on which the whole of that part of pure mathematics which deals with continuously varying quantity rests, the answer would not be too wide nor would it always imply too much.

It cannot be denied that the teaching of pure mathematics in this country has followed curiously restricted lines. While in geometry and the theory of forms the student has for many years past had the advantage of excellent English text-books, the general theory of functions has been entirely unrepresented till the appearance of the treatise whose title stands at the head of this notice. Of treatises on special classes of functions, if we omit those written purely with a view to applications, Cayley's "Elliptic Functions," published in 1876, is the sole representative; while till last year there was no work on the theory of numbers. The theory of groups, and its applications to the theory of equations, is still unrepresented in native English mathematical literature, though here we have the translations of Prof. Klein's "Vorlesungen über das Icosaeder," and Herr Netto's "Substitutionentheorie," published, the one in 1888, and the other last year. At Cambridge, and probably to a great extent in other centres, the teaching and the course of study of individual students have tended on the whole to follow the lines of the available English text-books, and where these have been incomplete or entirely wanting there has, till very recent years, been no sufficient introduction to the corresponding subjects.

Why a subject of such fundamental importance for the advancement of pure mathematics as the theory of functions should have happened to fall into this latter class, it is not easy to tell. It may be said to have been first put on a secure footing by Cauchy's great memoir on integrals taken between imaginary limits, which was published in 1825. Many advances were made by a number of eminent mathematicians in the following years, and the study of the subject received a great impetus from the new and very fascinating method of presenting it which Riemann gave in his famous memoirs on the theory of functions of a complex variable (1851), and on the theory of the Abelian functions (1857).

Weierstrass and his pupils, again, developed their theory from a standpoint which is essentially distinct from that of either Cauchy or Riemann. The growth of the subject during the last thirty years has been remarkable, and it is probably safe to say that the foreign literature of the subject is now more extensive than that of any other branch of pure mathematics. The number of text-books that have been published directly on the subject is wonderful in itself, and more so when it is remembered that almost every foreign treatise on the Differential and Integral Calculus contains some introduction to the theory of functions.

If there is any justice in the preceding remarks, the want of a treatise on this subject has too long caused a serious gap in our mathematical literature; and it may be at once said that Dr. Forsyth's book supplies that want so completely that it is not likely to be felt again for a long time to come.

Among the large number of foreign treatises above referred to are several which, in their own line, it would be difficult to improve upon; but they all, or nearly all, deal with the subject from a single point of view, being indeed written with that intention. Dr. Forsyth, on the other hand, has aimed at giving a complete introduction to the theory; and it may safely be said that, with his book as a guide, the task of the student who wishes to enable himself to follow its various recent developments will have lost half its difficulty.

The bringing of the various parts of the subject, and the different points of view from which they may be approached, into their proper connection with each other has here been done in the most masterly way; while though Dr. Forsyth expressly disclaims in the preface to have dealt at length with anything but the general theory, he has carried the developments of the subject in the direction of doubly-periodic and allied functions, Abelian integrals, and automorphic functions to a point from which the student can have no difficulty in passing on to the study of any recent work done in these branches.

It is impossible in the limits of a short article to give any complete account of a book extending to over 700 pages, but some attempt may be made to describe the order of treatment. The first four chapters are devoted to the simpler properties of uniform functions, their expansion in power-series and their integration. Chapters v., vi. and vii. deal with uniform transcendal functions, giving the principal results of the investigations of Weierstrass and Mittag-Leffler. In this connection the very remarkable result due to Weierstrass is given, which is expressed by him in the following words:—"Dass der Begriff einer monogenen Function einer complexen Veränderlichen mit dem Begriff einer durch arithmetische Grössenoperationen ausdrückbaren Abhängigkeit sich nicht vollständig deckt." The writer of a recent criticism in this journal would probably say that this statement deals only with the morbid pathology of mathematics; but the pure mathematician at all events should surely know, as far as possible, what is implied in the word function.

Non-uniform functions are introduced in chapter viii. They are regarded, to begin with, as arising from the various continuations of a power-series, the most general point of view that can be taken; Riemann's method of dealing with algebraic functions and their integrals not being introduced till considerably later. The following chapter deals with the integrals of non-uniform functions; and from the particular examples given arise some of the simplest singly- and doubly-periodic functions, whose properties, when uniform, are discussed in Chaps. x., xi., and xii. This part of the subject aptly ends with a demonstration, due to the author, of the theorem that if $f(u)$, $f(v)$, and $f(u+v)$ are connected by an algebraical equation with constant coefficients, $f(u)$ must be either an algebraic, a simply-periodic, or a doubly-periodic function of u. The proof of this important theorem by

Weierstrass, to whom it is due, has never been printed; and the only published proof, besides the one which Dr. Forsyth gives, appears in a paper by M. Phragmen in vol. vii. of the "Acta Mathematica," and is on entirely different lines. Whether either proof is entirely satisfactory is a point on which differences of opinion may conceivably occur, though of course there is no doubt as to the truth of the theorem itself.

Chap. xiv., which is headed "Connectivity of Surfaces," is purely geometrical, and strictly has nothing to do with the theory of functions. It was however necessary for the author to introduce such a digression if the following chapters dealing with Riemann's theory were to be understood, since there is no treatise to which reference could be made for the various theorems and results that have to be used. The chief properties of a Riemann's surface, regarded as arising from an algebraical equation between the variables, are discussed in Chap. xv. Though there is no difficulty in conceiving the geometrical nature of a Riemann's surface from a description, the relation between the surface and the set of functions (algebraische Gebilde) whose study it is intended to simplify is not so readily grasped at first by the student; and it would not perhaps have been amiss to have dealt with this relation in one or two simple cases, at some length, as an introduction to this part of the subject. In Chap. xvi., the surface still being regarded as defined by a given equation, the properties of uniform functions on the surface, and of their integrals, is investigated.

From this point to the end of the book we have to do, more or less directly, with the fundamentally new conception of Riemann which has been so wonderfully developed during the last ten or fifteen years. The Riemann's surface, as defined by a given equation, affords a most convenient means of study of a system of connected functions. Suppose, however, the surface to be given quite independently of any equation. The possibility at once suggests itself that the surface may serve as the definition of a set of connected functions. Riemann's own demonstration that this is the case has since been shown to be faulty, but the conception is an invaluable one, and it has been placed on a secure foundation by Schwartz (and others), by means of the so-called existence theorem. Chap. xvii. is entirely occupied with the proof of this theorem, and in Chap. xviii. follow the investigations with respect to the form and nature of the integrals and uniform functions, so shown to exist, on a Riemann's surface given arbitrarily.

Chaps. xix. and xx. deal at length with the theory of conformal representation. This forms one of the most obviously interesting parts of the subject, and is also one of those which lend themselves most readily to the purposes of application; and it is to be noted that, although owing to necessities of arrangement these chapters occur near the end of the book, the author suggests that, on a first reading, Chap. xix. should be taken at an early stage.

The last chapter in the book gives an introduction to the theory of automorphic functions, the previous one being taken up by a necessary digression on groups of linear substitutions. Dr. Forsyth follows M. Poincaré in actually obtaining analytical expressions for the functions in the form of the ratio of infinite series, analogous to the expressions for elliptic functions as ratios of the theta-functions. These analytical expressions, though of great interest, are too complicated in form to be readily used for deducing the properties of the functions they represent, so that their properties must be inferred from their quasi-geometrical definition by means of a "fundamental region"; and this is essentially the method of dealing with them used by Prof. Klein.

In thus shortly stating the contents, or rather the headings, of the successive chapters some risk is run of representing the book as a mere compilation. Nothing could possibly be further from the truth. From the nature of the case it is inevitable that the greater portion of the book should be taken up with detailing the results of other writers, but Dr. Forsyth has done this in a most independent way. The book is instinct all through with an original spirit; in numerous instances, where clearness or conciseness were to be gained, the author has modified or completely altered the usually-given proofs, while, as has been already stated, the various parts of the subject have been brought together, and the many different ways of dealing with them have been used, in such a way that the theory is presented to the reader as a connected and harmonious whole. Dr. Forsyth is to be warmly congratulated on having brought to so successful a conclusion what must have been an extremely arduous task. If it is not ungracious to "ask for more" so soon, we may express the hope that he will now go on to deal, as completely and successfully, with functions defined by differential equations.

The book itself is beautifully printed and the figures, many of which must have required careful drawing, are well reproduced. The table of contents is sufficiently complete to form a sort of *précis* of the whole; and lastly, we have to be grateful for three separate indices. The first of these, an index to all the technical terms used in the book, whether English or foreign, is a most useful addition; especially for those who wish to use the book without reading right through it.

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TINCTORIAL ART AND SCIENCE.

A Manual of Dyeing: for the use of Practical Dyers, Manufacturers, Students, and all interested in the Art of Dyeing. By Edmund Knecht, Ph.D., Christopher Rawson, F.I.C., and Richard Loewenthal, Ph.D. (London: Charles Griffin and Co., 1893.)

THE present work consists of three volumes, two of letterpress, interspersed with illustrations of plant, which run to over 900 pages, and a third volume containing specimens of dyed fabrics. It is a substantial contribution to an important branch of technology, and the authors have succeeded fairly well in meeting the requirements of the various classes of readers for whose use the work has been written. The first general impression produced on looking through the volumes is one of satisfaction that the subject is handled in a more scientific way than has hitherto been the case in such works. The only feeling of disappointment to which the consideration of the book gives rise is in no way attributable to the authors, but is due to the circumstance that so little is known about the scientific relationship between a colouring-matter and the fabric which is dyed thereby. All that is known about the theory of dyeing is ably stated in the introductory chapter, and one of the