tion of S and S′ (*e.g.*, if S be an ellipse or a parabola, S′ may be any pair of straight lines which meet within the conic), then Δ′ vanishes, and one value of κ is zero, the other two being given by the equation $\Delta\kappa^2+\Theta\kappa+\Theta'=0$, so that their difference is $\frac{\sqrt{\Theta^2-4\Delta\Theta'}}{\Delta}$, and, by (8), the area of the quadrangle formed by the points of intersection is

$$\frac{4\sqrt{-C'}}{\nu^2-4CC'}\sqrt{\Theta^2-4\Delta\Theta'}.$$

5. If, in this last case (4), the pair of straight lines S′ be supposed to meet on S, then also $\Theta'=0$, and the area of the triangle formed is $-\frac{\Theta}{\delta}\sqrt{-C'}$.

6. If $S=0$, $S'=0$ each represent a pair of parallel straight lines, then C, C′ vanish as well as Δ, Δ′; so that, by (7), the area of the parallelogram formed is $\sqrt{\frac{\Theta\Theta'}{\nu^3}}$.

Geometrical Illustration of a Theorem relating to an Irrational Function of an Imaginary Variable. By Prof. CAYLEY.

[*Read May* 11*th*, 1876.]*

If we have v, a function of u, determined by an equation $f(u, v)=0$, then to any given imaginary value $x+iy$ of u there belong two or more values, in general imaginary, $x'+iy'$ of v: and for the complete understanding of the relation between the two imaginary variables, we require to know the series of values $x'+iy'$ which correspond to a given series of values $x+iy$, of v, u respectively. We must for this purpose take x, y as the coordinates of a point P in a plane Π, and x', y' as the coordinates of a corresponding point P′ in another plane Π′. The series of values $x+iy$ of u is then represented by means of a curve in the first plane, and the series of values $x'+iy'$ of v by means of a corresponding curve in the second plane. The correspondence between the two points P and P′ is of course established by the two equations into which the given equation $f(x+iy, x'+iy')=0$ breaks up, on the assumption that x, y, x', y' are all of them real. If we assume that the coefficients in the equation are real, then the two equations are

$$f(x+iy,\ x'+iy')+f(x-iy,\ x'-iy')=0,$$
$$f(x+iy,\ x'+iy')-f(x-iy,\ x'-iy')=0;$$

viz., if in these equations we regard either set of coordinates, say (x, y),

* Not received in time to be placed earlier in the Proceedings.

as constants, then the other set (x', y') are the coordinates of any real point of intersection of the curves represented by these equations respectively.

I consider the particular case where the equation between u, v is $u^2+v^2=a^2$: we have here $(x+iy)^2+(x'+iy')^2=a^2$: so that to a given point P in the first plane there correspond in general two points P'_1, P'_2 in the second plane: but to each of the points A and B, coordinates $(a, 0)$ and $(-a, 0)$, there corresponds only a single point in the second plane.

We have here a particular case of a well known theorem: viz., if from a given point P we pass by a closed curve not containing within it either of the points A or B, back to the initial point P, we pass in the other plane from P'_1 by a closed curve back to P'_1; and similarly from P'_2 by a closed curve back to P'_2: but if the closed curve described by P contain within it A or B, then, in the other plane, we pass continuously from P'_1 to P'_2; and also continuously from P'_2 to P'_1.

The relations between (x, y), (x', y') are

$$x'^2-y'^2=a^2-(x^2-y^2),$$
$$x'y'=-xy,$$

whence also $\quad (x'^2+y'^2)=a^4-2a^2(x^2-y^2)+(x^2+y^2)^2.$

And if the point (x, y) describe a curve $x^2+y^2=\phi(x^2-y^2)$, then will the point (x', y') describe a curve $x'^2+y'^2=\psi(x'^2-y'^2)$ obtained by the elimination of x^2-y^2 from the two equations

$$x'^2-y'^2=\quad a^2-\quad (x^2-y^2),$$
$$(x'^2+y'^2)^2=\quad a^4-2a^2(x^2-y^2)+\phi(x^2-y^2),$$

viz., this is $\quad (x'^2+y'^2)^2=-a^4+2a^2(x'^2-y'^2)+\phi\{a^2-(x'^2-y'^2)\}.$

In particular, if the one curve be $(x^2+y^2)^2=\alpha+\beta(x^2-y^2)$; then the other curve is $\quad (x'^2+y'^2)^2=-a^4+2a^2(x'^2-y'^2)+\alpha+\beta\{a^2-(x'^2-y'^2)\},$

that is, $\quad (x'^2+y'^2)^2=\alpha'+\beta'(x'^2-y'^2),$

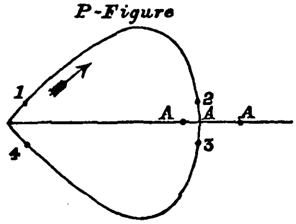
where $\quad \alpha'=-a^4+\beta a^2+\alpha,\quad \beta'=2a^2-\beta;$

or, writing for greater simplicity $a=1$, then $\alpha'=-1+\alpha+\beta$, $\beta'=2-\beta$, or, in particular, if $\alpha=0$, then $\alpha'=-1+\beta$, $\beta'=2-\beta$.

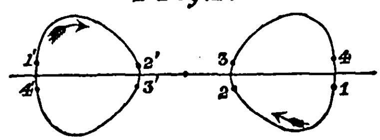
Supposing successively $\beta<1$, $\beta=1$, and $\beta>1$, then in each case P describes a closed curve or half figure-of-eight, as shown in the annexed P-figure; but in the first case the point A is inside the curve, in the second case on it, and in the third case outside it, as shown by the letters A, A, A of the figure; and, corresponding to the three cases respectively, we have the three P′-figures, the curve in the first of them consisting of two ovals, in the second of them being a figure of eight, and in the third a twice-indented or pinched oval: the small figures

1, 2, 3, 4 in the P-figure, and 1, 2, 3, 4 and 1′, 2′, 3′, 4′ in the P′-figures serve to show the corresponding positions of the points P and P'_1, P'_2

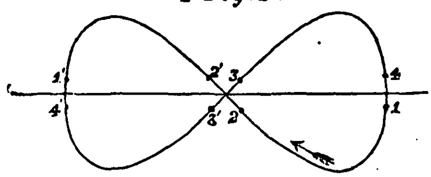
P-Figure



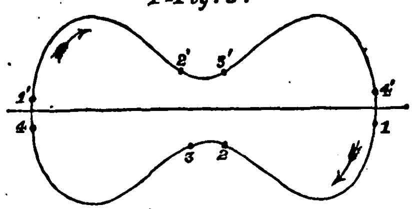
P′-Fig. 1.



P′-Fig. 2.



P′-Fig. 3.



respectively; and the courses are further indicated by the arrows. And we thus see how the two separate closed curves described by P'_1 and P'_2, as in figure 1, change into the single closed curve described one half of it by P'_1 and the other half of it by P'_2 as in figure 3.

March 8th, 1877.

C. W. MERRIFIELD, Esq., F.R.S., Vice-President, in the Chair.

Mr. C. Pendlebury, B.A., St. John's College, Cambridge, and Mathematical Master in St. Paul's School, was proposed for election; and Mr. R. F. Davis was admitted into the Society.

The following communications were made:—

"On a New View of Pascal's Hexagram": Mr. T. Cotterill.

"On a Class of Integers expressible as the sum of Two Integral Squares": Mr. T. Muir.

"Some Properties of the Double-Theta Functions" (founded on papers by Goepel and Rosenhain): Prof. Cayley.

"A Property of an Envelope": Mr. J. J. Walker.

The following presents were received:

"Elementary Treatise on the Differential Calculus," by B. Williamson, F.R.S., 1877: from the Author.

"Educational Times," March.